

Joint universality of the Riemann zeta-function and Lerch zeta-functions

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Abstract. In the paper, we prove a joint universality theorem for the Riemann zeta-function and a collection of Lerch zeta-functions with parameters algebraically independent over the field of rational numbers.

Keywords: Lerch zeta-function, Riemann zeta-function, limit theorem, universality.

1 Introduction

Let $\lambda \in \mathbb{R}$ and α , $0 < \alpha \leq 1$, be fixed parameters. The Lerch zeta-function $L(\lambda, \alpha, s)$, $s = \sigma + it$, is defined, for $\sigma > 1$, by

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m + \alpha)^s}.$$

For $\lambda \in \mathbb{Z}$, the function $L(\lambda, \alpha, s)$ reduces to the Hurwitz zeta-function $\zeta(s, \alpha)$ which is a meromorphic function with a unique simple pole at the point $s = 1$ with residue 1. If $\lambda \notin \mathbb{Z}$, then the Lerch zeta-function has analytic continuation to an entire function. In view of the periodicity of $e^{2\pi i \lambda m}$, we can suppose that $0 < \lambda \leq 1$.

It is well known that the Lerch zeta-function $L(\lambda, \alpha, s)$ with transcendental parameter α is universal (see [1], also [2]). Let $D = \{s \in \mathbb{C}: 1/2 < \sigma < 1\}$. Denote by \mathcal{K} the class of compact subsets of the strip D with connected complements, and, for $K \in \mathcal{K}$, denote by $H(K)$ the set of continuous functions on K which are analytic in the interior of K .

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Moreover, we use the notation $\text{meas}\{A\}$ for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then the universality of $L(\lambda, \alpha, s)$ is contained in the following theorem.

Theorem 1. *Suppose that α is transcendental. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\epsilon > 0$,*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T]: \sup_{s \in K} |L(\lambda, \alpha, s + i\tau) - f(s)| < \epsilon \right\} > 0.$$

Thus, the universality of $L(\lambda, \alpha, s)$ means that the shifts $L(\lambda, \alpha, s + i\tau)$ approximate with a given accuracy a wide class of analytic functions.

The functions $\zeta(s, \alpha)$, $\alpha \neq 1, 1/2$, and $L(\lambda, \alpha, s)$ with rational λ are also universal in the above sense with rational parameter α . The case of $\zeta(s, \alpha)$ has been examined in [3]. The universality of $L(\lambda, \alpha, s)$ follows from its expression by a linear combination of Hurwitz zeta-functions.

Also, in [4–6] and [7], the joint universality of Lerch zeta-functions has been considered. We state a general result from [7].

Theorem 2. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over the field of rational numbers \mathbb{Q} . For $j = 1, \dots, r$, let $\lambda_j \in (0, 1]$, $K_j \in \mathcal{K}$, and $f_j(s) \in H(K_j)$. Then, for every $\epsilon > 0$,*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T]: \sup_{1 \leq j \leq r} \sup_{s \in K_j} |L(\lambda_j, \alpha_j, s + i\tau) - f_j(s)| < \epsilon \right\} > 0.$$

We note that the algebraic independence of the numbers $\alpha_1, \dots, \alpha_r$ can be replaced by a more general hypothesis that the set

$$L(\alpha_1, \dots, \alpha_r) = \{ \log(m + \alpha_j): m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, j = 1, \dots, r \}$$

is linearly independent over \mathbb{Q} . In the case $\lambda_j \in \mathbb{Z}$, $j = 1, \dots, r$, this was done in [8].

In [9], a joint universality theorem for the Riemann zeta-function $\zeta(s)$ and periodic Hurwitz zeta-functions has been obtained. Let $\mathfrak{A} = \{a_m: m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers with minimal period $k \in \mathbb{N}$. We remind that the periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{A})$ with parameter α , $0 < \alpha \leq 1$, is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s, \alpha; \mathfrak{A}) = \sum_{m=0}^{\infty} \frac{a_m}{(m + \alpha)^s},$$

and is meromorphically continued to the whole complex plane with a unique possible pole at the point $s = 1$ with residue

$$a \stackrel{\text{def}}{=} \frac{1}{k} \sum_{m=0}^{k-1} a_m.$$

If $a = 0$, then $\zeta(s, \alpha; \mathfrak{A})$ is an entire function.

