Multivalued generalizations of fixed point results in fuzzy metric spaces

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Received: October 22, 2014 / Revised: July 10, 2015 / Published online: December 28, 2015

Abstract. This paper attempts to prove fixed and coincidence point results in fuzzy metric space using multivalued mappings. Altering distance function and multivalued strong \{b_n\}-fuzzy contraction are used in order to do that. Presented theorems are generalization of some well known single valued results. Two examples are given to support the theoretical results.

Keywords: fixed point, coincidence point, fuzzy metric space, multivalued mappings, altering distance.

1 Introduction

Banach contraction principle [1] was motivation for many fixed point studies in various spaces [2,3,4,5,6,7,10,12,13,14,15,16,17,18,22,24,25,26,30]. In particular, multivalued generalization of this principle in metric space \((X, d)\) is done by Nadler [27] on the following way: there exist \(k \in (0, 1)\) so that, for every \(x, y \in X\),

\[H(fx, fy) \leq kd(x, y),\]    \hspace{1cm} (1)

where \(H\) is Hausdorff–Pompeiu metric and \(f\) is multivalued mapping from \(X\) to the family of its non-empty, closed and bounded subsets. Later on, the probabilistic versions

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*This research was supported by grant No. MNTRRS-174009, III 44006 and PSNTR project No. 114-451-1084.

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of condition (1) are given in [12, 13, 15, 16], where notions of weakly demicompact mapping, $f$-strongly demicompact and weakly commuting mapping are introduced. Further, Hausdorff distance between sets in fuzzy metric spaces is introduced [22] and used in [7] for study of existence of coincidence point using two multivalued and one single valued mappings.

Also, Banach contraction principle in metric spaces is improved by Khan, Swaleh and Sessa [19], where control function, called altering distance function, is introduced. This type of function is used in [30] in fuzzy metric space $(X, M, T)$ with the following condition:

$$\varphi(M(fx, fy, t)) \leq k(t) \cdot \varphi(M(x, y, t)), \quad x, y \in X, \quad t > 0, \quad 0 < k(t) < 1,$$

where $\varphi$ is altering distance function. Note that condition (2) is improved in [5]. Moreover, many functions of this type are used in the study of fixed point problems [3, 4, 26].

Another classes of contraction, so called (strong) $\{b_n\}$-probabilistic contraction, are introduced in [6, 24] and used in the study of fixed point problems in multivalued case in probabilistic spaces [25].

Our aim in present paper is to study the multivalued generalization in fuzzy metric spaces of results given in [6, 19, 30]. First, we use altering distance function in the style of condition (2) to obtain coincidence point results. That is realized through two theorems using strong fuzzy metric space with $t$-norm of $H$-type in the first and $f$-strongly demicompact mappings in the second one. On the other side, result given in [6] is transferred to multivalued case by introducing multivalued strong $\{b_n\}$-fuzzy contraction.

2 Preliminaries

In order to make paper more readable, first, we list the definitions of basic notions important to further work. Using the results of Menger and Zadeh [23, 31], Kramosil and Michalek [21] introduced the notion of fuzzy metric space. Later, George and Veermani [8, 9] modified their definition in way to associate each fuzzy metric to a Hausdorff topology.

Definition 1. (See [29].) A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm ($t$-norm) if the following conditions are satisfied:

(T1) $T(a, 1) = a$, $a \in [0, 1]$,
(T2) $T(a, b) = T(b, a)$, $a, b \in [0, 1]$,
(T3) $a \geq b$, $c \geq d \Rightarrow T(a, c) \geq T(b, d)$, $a, b, c, d \in [0, 1]$,
(T4) $T(a, T(b, c)) = T(T(a, b), c)$, $a, b, c \in [0, 1]$.

Definition 2. (See [21].) The 3-tuple $(X, M, T)$ is said to be a KM fuzzy metric space in the sense of Kramosil and Michalek if $X$ is an arbitrary set, $T$ is a $t$-norm and $M$ is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions:

(KM1) $M(x, y, 0) = 0$, $x, y \in X$,
(KM2) $M(x, y, t) = 1$, $t > 0 \iff x = y$. 