For $j = 1, \dots, r$, let $l_j \in \mathbb{N}$. In [9], the joint universality for the functions

$$\zeta(s), \zeta(s, \alpha_1; \mathfrak{A}_{11}), \dots, \zeta(s, \alpha_1; \mathfrak{A}_{1l_1}), \dots, \zeta(s, \alpha_r; \mathfrak{A}_{r1}), \dots, \zeta(s, \alpha_r; \mathfrak{A}_{rl_r}) \quad (1)$$

has been proved. Here a collection of periodic sequences \mathfrak{A}_{jl} , $\mathfrak{A}_{jl} = \{a_{mjl}: m \in \mathbb{N}_0\}$, with minimal period $k_{jl} \in \mathbb{N}$, $l = 1, \dots, l_j$, corresponds the parameter α_j , $0 < \alpha_j \leq 1$, $j = 1, \dots, r$. For $K \in \mathcal{K}$, denote by $H_0(K)$ the class of continuous non-vanishing functions on K which are analytic in the interior of K . Let k_j be the least common multiple of the periods k_{j1}, \dots, k_{jl_j} , and

$$A_j = \begin{pmatrix} a_{1j1} & a_{1j2} & \dots & a_{1jl_j} \\ a_{2j1} & a_{2j2} & \dots & a_{2jl_j} \\ \dots & \dots & \dots & \dots \\ a_{k_j j1} & a_{k_j j2} & \dots & a_{k_j jl_j} \end{pmatrix}, \quad j = 1, \dots, r.$$

Then the main result of [9] is of the form.

Theorem 3. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over \mathbb{Q} , and that $\text{rank}(A_j) = l_j$, $j = 1, \dots, r$. For $j = 1, \dots, r$ and $l = 1, \dots, l_j$, let $K_{jl} \in \mathcal{K}$ and $f_{jl} \in H(K_{jl})$. Moreover, let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\epsilon > 0$,*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0; T]: \sup_{1 \leq j \leq r} \sup_{1 \leq l \leq l_j} \sup_{s \in K_{jl}} |\zeta(s + i\tau, \alpha_j; \mathfrak{A}_{jl}) - f_{jl}(s)| < \epsilon, \right. \\ \left. \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \epsilon \right\} > 0.$$

We call the approximation property of the functions (1) in Theorem 3 a mixed joint universality because the function $\zeta(s)$ and the functions $\zeta(s, \alpha_j; \mathfrak{A}_{jl})$ are of different types: the function $\zeta(s)$ has Euler product, while the functions $\zeta(s, \alpha_j; \mathfrak{A}_{jl})$ with transcendental α_j do not have Euler product over primes. This is reflected in the approximated functions: the function $f(s)$ must be non-vanishing on K , while the functions f_{jl} are arbitrary continuous functions on K_{jl} .

The first mixed joint universality theorem has been obtained by Mishou [10] for the Riemann zeta-function and Hurwitz zeta-function $\zeta(s, \alpha)$ with transcendental parameter α . This result in [11] has been generalized for a periodic zeta-function and a periodic Hurwitz zeta-function. In [12], the latter mixed joint universality theorem has been extended for several periodic zeta-functions and periodic Hurwitz zeta-functions.

Universality theorems for zeta-functions have a series of interesting applications. From them, for example, various denseness results of Bohr's type for values of zeta-functions follow. The universality implies the functional independence of zeta-functions. This property of zeta-functions is applied to the zero-distribution of those zeta-functions. In [13], the universality has been applied to the famous class number problem. Universality theorems find applications even in solving some problems of physics [14]. For the above mentioned and other facts related to universality and references, we refer to [2, 15–20].

Thus, the universality of zeta-functions is a very interesting and useful property which motivates to continue investigations in the field.

The aim of this paper is to replace the zeta-functions $\zeta(s, \alpha_j; \mathfrak{A}_{j,l})$ with periodic coefficients in Theorem 3 by Lerch zeta-functions $L(\lambda_j, \alpha_j, s)$ with arbitrary $\lambda_j \in (0, 1]$ whose coefficients, in general, are not periodic. This is the novelty of the paper.

Theorem 4. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over \mathbb{Q} . For $j = 1, \dots, r$, let $\lambda_j \in (0, 1]$, $K_j \in \mathcal{K}$ and $f_j \in H(K_j)$. Moreover, let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\epsilon > 0$,*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0; T]: \sup_{1 \leq j \leq r} \sup_{s \in K_j} |L(\lambda_j, \alpha_j, s + i\tau) - f_j(s)| < \epsilon, \right. \\ \left. \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \epsilon \right\} > 0.$$

We note that the linear independence of the set $L(\alpha_1, \dots, \alpha_r)$ is not sufficient for the proof of Theorem 4 because we need the linear independence of the set

$$L \stackrel{\text{def}}{=} \{(\log p: p \in \mathcal{P}), L(\alpha_1, \dots, \alpha_r)\},$$

where \mathcal{P} is the set of all prime numbers. This set consists of logarithms of all prime numbers and of all logarithms $\log(m + \alpha_j)$, $m \in \mathbb{N}$, $j = 1, \dots, r$. Really, L is a multiset. For example, if L has two identical elements, then it is linearly dependent over \mathbb{Q} . The proof of Theorem 4 is based on a joint limit theorem on weakly convergent probability measures in the space of analytic functions.

2 Joint limit theorem

Denote by $\mathcal{B}(S)$ the σ -field of Borel sets of the space S , and by γ the unit circle on the complex plane. Define

$$\hat{\Omega} = \prod_p \gamma_p \quad \text{and} \quad \Omega = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_p = \gamma$ for all $p \in \mathcal{P}$, and $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. By the Tikhonov theorem, with the product topology and pointwise multiplication the tori $\hat{\Omega}$ and Ω are compact topological Abelian groups. Moreover, let

$$\underline{\Omega} = \hat{\Omega} \times \Omega_1 \times \dots \times \Omega_r,$$

where $\Omega_j = \Omega$ for all $j = 1, \dots, r$. Then $\underline{\Omega}$ again is a compact topological Abelian group. This gives the probability spaces $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}), \hat{m}_H)$, $(\Omega_j, \mathcal{B}(\Omega_j), m_{jH})$ and $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), \underline{m}_H)$, where \hat{m}_H , m_{jH} and \underline{m}_H are the probability Haar measures on $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}))$, $(\Omega_j, \mathcal{B}(\Omega_j))$ and $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}))$, respectively, $j = 1, \dots, r$. We note that the

measure \underline{m}_H is the product of the measures $\hat{m}_H, m_{1H}, \dots, m_{rH}$. Denote by $\hat{\omega}(p)$ the projection of $\hat{\omega} \in \hat{\Omega}$ to γ_p , $p \in \mathcal{P}$, and by $\omega_j(m)$ the projection of $\omega_j \in \Omega_j$ to γ_m , $m \in \mathbb{N}_0$. For brevity, we set $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$, $\underline{\lambda} = (\lambda_1, \dots, \lambda_r)$ and $\underline{\omega} = (\hat{\omega}, \omega_1, \dots, \omega_r) \in \underline{\Omega}$.

Let $H(D)$ be the space of analytic functions on D endowed with the topology of uniform convergence on compacta, and let $r_1 = r + 1$. On the probability space $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), \underline{m}_H)$, define the $H^{r_1}(D)$ -valued random element $\underline{\zeta}(s, \underline{\alpha}, \underline{\lambda}, \underline{\omega})$ by the formula

$$\underline{\zeta}(s, \underline{\alpha}, \underline{\lambda}, \underline{\omega}) = (\zeta(s, \hat{\omega}), L(\lambda_1, \alpha_1, s, \omega_1), \dots, L(\lambda_r, \alpha_r, s, \omega_r)),$$

where

$$\zeta(s, \hat{\omega}) = \prod_p \left(1 - \frac{\hat{\omega}(p)}{p^s}\right)^{-1}$$

and

$$L(\lambda_j, \alpha_j, s, \omega_j) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda_j m} \omega_j(m)}{(m + \alpha_j)^s}, \quad j = 1, \dots, r.$$

Let $P_{\underline{\zeta}}$ stand for the distribution of the random element $\underline{\zeta}(s, \underline{\alpha}, \underline{\lambda}, \underline{\omega})$, i.e., $P_{\underline{\zeta}}$ is the probability measure on $(H^{r_1}(D), \mathcal{B}(H^{r_1}(D)))$ given by

$$P_{\underline{\zeta}}(A) = \underline{m}_H(\underline{\omega} \in \underline{\Omega}: \underline{\zeta}(s, \underline{\alpha}, \underline{\lambda}, \underline{\omega}) \in A).$$

We set

$$\underline{\zeta}(s, \underline{\alpha}, \underline{\lambda}) = (\zeta(s), L(\lambda_1, \alpha_1, s), \dots, L(\lambda_r, \alpha_r, s)).$$

Now we state a limit theorem on the space $(H^{r_1}(D), \mathcal{B}(H^{r_1}(D)))$.