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Fixed point results in fuzzy metric spaces

(KM3) $M(x, y, t) = M(y, x, t)$, $x, y \in X$, $t > 0$.

(KM4) $T(M(x, y, t), M(y, z, s)) \leq M(x, z, t + s)$, $x, y, z \in X$, $t, s > 0$.

(KM5) $M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is left-continuous for every $x, y \in X$.

**Definition 3.** (See [8, 9].) The 3-tuple $(X, M, T)$ is said to be a fuzzy metric space in the sense of George and Veeramani if $X$ is an arbitrary set, $T$ is a continuous t-norm and $M$ is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions:

(GV1) $M(x, y, t) > 0$, $x, y \in X$, $t > 0$.

(GV2) $M(x, y, t) = 1$, $t > 0$ $\Leftrightarrow$ $x = y$.

(GV3) $M(x, y, t) = M(y, x, t)$, $x, y \in X$, $t > 0$.

(GV4) $T(M(x, y, t), M(y, z, s)) \leq M(x, z, t + s)$, $x, y, z \in X$, $t, s > 0$.

(GV5) $M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is continuous for every $x, y \in X$.

If (GV4) is replaced by condition

(GV4') $T(M(x, y, t), M(y, z, t)) \leq M(x, z, t)$, $x, y, z \in X$, $t > 0$,

then $(X, M, T)$ is called a strong fuzzy metric space [11].

Moreover, if $(X, M, T)$ is a fuzzy metric space, then $M$ is a continuous function on $X \times X \times (0, \infty)$ [28] and $M(x, y, \cdot)$ is non-decreasing for all $x, y \in X$ [10].

If $(X, M, T)$ is a fuzzy metric space, then $M$ generates the Hausdorff topology on $X$ (see [8, 9]) with base of open sets $\{U(x, r, t) : x \in X, r \in (0, 1), t > 0\}$, where $U(x, r, t) = \{y : y \in X, M(x, y, t) > 1 - r\}$.

A function $\varphi : [0, 1] \rightarrow [0, 1]$ is called an altering distance function [26, 30] if it satisfies the following properties:

(AD1) $\varphi$ is strictly decreasing and left continuous;

(AD2) $\varphi(\lambda) = 0$ if and only if $\lambda = 1$.

It is obvious that $\lim_{\lambda \rightarrow 1^{-}} \varphi(\lambda) = \varphi(1) = 0$.

**Definition 4.** (See [8, 9]) Let $(X, M, T)$ be a fuzzy metric space.

(a) A sequence $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $(X, M, T)$ if, for every $\varepsilon \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$, $n, m \geq n_0$, $t > 0$.

(b) A sequence $\{x_n\}_{n \in \mathbb{N}}$ converges to $x$ in $(X, M, T)$ if, for every $\varepsilon \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - \varepsilon$, $n \geq n_0$, $t > 0$. Then we say that $\{x_n\}_{n \in \mathbb{N}}$ is convergent. Every convergent sequence is a Cauchy sequence.

(c) A fuzzy metric space $(X, M, T)$ is complete if every Cauchy sequence in $(X, M, T)$ is convergent.

**Definition 5.** (See [14].) Let $T$ be a t-norm and $T_n : [0, 1] \rightarrow [0, 1]$, $n \in \mathbb{N}$, be defined in the following way:

$$T_1(x) = T(x, x), \quad T_{n+1}(x) = T(T_n(x), x), \quad n \in \mathbb{N}, \quad x \in [0, 1].$$

We say that t-norm $T$ is of $H$-type if the family $\{T_n(x)\}_{n \in \mathbb{N}}$ is equicontinuous at $x = 1$. 
Each t-norm $T$ can be extended (see [20]) (by associativity) in a unique way to an $n$-ary operation taking for $(x_1, \ldots, x_n) \in [0,1]^n$ the values 

$$T_1^n x_i = 1, \quad T^n_{i=1} x_i = T(T^n_{i=1} x_i, x_n).$$

A t-norm $T$ can be extended to a countable infinite operation taking for any sequence $(x_n)_{n \in \mathbb{N}}$ from $[0,1]$ the value 

$$T_1^\infty x_i = \lim_{n \to \infty} T^n_{i=1} x_i.$$ 

The sequence $(T^n_{i=1} x_i)_{n \in \mathbb{N}}$ is non-increasing and bounded from below. Hence, the limit $T_1^\infty x_i$ exists.

In the fixed point theory (see [15, 17]), it is of interest to investigate the classes of t-norms $T$ and sequences $(x_n)_{n \in \mathbb{N}}$ from the interval $[0,1]$ such that 

$$\lim_{n \to \infty} x_n = 1$$

and 

$$\lim_{n \to \infty} T_{i=n}^\infty x_i = \lim_{n \to \infty} T_{i=1}^\infty x_{n+1} = 1. \quad (3)$$

In [15], the following proposition is obtained.

**Proposition 1.** Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of numbers from $[0,1]$ such that $\lim_{n \to \infty} x_n = 1$ and t-norm $T$ is of $H$-type. Then $\lim_{n \to \infty} T_{i=n}^\infty x_i = \lim_{n \to \infty} T_{i=1}^\infty x_{n+1} = 1.$

**Definition 6.** (See [12,15].) Let $(X, M, T)$ be a fuzzy metric space, $A$ a non-empty subset of $X$ and $f : A \to 2^X \setminus \{\emptyset\}$. The mapping $f$ is weakly demicompact if, for every sequence $(x_n)_{n \in \mathbb{N}}$ from $A$ such that $x_{n+1} \in f x_n$, $n \in \mathbb{N}$, and $\lim_{n \to \infty} M(x_{n+1}, x_n, t) = 1$, $t > 0$, there exists a convergent subsequence $(x_{n_k})_{k \in \mathbb{N}}$.

Throughout the paper by $C(X)$ is denoted a family of all non-empty and closed subsets of $X$.

**Definition 7.** (See [22].) Let $(X, M, T)$ be a fuzzy metric space, $A$ a non-empty subset of $X$, $f : A \to A$ and $F : A \to C(A)$. The mapping $F$ is a $f$-strongly demicompact if, for every sequence $(x_n)_{n \in \mathbb{N}}$ from $A$ such that $x_{n+1} \in f x_n$, $n \in \mathbb{N}$, and $\lim_{n \to \infty} M(f x_n, y_n, t) = 1$, $t > 0$, for some sequence $(y_n)_{n \in \mathbb{N}}$, $y_n \in F x_n$, $n \in \mathbb{N}$, there exists a convergent subsequence $(f x_{n_k})_{k \in \mathbb{N}}$.

**Definition 8.** (See [13,15].) A mapping $F : X \to C(X)$ is weakly commuting with $f : X \to X$ if, for all $x \in X$, it holds $f(F x) \subseteq F(f x)$.

### 3 Main results

#### 3.1 Multivalued mappings using altering distance

Main result of this section is an extension of results given in [30] to the case of multivalued mappings.
Let $A$ and $B$ be two nonempty subsets of $X$, define the Hausdorff–Pompeiu fuzzy metric as

$$\widetilde{M}(A, B, t) = \min \left\{ \inf_{x \in A} E(x, B, t), \inf_{y \in B} E(y, A, t) \right\}, \quad t > 0,$$

where $E(x, B, t) = \sup_{y \in B} M(x, y, t)$.