Theorem 5. Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over \mathbb{Q} , and $\lambda_j \in (0, 1]$, $j = 1, \dots, r$. Then

$$P_T(A) \stackrel{\text{def}}{=} \frac{1}{T} \text{meas}\{\tau \in [0, T]: \underline{\zeta}(s + i\tau, \underline{\alpha}; \underline{\lambda}) \in A\}, \quad A \in \mathcal{B}(H^{r_1}(D)),$$

converges weakly to the measure $P_{\underline{\zeta}}$ as $T \rightarrow \infty$.

We divide the proof of Theorem 5 into lemmas. The first lemma is a limit theorem on the torus $\underline{\Omega}$. For $A \in \mathcal{B}(\underline{\Omega})$, define

$$Q(A) = \frac{1}{T} \text{meas}\{((p^{-i\tau}: p \in \mathcal{P}), ((m + \alpha_j)^{-i\tau}: m \in \mathbb{N}_0, j = 1, \dots, r)) \in A\}.$$

Lemma 1. Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over \mathbb{Q} . Then Q_T converges weakly to the Haar measure \underline{m}_H as $T \rightarrow \infty$.

Proof. The proof of the lemma is given in [9, Lemma 1]. □

Let $\sigma_1 > 1/2$ be a fixed number, and

$$u_n(m) = \exp\left\{-\left(\frac{m}{n}\right)^{\sigma_1}\right\}, \quad m, n \in \mathbb{N},$$

$$u_n(m, \alpha_j) = \exp\left\{-\left(\frac{m + \alpha_j}{n + \alpha_j}\right)^{\sigma_1}\right\}, \quad m \in \mathbb{N}_0, \quad n \in \mathbb{N}.$$

Define the series

$$\zeta_n(s) = \sum_{m=1}^{\infty} \frac{u_n(m)}{m^s},$$

and

$$L_n(\lambda_j, \alpha_j, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda_j m} u_n(m, \alpha_j)}{(m + \alpha_j)^s}, \quad j = 1, \dots, r,$$

and, for $\underline{\omega} \in \underline{\Omega}$,

$$\zeta_n(s, \hat{\omega}) = \sum_{m=1}^{\infty} \frac{\hat{\omega}(m) u_n(m)}{m^s},$$

$$L_n(\lambda_j, \alpha_j, \omega_j, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda_j m} \omega_j(m) u_n(m, \alpha_j)}{(m + \alpha_j)^s}, \quad j = 1, \dots, r.$$

It is known, see, for example, [2, 16], that all above series converge absolutely for $\sigma > 1/2$. Let

$$\underline{\zeta}_n(s, \underline{\alpha}, \underline{\lambda}) = (\zeta_n(s), L_n(\lambda_1, \alpha_1, s), \dots, L_n(\lambda_r, \alpha_r, s))$$

and

$$\underline{\zeta}_n(s, \underline{\alpha}, \underline{\lambda}, \underline{\omega}) = (\zeta_n(s, \hat{\omega}), L_n(\lambda_1, \alpha_1, \omega_1, s), \dots, L_n(\lambda_r, \alpha_r, \omega_r, s)).$$

Lemma 2. Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over \mathbb{Q} , and $\underline{\omega} \in \underline{\Omega}$. Then

$$\frac{1}{T} \text{meas}\{\tau \in [0, T]: \underline{\zeta}_n(s + i\tau, \underline{\alpha}, \underline{\lambda}) \in A\}, \quad A \in \mathcal{B}(H^{r_1}(D)),$$

and

$$\frac{1}{T} \text{meas}\{\tau \in [0, T]: \underline{\zeta}_n(s + i\tau, \underline{\alpha}, \underline{\lambda}, \underline{\omega}) \in A\}, \quad A \in \mathcal{B}(H^{r_1}(D)),$$

converges weakly to the same probability measure P_n on $(H^{r_1}(D), \mathcal{B}(H^{r_1}(D)))$ as $T \rightarrow \infty$.

Proof. The proof uses Lemma 1 and does not depend on the coefficients of the functions $L_n(\lambda_j, \alpha_j, s)$, $j = 1, \dots, r$. Therefore, it coincides with the proof of [9, Lemma 2]. \square

Now we define a metric on $H^{r_1}(D)$ which induces the topology of uniform convergence on compacta. For $g_1, g_2 \in H(D)$, we define

$$\rho(g_1, g_2) = \sum_{m=1}^{\infty} 2^{-m} \frac{\sup_{s \in K_m} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_m} |g_1(s) - g_2(s)|},$$

where $\{K_m: m \in \mathbb{N}\}$ is a sequence of compact subsets of the strip D such that

$$D = \bigcup_{m=1}^{\infty} K_m,$$

$K_m \subset K_{m+1}$ for all $m \in \mathbb{N}$, and, if $K \subset D$ is a compact set, then $K \subset K_m$ for some $m \in \mathbb{N}$. The existence of the sequence $\{K_m\}$ follows from a general theorem, see, for example, [21], however, in the case of the region D , it is easily seen that we can take closed rectangles. Clearly, ρ is a metric on $H(D)$ inducing its topology. For $\underline{g}_j = (g_j, g_{j1}, \dots, g_{jr}) \in H^{r_1}(D)$, $j = 1, 2$, we put

$$\rho(\underline{g}_1, \underline{g}_2) = \max\left(\rho(g_1, g_2), \max_{1 \leq j \leq r} \rho(g_{1j}, g_{2j})\right).$$

Then we have that ρ is a desired metric on $H^{r_1}(D)$. Using this metric, we approximate $\zeta(s, \underline{\alpha}, \underline{\lambda})$ and $\zeta(s, \underline{\alpha}, \underline{\lambda}, \underline{\omega})$ by $\zeta_n(s, \underline{\alpha}, \underline{\lambda})$ and $\zeta_n(s, \underline{\alpha}, \underline{\lambda}, \underline{\omega})$, respectively.

Lemma 3. *We have*

$$\lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \rho(\zeta(s + i\tau, \underline{\alpha}, \underline{\lambda}), \zeta_n(s + i\tau, \underline{\alpha}, \underline{\lambda})) d\tau = 0.$$

Moreover, suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over \mathbb{Q} . Then, for almost all $\underline{\omega} \in \underline{\Omega}$,

$$\lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \rho(\zeta(s + i\tau, \underline{\alpha}, \underline{\lambda}, \underline{\omega}), \zeta_n(s + i\tau, \underline{\alpha}, \underline{\lambda}, \underline{\omega})) d\tau = 0.$$

Proof. In [16], it is proved that

$$\lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \rho(\zeta(s + i\tau), \zeta_n(s + i\tau)) d\tau = 0,$$

and, for almost all $\hat{\omega} \in \hat{\Omega}$

$$\lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \rho(\zeta(s + i\tau, \hat{\omega}), \zeta_n(s + i\tau, \hat{\omega})) d\tau = 0.$$

Since the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over \mathbb{Q} , each number α_j is transcendental. Therefore, in [2], it was obtained that, for $j = 1, \dots, r$,

$$\lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \rho(L(\lambda_j, \alpha_j, s + i\tau), L_n(\lambda_j, \alpha_j, s + i\tau)) d\tau = 0,$$

and, for almost all $\omega_j \in \Omega_j$,

$$\lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \rho(L(\lambda_j, \alpha_j, \omega_j, s + i\tau), L_n(\lambda_j, \alpha_j, \omega_j, s + i\tau)) d\tau = 0.$$

All these equalities together with the definition of the metric ρ prove the lemma. \square

On $(H^{r_1}(D), \mathcal{B}(H^{r_1}(D)))$, define one more probability measure

$$\hat{P}_T(A) = \frac{1}{T} \text{meas}\{\tau \in [0, T]: \zeta(s + i\tau, \underline{\alpha}, \underline{\lambda}, \underline{\omega}) \in A\}.$$

Lemma 4. Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over \mathbb{Q} . Then P_T and \hat{P}_T both converge weakly for almost all $\underline{\omega} \in \underline{\Omega}$ to the same probability measure P on $(H^{r_1}(D), \mathcal{B}(H^{r_1}(D)))$ as $T \rightarrow \infty$.