**Theorem 1.** Let $(X, M, T)$ be a complete strong fuzzy metric space and $T$ is $t$-norm of $H$-type. Let $f : X \to X$ be a continuous mapping and $F, G : X \to C(X)$ are weakly commuting with $f$. If there exist $k : (0, \infty) \to (0, 1)$ and altering distance function $\varphi$ such that the following condition is satisfied:

$$\varphi(\widetilde{M}(Fx, Gy, t)) \leq k(t) \cdot \varphi(M(fx, fy, t)), \quad x, y \in X, \ x \neq y, \ t > 0,$$

then there exists $x \in X$ such that $f x \in F x \cap G x$.

**Proof.** Let $x_0 \in X$. Since $Fx_0$ is a non-empty subset of $X$, there exist $x_1 \in X$ such that $fx_1 \in Fx_0$. Let $t_0 > 0$ be arbitrary. Continuity of $M$ and the fact that $k(t) < 1, t > 0$, implies that, for $\varepsilon_1 > 0$, the following inequality holds:

$$k(t_0) \cdot \varphi(M(fx_0, fx_1, t_0)) < \varphi(M(fx_0, fx_1, t_0) + \varepsilon_1).$$

By definition of Hausdorff fuzzy metric, for $\varepsilon_1 > 0$ given in (5), there exist $x_2 \in X$, $fx_2 \in Gx_1$ and $l_1 \in \mathbb{N} \setminus \{0\}$ such that

$$\widetilde{M}(Fx_0, Gx_1, t_0) \leq M(fx_1, fx_2, t_0) + \frac{\varepsilon_1}{2^l_1}.$$  \hspace{1cm} (6)

Now, by (4), (5) and (6), using that $\varphi$ is strictly decreasing, we conclude that

$$M(fx_0, fx_1, t_0) < M(fx_1, fx_2, t_0).$$ \hspace{1cm} (7)

Similarly, we can find $x_3 \in X$, $fx_3 \in Fx_2$, and $l_2 \in \mathbb{N}$, $l_2 > l_1$ such that

$$k(t) \cdot \varphi(M(fx_1, fx_2, t_0)) < \varphi(M(fx_1, fx_2, t_0) + \varepsilon_1)$$ \hspace{1cm} (8)

and

$$\widetilde{M}(Gx_1, Fx_2, t_0) \leq M(fx_2, fx_3, t_0) + \frac{\varepsilon_1}{2^l_2}.$$ \hspace{1cm} (9)

By (8) and (9) we have

$$M(fx_1, fx_2, t_0) < M(fx_2, fx_3, t_0).$$ \hspace{1cm} (10)

Repeating the procedure presented above, we define a sequence $\{x_n\}_{n \in \mathbb{N}}$ from $X$ and strictly increasing sequence $\{l_n\}_{n \in \mathbb{N}}$ from $\mathbb{N}$ such that the following conditions are satisfied:

(i) $fx_{2n+1} \in Fx_{2n}, \ fx_{2n+2} \in Gx_{2n+1}, \ n \in \mathbb{N},$

(ii) $M(fx_{n-1}, fx_n, t) < M(fx_n, fx_{n+1}, t), \ t > 0, \ n \in \mathbb{N},$

where
\[
\bar{M}(Fx_{2n}, Gx_{2n+1}, t) \leq M(fx_{2n+1}, fx_{2n+2}, t) + \frac{\varepsilon_1}{2^n}, \quad t > 0, \ n \in \mathbb{N}.
\] (11)

Hence, the sequence \(\{M(fx_n, fx_{n+1}, t)\}_{n \in \mathbb{N}}, t > 0\), is non-decreasing and bounded, so there exist \(a : (0, \infty) \to [0, 1]\) such that
\[
\lim_{n \to \infty} M(fx_n, fx_{n+1}, t) = a(t), \quad t > 0.
\] (12)

By (4), (11) and (12), for \(n \in \mathbb{N}, t > 0\), we have
\[
\varphi(M(fx_{2n+1}, fx_{2n+2}, t) + \frac{\varepsilon_1}{2^n}) < \varphi(\bar{M}(Fx_{2n}, Gx_{2n+1}, t)) < k(t) \cdot \varphi(M(fx_{2n}, fx_{2n+1}, t)).
\] (13)

Letting \(n \to \infty\) in (13), we get
\[
\varphi(a(t)) \leq k(t) \cdot \varphi(a(t)), \quad t > 0,
\] (14)

and we conclude that \(\varphi(a(t)) = 0\) for all \(t > 0\) so that \(a \equiv 1\).