Proof. We give a shortened proof because we apply similar arguments as in [9]. Let θ be a random variable defined on a certain probability space $(\Omega_0, \mathcal{A}, \mathbf{P})$ and uniformly distributed on $[0, 1]$. Let

$$\underline{X}_{T,n}(s) = \zeta_n(s + i\theta T, \underline{\alpha}, \underline{\lambda}). \quad (2)$$

Then, in view of Lemma 2, $\underline{X}_{T,n} \xrightarrow[T \rightarrow \infty]{\mathcal{D}} \underline{X}_n$, where \underline{X}_n is the random element with the distribution P_n (P_n is the limit measure in Lemma 2), and $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution. Using the absolute convergence of series for $\zeta_n(s)$ and $L_n(\lambda_j, \alpha_j, s)$, $j = 1, \dots, r$, we prove without difficulties that the family of probability measures $\{P_n: n \in \mathbb{N}\}$ is tight. Hence, by the Prokhorov theorem, this family is relatively compact. Thus, we have a subsequence $\{P_{n_k}\}$ such that P_{n_k} converges weakly to some probability measure P as $k \rightarrow \infty$. Hence,

$$\underline{X}_{n_k} \xrightarrow[k \rightarrow \infty]{\mathcal{D}} P.$$

Define

$$\underline{X}_T(s) = \zeta(s + i\theta T, \underline{\alpha}, \underline{\lambda}). \quad (3)$$

Then Lemma 3 implies that, for every $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \mathbf{P}(\rho(\underline{X}_T(s), \underline{X}_{T,n}(s)) \geq \epsilon) = 0.$$

This, (2), (3) and Theorem 4.2 of [22] show that

$$\underline{X}_T \xrightarrow[T \rightarrow \infty]{\mathcal{D}} P,$$

and this is equivalent to the weak convergence of P_T to P as $T \rightarrow \infty$.

Repeating the above arguments for the random elements

$$\hat{X}_{T,n}(s) = \zeta_n(s + i\theta T, \underline{\alpha}, \underline{\lambda}, \underline{\omega})$$

and

$$\hat{X}_T(s) = \zeta(s + i\theta T, \underline{\alpha}, \underline{\lambda}, \underline{\omega}),$$

and using Lemmas 2 and 3, we find that the measure \hat{P}_T also converges weakly to P as $T \rightarrow \infty$ for almost all $\underline{\omega} \in \underline{\Omega}$. \square

Proof of Theorem 5. In virtue of Lemma 4, it suffices to check that the measure P in Lemma 4 coincides with $P_{\underline{\zeta}}$.

Let, for $\tau \in \mathbb{R}$,

$$a_\tau = ((p^{-i\tau}: p \in \mathcal{P}), ((m + \alpha_j)^{-i\tau}: m \in \mathbb{N}_0, j = 1, \dots, r)),$$

and

$$\Phi_\tau(\underline{\omega}) = a_\tau \underline{\omega}, \quad \underline{\omega} \in \underline{\Omega}.$$

Then $\{\Phi_\tau: \tau \in \mathbb{R}\}$ is an ergodic group of measurable measure preserving transformations on $\underline{\Omega}$ (see [12]).

Let ξ be a random variable on $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), \underline{m}_H)$ given by

$$\xi(\underline{\omega}) = \begin{cases} 1 & \text{if } \underline{\zeta}(s, \underline{\alpha}, \underline{\lambda}, \underline{\omega}) \in A, \\ 0 & \text{if } \underline{\zeta}(s, \underline{\alpha}, \underline{\lambda}, \underline{\omega}) \notin A, \end{cases}$$

where A is a fixed continuity set of the measure P .

By Lemma 4, for almost all $\underline{\omega} \in \underline{\Omega}$,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \text{meas}\{\tau \in [0, T]: \underline{\zeta}(s + i\tau, \underline{\alpha}, \underline{\lambda}, \underline{\omega}) \in A\} = P(A). \quad (4)$$