Further, we will prove that \(\{fx_n\}_{n \in \mathbb{N}}\) is a Cauchy sequence. Let \(\varepsilon > 0\) and \(s \in \mathbb{N}\).

Since t-norm \(T\) is of \(H\)-type, using (12) and Proposition 1, we have that there exist \(n_0 \in \mathbb{N}\) such that
\[
T_{i=n}^{\infty} M(fx_i, fx_{i+1}, t) > 1 - \varepsilon, \quad t > 0, \ n \geq n_0.
\] (15)

Since \((X, M, T)\) is strong fuzzy metric space and \(\{T_{i=1}^{n} M(fx_i, fx_{i+1}, t)\}_{n \in \mathbb{N}}\) is nonincreasing sequence, by (15), we have that
\[
M(fx_{n+s+1}, fx_n, t) \geq T_{i=n}^{n+s} M(fx_i, fx_{i+1}, t) > 1 - \varepsilon, \quad t > 0, \ n \geq n_0.
\] (16)

So, \(\{fx_n\}_{n \in \mathbb{N}}\) is a Cauchy sequence and, since the space \((X, M, T)\) is complete, there exist \(x \in X\) such that
\[
x = \lim_{n \to \infty} fx_n.
\] (17)

It remains to prove that \(fx \in Fx \cap GX\). As \(Fx \cap GX = \overline{Fx \cap GX}\), we need to show that, for every \(t > 0\) and \(\lambda \in (0, 1)\), there exists \(r_1 = r_1(t, \lambda) \in Fx\) and \(r_2 = r_2(t, \lambda) \in GX\) such that \(r_1, r_2 \in U(fx, t, \lambda)\), i.e. \(M(fx, r_1, t) > 1 - \lambda\) and \(M(fx, r_2, t) > 1 - \lambda\).

Let \(t_0 > 0\) and \(\lambda \in (0, 1)\). Since t-norm \(T\) is continuous, it follows that there exist \(\delta = \delta(\lambda) \in (0, 1)\) such that
\[
T(1 - \delta, T(1 - \delta, 1 - \delta)) > 1 - \lambda.
\] (18)

By the continuity of \(f\) and (17) there exist \(n_1 \in \mathbb{N}\) such that
\[
M\left(fx, \frac{fx_{2n} + t_0}{3}\right) > 1 - \delta, \quad n \geq n_1.
\] (19)
By (12) there exists $n_2 \in \mathbb{N}$ such that
\[
M\left(\frac{f f x_{2n_1} + f f x_{2n_1 + 1}}{3}, t_0\right) > 1 - \delta, \quad n \geq n_2.
\]
Since $f$ is weakly commuting with $F$, we have
\[
ff x_{2n+1} \in f(F x_{2n}) \subseteq F(f x_{2n}). \quad (20)
\]
Also, there exist $\varepsilon^* \in (0, 1)$ such that
\[
k\left(\frac{t_0}{3}\right) \cdot \varphi\left(M\left(\frac{f x, f f x_{2n_0}}{3}, \frac{t_0}{3}\right)\right) < \varphi\left(M\left(\frac{f x, f f x_{2n_0}}{3}, \frac{t_0}{3}\right) + \varepsilon^*\right) \quad (21)
\]
for arbitrary $n_0 \geq \max\{n_1, n_2\}$. By (20) and definition of Hausdorff fuzzy metric there exist $r_2 \in G x$ such that, for $\varepsilon^* > 0$ (defined in (21)), the following is satisfied:
\[
\tilde{M}\left(G x, F(f x_{2n_0}), \frac{t_0}{3}\right) \leq M\left(r_2, f f x_{2n_0 + 1}, \frac{t_0}{3}\right) + \varepsilon^*. \quad (22)
\]
By (4), (20) and (21) we have:
\[
\varphi\left(M\left(r_2, f f x_{2n_0 + 1}, \frac{t_0}{3}\right) + \varepsilon^*\right)
\leq \varphi\left(M\left(G x, F(f x_{2n_0}), \frac{t_0}{3}\right)\right) \leq k\left(\frac{t_0}{3}\right) \cdot \varphi\left(M\left(f x, f f x_{2n_0}, \frac{t_0}{3}\right)\right)
\leq \varphi\left(M\left(f x, f f x_{2n_0}, \frac{t_0}{3}\right) + \varepsilon^*\right).
\]
Now, by (19) follows that
\[
M\left(r_2, f f x_{2n_0}, \frac{t_0}{3}\right) > M\left(f x, f f x_{2n_0}, \frac{t_0}{3}\right) > 1 - \delta.
\]
Finally, using (18), we get
\[
M(f x, r_2, t_0) \geq T\left(M\left(f x, f f x_{2n_0}, \frac{t_0}{3}\right), M\left(f f x_{2n_0 + 1}, \frac{t_0}{3}\right), M\left(f f x_{2n_0 + 1}, r_2, \frac{t_0}{3}\right)\right)
\geq T(1 - \delta, T(1 - \delta, 1 - \delta)) > 1 - \lambda.
\]
So, $r_2 \in U(f x, t_0, \lambda)$ for arbitrary $t_0 > 0$ and $\lambda \in (0, 1)$, i.e. $f x \in G x$. Similarly, it can be shown that $r_1 \in U(f x, t, \lambda)$, $t > 0$, $\lambda \in (0, 1)$, which implies that $f x \in F x$, too. \hfill \square
Theorem 2. Let \((X, M, T)\) be a complete fuzzy metric space and \(f : X \to X\) be a continuous mapping. Let \(F, G : X \to C(X)\) are weakly commuting with \(f\) and \(F\) or \(G\) is \(f\)-strongly demicompact. If, for some \(k : (0, \infty) \to (0, 1)\) and altering distance function \(\varphi\), the following condition is satisfied:

\[
\varphi(M(Fx, Gx, t)) \leq k(t) \cdot \varphi(M(fx, fy, t)), \quad x, y \in X, \quad x \neq y, \quad t > 0,
\]

then there exists \(x \in X\) such that \(fx \in Fx \cap Gx\).

Proof. The proof is similar with that of the Theorem 1, except in the part related to Cauchy sequence. Namely, since \(F\) or \(G\) is \(f\)-strongly demicompact, \(fx_{2n+1} \in Fx_{2n}\) or \(fx_{2n+2} \in Gx_{2n+1}\) and \(\lim_{n \to \infty} M(fx_{2n}, fx_{2n+1}, t) = 1, t > 0\), we conclude that there exist convergent subsequence \(\{fx_{2n_p}\}_{p \in \mathbb{N}}\) or \(\{fx_{2n_p+1}\}_{p \in \mathbb{N}}\), respectively, such that

\[
\lim_{p \to \infty} fx_{2n_p} = x.
\]

The last part of the proof is analogous as in Theorem 1, where instead of sequence \(\{fx_n\}_{n \in \mathbb{N}}\), we deal with subsequences \(\{fx_{2n_p}\}_{p \in \mathbb{N}}\) and \(\{fx_{2n_p+1}\}_{p \in \mathbb{N}}\).

If in Theorems 1 and 2, we take that \(F = G\) and that \(f\) is the identity mapping, we get the following corollary.

Corollary 1. Let \((X, M, T)\) be a complete fuzzy metric space, \(F : X \to C(X)\), and one of the following conditions is satisfied:

(a) \(F\) is weakly demicompact mapping,

or

(b) \((X, M, T)\) is strong fuzzy metric space and \(T\) is \(t\)-norm of \(H\)-type.