The ergodicity of the group $\{\Phi_\tau: \tau \in \mathbb{R}\}$ implies that of the process $\xi(\Phi_\tau(\underline{\omega}))$. Therefore, the classical Birkhoff–Khinchine theorem shows that, for almost all $\underline{\omega} \in \underline{\Omega}$,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \xi(\Phi_\tau(\underline{\omega})) \, d\tau = \mathbf{E}\xi, \quad (5)$$

where $\mathbf{E}\xi$ denotes the expectation of ξ . The definitions of ξ and of Φ_τ give the equalities

$$\begin{aligned} \mathbf{E}\xi &= \int_{\underline{\Omega}} \xi \, d\underline{m}_H = \underline{m}_H(\underline{\omega} \in \underline{\Omega}: \underline{\zeta}(s, \underline{\alpha}, \underline{\lambda}, \underline{\omega}) \in A) = P_{\underline{\zeta}}(A), \\ \frac{1}{T} \int_0^T \xi(\Phi_\tau(\underline{\omega})) \, d\tau &= \frac{1}{T} \text{meas}\{\tau \in [0, T]: \underline{\zeta}(s + i\tau, \underline{\alpha}, \underline{\lambda}, \underline{\omega}) \in A\}. \end{aligned} \quad (6)$$

Thus, by (5) and (6),

$$\lim_{T \rightarrow \infty} \frac{1}{T} \text{meas} \{ \tau \in [0, T]: \zeta(s + i\tau, \underline{\alpha}, \underline{\lambda}, \omega) \in A \} = P_{\underline{\zeta}}(A).$$

This and (4) show that $P(A) = P_{\underline{\zeta}}(A)$ for all continuity sets of P . Hence, $P = P_{\underline{\zeta}}$. The theorem is proved. \square

3 Support

A proof of Theorem 4 is based on Theorem 5 and the support of the limit measure $P_{\underline{\zeta}}$ in it. We remind that the support of $P_{\underline{\zeta}}$ is a minimal closed set $S_{P_{\underline{\zeta}}} \subset H^{r_1}(D)$ such that $P_{\underline{\zeta}}(S_{P_{\underline{\zeta}}}) = 1$. The set $S_{P_{\underline{\zeta}}}$ consists of all elements $\underline{g} \in H^{r_1}(D)$ such that, for every open neighbourhood G of \underline{g} , the inequality $P_{\underline{\zeta}}(G) > 0$ is satisfied.

Define

$$S = \{g \in H(D): g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

Theorem 6. *The support of the measure $P_{\underline{\zeta}}$ is the set $\underline{S} = S \times H^r(D)$.*

Proof. We write

$$H^{r_1}(D) = H(D) \times \underbrace{H(D) \times \cdots \times H(D)}_r.$$

The space $H(D)$ is separable, therefore, it follows from [22] that

$$\mathcal{B}(H^{r_1}(D)) = \mathcal{B}(H(D)) \times \underbrace{\mathcal{B}(H(D)) \times \cdots \times \mathcal{B}(H(D))}_r.$$

Thus, it suffices to consider the measure $P_{\underline{\zeta}}$ on the sets of the form

$$B = A \times A_1 \times \cdots \times A_r, \quad A, A_j \in \mathcal{B}(H(D)), \quad j = 1, \dots, r.$$

Since the measure \underline{m}_H is the product of the measures $\hat{m}_H, m_{1H}, \dots, m_{rH}$, the definition of $P_{\underline{\zeta}}$ gives the equality

$$P_{\underline{\zeta}}(B) = \underline{m}_H(A \times A_1 \times \cdots \times A_r) = \hat{m}_H(A) m_{1H}(A_1) \cdots m_{rH}(A_r). \quad (7)$$

In [16], it is proved that the support of the random element $\zeta(s, \hat{\omega})$ is the set S . The algebraic independence of the numbers $\alpha_1, \dots, \alpha_r$ implies their transcendence. Therefore, by [2] the support the random element $L(\lambda_j, \alpha_j, s, \omega_j)$ is the space $H(D)$, $j = 1, \dots, r$. On the other hand, the distribution $P_{\underline{\zeta}}$ of $\zeta(s, \hat{\omega})$ is

$$P_{\underline{\zeta}}(A) = \hat{m}_H(\hat{\omega} \in \hat{\Omega}: \zeta(s, \hat{\omega}) \in A), \quad A \in \mathcal{B}(H(D)),$$

and the distribution P_{L_j} of $L(\lambda_j, \alpha_j, s, \omega_j)$, $j = 1, \dots, r$, is

$$P_{L_j}(A_j) = m_{jH}(\omega_j \in \Omega_j: L(\lambda_j, \alpha_j, s, \omega_j) \in A_j), \quad A_j \in \mathcal{B}(H(D)).$$