If there exist \(k : (0, \infty) \to (0, 1)\) and altering distance function \(\varphi\) such that:

\[
\varphi(M(Fx, Fy, t)) \leq k(t) \cdot \varphi(M(x, y, t)), \quad x, y \in X, \quad t > 0,
\]

then there exists \(x \in X\) such that \(x \in Fx\).

Moreover, if the mapping \(F\) in Corollary 1 is single-valued we got the result in [30].

Example 1.

(a) Let \(X = [0, 2], T = T_p, M(x, y, t) = t/(t + d(x, y)), \) where \(d\) is Euclidian metric. Then \((X, M, T)\) is a fuzzy metric space. Let \(F(x) = \{1, 2\}, x \in X\). Since \(F\) is weakly demicompact and condition (25) is satisfied, by Corollary 1(a) follows that there exists \(x \in X\) such that \(x \in Fx\).

(b) Let \(X = [0, 2], T = T_M, M^*(x, y, t) = t/(t + d^*(x, y)), \) where \(d^*\) is ultrametric. Ultrametric space is metric space, where instead of triangle inequality condition, the following is satisfied: \(d^*(x, z) \leq \max\{d^*(x, y), d^*(y, z)\}\). Then \((X, M^*, T)\) is a strong fuzzy metric space [11]. For \(F(x) = \{1, 2\}, x \in X\), condition (25) is satisfied and by Corollary 1(b) follows that there exists \(x \in X\) such that \(x \in Fx\).
3.2 Multivalued strong \( \{b_n\} \)-fuzzy contraction

In this part, we present multivalued extension of results given in [6] using multivalued strong \( \{b_n\}\)-fuzzy contraction.

**Definition 9.** Let \((X, M, T)\) be a fuzzy metric space and \(\{b_n\}_{n \in \mathbb{N}}\) a sequence from \((0,1)\) such that \(\lim_{n \to \infty} b_n = 1\). The mapping \(F : X \to C(X)\) is a multivalued strong \(\{b_n\}\)-fuzzy contraction if there exist \(q \in (0,1)\) such that

\[
M(x, y, t) > b_n \implies \overline{M}(Fx, Fy, qt) > b_{n+1}, \quad x, y \in X, \quad t > 0, \quad n \in \mathbb{N}.
\]

**Theorem 3.** Let \((X, M, T)\) be a complete KM fuzzy metric space such that \(\lim_{n \to \infty} M(x, y, t) = 1, x, y \in X, \sup_{a \in [0,1]} T(a, a) = 1\). Let \(\{b_n\} \subset (0,1)\) be a sequence such that \(\lim_{n \to \infty} b_n = 1\) and \(F : X \to C(X)\) be a multivalued strong \(\{b_n\}\)-fuzzy contraction. If \(t\)-norm \(T\) satisfies the following condition:

\[
\lim_{n \to \infty} T_{i=n}^\infty b_i = 1,
\]

then there exists \(x \in X\) such that \(x \in Fx\).

**Proof.** Let \(x_0, x_1 \in X\), where \(x_1 \in Fx_0\). By (27), for arbitrary \(\varepsilon > 0\), there exist \(n_0 \in \mathbb{N}\) and \(t_0 > 0\) such that

\[
T_{i=n_0}^\infty b_i > 1 - \varepsilon \quad \text{and} \quad M(x_0, x_1, t_0) > b_{n_0}.
\]

Then by condition (26), for some \(q \in (0,1)\) and \(\varepsilon_0 > 0\), we have

\[
\overline{M}(Fx_0, Fx_1, qt_0) > b_{n_0+1} + \varepsilon_0.
\]

Keeping the same \(\varepsilon_0\) and using definition of Hausdorff metric, we can find \(x_2 \in Fx_1\) such that

\[
\overline{M}(Fx_0, Fx_1, qt_0) \leq M(x_1, x_2, qt_0) + \varepsilon_0.
\]