In view of (7),

$$P_{\underline{S}}(B) = P_{\underline{S}}(A)P_{L_1}(A_1) \cdots P_{L_r}(A_r).$$

Hence, obviously, $P_{\underline{S}}(\underline{S}) = 1$. Moreover, if $A \in \mathcal{B}(H(D))$ with $A \not\subseteq S$, or $A_j \in \mathcal{B}(H(D))$ with $A_j \not\subseteq H(D)$, for some j , then, in view of the minimality of S and $H(D)$ for $P_{\underline{S}}(A)$ and $P_{L_j}(A_j)$, respectively, we have that $P_{\underline{S}}(A) < 1$ or $P_{L_j}(A_j) < 1$. Thus, then $P_{\underline{S}}(B) < 1$. Hence, the minimality of \underline{S} follows. \square

4 Universality theorem

In this section, we will prove Theorem 4. Its proof is based on Theorems 5 and 6 as well as on the Mergelyan theorem on the approximation of analytic functions by polynomials. We state this theorem as the next lemma.

Lemma 5. *Let $K \subset \mathbb{C}$ be a compact set with connected complement, and $f(s)$ be a continuous function on K which is analytic in the interior of K . Then, for every $\epsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s \in K} |f(s) - p(s)| < \epsilon.$$

Proof. The proof of the lemma can be found in [23], see also [24]. \square

Proof of Theorem 4. By Lemma 5, there exists a polynomial $p(s)$ such that

$$\sup_{s \in K} |f(s) - p(s)| < \frac{\epsilon}{4}. \quad (8)$$

Since $f(s) \neq 0$ on K , $p(s) \neq 0$ on K as well provided ϵ is small enough. Thus, we can define on K a continuous branch of $\log p(s)$ which will be analytic in the interior of K . Applying Lemma 5 once more, we obtain that there exists a polynomial $q(s)$ such that

$$\sup_{s \in K} |p(s) - e^{q(s)}| < \frac{\epsilon}{4}.$$

This together with (8) shows that

$$\sup_{s \in K} |f(s) - e^{q(s)}| < \frac{\epsilon}{2}. \quad (9)$$

Again, by Lemma 5, there exist polynomials $p_j(s)$ such that

$$\sup_{1 \leq j \leq r} \sup_{s \in K_j} |f_j(s) - p_j(s)| < \frac{\epsilon}{2}. \quad (10)$$

Define

$$G = \left\{ (g, g_1, \dots, g_r) \in H^{r+1}(D): \sup_{s \in K} |g(s) - e^{q(s)}| < \frac{\epsilon}{2}, \right. \\ \left. \sup_{1 \leq j \leq r} \sup_{s \in K_j} |g_j(s) - p_j(s)| < \frac{\epsilon}{2} \right\}.$$

Then G is an open set, and, in view of Theorem 6, $e^{q(s)}, p_1(s), \dots, p_r(s)$ is an element of the support of the measure $P_{\underline{\zeta}}$. Therefore, an equivalent of the weak convergence of probability measures in terms of open sets, see Theorem 2.1 of [22], together with Theorem 5 and properties of the support give the inequality

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas}\{\tau \in [0, T]: \zeta(s + i\tau, \underline{\alpha}, \underline{\lambda}) \in G\} \geq P_{\underline{\zeta}}(G) > 0.$$

Hence, by the definition of G , we find that

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas}\left\{\tau \in [0, T]: \sup_{s \in K} |\zeta(s + i\tau) - e^{q(s)}| < \frac{\epsilon}{2}, \right. \\ \left. \sup_{1 \leq j \leq r} \sup_{s \in K_j} |L(\lambda_j, \alpha_j, s + i\tau) - p_j(s)| < \frac{\epsilon}{2}\right\} > 0. \quad (11)$$

Inequalities (9) and (10) show that

$$\left\{\tau \in [0, T]: \sup_{s \in K} |\zeta(s + i\tau) - e^{q(s)}| < \frac{\epsilon}{2}, \right. \\ \left. \sup_{1 \leq j \leq r} \sup_{s \in K_j} |L(\lambda_j, \alpha_j, s + i\tau) - p_j(s)| < \frac{\epsilon}{2}\right\} \\ \subset \left\{\tau \in [0, T]: \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \epsilon, \right. \\ \left. \sup_{1 \leq j \leq r} \sup_{s \in K_j} |L(\lambda_j, \alpha_j, s + i\tau) - f_j(s)| < \epsilon\right\}.$$

Combining this with (11) gives the assertion of the theorem. \square

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