By (26), (29) and (30) we obtain

\[
M(x_1, x_2, qt_0) > b_{n_0+1} \implies \overline{M}(Fx_1, Fx_2, q^2t_0) > b_{n_0+2}.
\]

Repeating the same procedure, we get

\[
M(x_k, x_{k+1}, q^k t_0) > b_{n_0+k}, \quad k \in \mathbb{N}.
\]

Let \(\varepsilon > 0\) and \(t > 0\). If we choose \(k_0 \in \mathbb{N}, k_0 > n_0\), such that \(\sum_{k=k_0}^{\infty} q^k < t/t_0\), then, for every \(l, r \in \mathbb{N}, r > 1\), we have

\[
M(x_{k_0+l}, x_{k_0+l+r}, t)
\]

\[
\geq T(T \cdots T \left( M(x_{k_0+l}, x_{k_0+l+1}, t_0 q^{k_0+l}), \ldots \right),)
\]

\[
\geq T_{i=n_0}^\infty b_i > 1 - \varepsilon,
\]
where is used (28) and (31). So, \( \{x_n\}_{n\in\mathbb{N}} \) is a Cauchy sequence and, since \((X,M,T)\) is complete, there exist \( x \in X \) so that
\[
\lim_{n\to\infty} x_n = x. \tag{32}
\]

It is remain to prove that \( x \in FX \). As \( FX = \overline{FX} \), it is enough to show that, for every \( \lambda \in (0, 1) \) and \( t > 0 \), there exists \( r = r(t, \lambda) \in FX \) such that \( M(x, r, t) > 1 - \lambda \).

Let \( t_0 > 0 \) and \( \lambda \in (0, 1) \). Since \( \sup_{a<1} T(a, a) = 1 \), there exist \( \delta = \delta(\lambda) \in (0, 1) \) such that
\[
T(T(1 - \delta, 1 - \delta), 1 - \delta) > 1 - \lambda. \tag{33}
\]
From \( \lim_{n \to \infty} b_n = 1 \), for \( \delta \) defined in (33), there exist \( p_0 \in \mathbb{N} \) such that
\[
b_p > 1 - \delta, \quad p \geq p_0. \tag{34}
\]
By (32), for \( p_0 \) given above, it is possible to find \( n_0 \in \mathbb{N} \) such that
\[
M(x, x_n, \frac{t_0}{3}) > b_{p_0} > 1 - \delta, \quad n \geq n_0, \tag{35}
\]
and
\[
M(x, x_{n+1}, \frac{t_0}{3}) > b_{p_0} > 1 - \delta, \quad n \geq n_0. \tag{36}
\]
Now, by (26) there exist \( \varepsilon^* > 0 \) such that
\[
\widetilde{M}(Fx_n, FX, \frac{q_{t_0}}{3}) > b_{p_{n+1}} + \varepsilon^*, \quad n \geq n_0.
\]
For the same \( \varepsilon^* \) there exist \( r \in FX \) such that
\[
M(x_{n+1}, r, \frac{q_{t_0}}{3}) + \varepsilon^* \geq \widetilde{M}(Fx_n, FX, \frac{q_{t_0}}{3}) > b_{p_{n+1}} + \varepsilon^*,
\]
i.e.
\[
M(x_{n+1}, r, \frac{q_{t_0}}{3}) > M(x_{n+1}, r, \frac{q_{t_0}}{3}) > b_{p_{n+1}} > 1 - \delta, \quad n \geq n_0. \tag{37}
\]
Finally, by (33), (35), (36) and (37) we get
\[
M(x, r, t_0) \geq T(T(M(x, x_n, \frac{t_0}{3}), M(x_n, x_{n+1}, \frac{t_0}{3})), M(x_{n+1}, r, \frac{t_0}{3})) > 1 - \lambda,
\]
which means \( x \in FX \).

4 Conclusion

In this paper we prove several fixed point and coincidence point results, which presented fuzzy generalization of Nadler fixed point result using altering distance function, as well as a multivalued generalizations of strong fuzzy \( \{b_n\} \)-contractions.
References


