On the Sojourn Time of the Brownian Process in a Multidimensional Sphere

S. Steišūnas
Institute of Mathematics and Informatics
Akademijos str. 4, LT-08663 Vilnius, Lithuania
stst@ktl.mii.lt

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Abstract. We consider the Brownian motion process $B^m(s)$ in the $m$-space and the distribution

$$F^m(t, x, a) = P \left\{ \sup_{0 \leq s \leq t} |B^m(s) + x| < a \right\}, \quad \text{where} \quad a > 0, \ x \in R^m, \ |x| < a.$$ 

There is a probability that a particle starting from the point $x$ on the sphere $S_m^r$ with the radius $r = |x| < a$ will not be absorbed by the sphere $S_m^a$ with a radius $a$ before the epoch $t$.

Keywords: Brownian motion process, distribution, random variable.

1 Introduction

The most important stochastic process is a Brownian or Wiener process. It was first discussed by Louis Bachelier (1900), who was interested in modelling fluctuations of prices in financial markets, and by Albert Einstein (1905), who gave a mathematical model for the irregular motion of colloidal particles, first observed by the Scottish botanist, Robert Brown, in 1827.

Let there be an $m$-dimensional Euclidean space and $e_1, e_2, \ldots, e_m$ be a fixed basis in $R^m$, where $x_1, x_2, \ldots, x_m$, are coordinates of the vector from $R^m$ in the basis. A scalar product of the elements $x$ and $y \in R^m$ is the number $(x \cdot y) = \sum_{i=1}^{m} x_i \cdot y_i$, and the norm of the element $x \in R^m$ is a (non-negative) number $|x| = \sqrt{(x \cdot x)}$. Let $S_m^a$ be an $m$-dimensional sphere with the center at the beginning of coordinates and the radius $a$.

Distribution of the random variable $B^m(s)$ is defined by density of the distribution

$$p(s, x) = \frac{(2\pi s)^{\frac{m}{2}}}{2^s} \exp \left( -\frac{|x|^2}{2s} \right).$$

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so for every Borel set \( A \in \mathbb{R}^m \) we get

\[
P\{ B^m(s) \in A \} = (2\pi s)^{-m/2} \int_A \exp \left( -\frac{|x|^2}{2s} \right) dx. \tag{1}\]

We have examined the distribution

\[
F^m(t, x, a) = P\left\{ \sup_{0 \leq s \leq t} |B^m(s) + x| < a \right\}, \tag{2}
\]

where \( a > 0, x \in \mathbb{R}^m \) and \(|x| < a\).

There is a probability that a particle starting from the point \( x \) on the sphere \( S^m_r \) with the radius \( r = |x| < a \) will not be absorbed by the sphere \( S^m_a \) with a radius \( a \) before the epoch \( t \).

In a one-dimensional case, the probability distribution function

\[
F^1(t, 0, a) = P\left\{ \sup_{0 \leq s \leq t} |B(s)| < a \right\}
\]

has a complicated expression and different authors obtained several forms of this function in [1–10]. The author [11] has proved that all the expressions are equivalent.

P. Levy [7] examined one-dimensional Brownian motion starting at the point \( -a_1 < x < a_2 \), impeded by two absorbing barriers at \(-a_1 < 0 < a_2\), and obtained the general formula

\[
P\{-a_1 < B(s) + x < a_2, 0 \leq s \leq t\} = \frac{1}{\sqrt{2\pi t}} \sum_{k=-\infty}^{\infty} \int_{x_k'}^{x_k''} \left[ e^{-\frac{(x-y)^2}{2t}} - e^{-\frac{(x+y)^2}{2t}} \right] dy, \tag{3}\]

where \( x_k' = 2dk \), \( x_k'' = 2a_2 - 2dk \), \( d = a_1 + a_2 \) and \( k = \ldots, -1, 0, 1, \ldots \).

If \( a_1 = a_2 = a \), \( d = 2a \), then it follows that

\[
F^1(t, x, a) = P\left\{ \sup_{0 \leq s \leq t} |B(s)| < a \right\} = \frac{1}{\sqrt{2\pi t}} \sum_{k=-\infty}^{\infty} \int_{-a}^a \left( e^{-\frac{(x-4ka)^2}{2t}} - e^{-\frac{(x+4ka-2a+y)^2}{2t}} \right) dy. \tag{4}\]

W. Feller [4] considered one-dimensional Brownian motion starting at the point \( 0 < x < a \), impeded by two absorbing barriers at 0 and \( a > 0 \) and has obtained two very different representations for the same distribution function \( \lambda_a(t, x) \) (see [4, Chapter X]):

\[
\lambda_a(t, x) = P\{0 < B(s) + x < a, 0 \leq s \leq t\} = \sum_{k=-\infty}^{\infty} \left\{ \Phi \left( \frac{2ka + a - x}{\sqrt{t}} \right) - \Phi \left( \frac{2ka - x}{\sqrt{t}} \right) - \Phi \left( \frac{2ka + a + x}{\sqrt{t}} \right) + \Phi \left( \frac{2ka + x}{\sqrt{t}} \right) \right\}. \tag{5}\]

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and

\[ \lambda_a(t, x) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \exp \left( -\frac{(2k+1)^2 \pi^2 t}{2a^2} \right) \sin \left( -\frac{(2k+1) \pi x}{a} \right), \]  

(6)

where \( \Phi(x) \) is standard normal distribution function.

Fortunately, the series in (5) converges reasonably only when \( t \) is small, whereas (6) is applicable to large \( t \).

In [11], the author derived an other different representation for the same distribution function (4) \( F^1(t, x, a) \)

\[ F^1(t, x, a) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp \left( -\frac{(2k+1)^2 \pi^2 t}{8a^2} \right) \cos \left( \frac{(2k+1) \pi x}{2a} \right), \]  

(7)

where \(-a < x < a\). This formula gives a probability that the Brownian motion leaving the point \( x \), will not be absorbed till the moment \( t \).

The authors in [12, 13] examined the distribution \( F^m(t, 0, a) \). They considered the Brownian motion \( B^m(t) \) starting from the origin. Definition of such probabilities is one of the most important problems in the theory of random processes. Following the results of A.V. Skorokhod [9], the probability \( F^m(t, x, a) \), we are interested in, satisfies a differential equation of diffusion. In the case of an \( m \)-dimensional Brownian motion, we impose a condition of a circular symmetry which leads to the equation

\[ \frac{2 \partial F^m(t, x, a)}{\partial t} = \frac{\partial^2 F^m(t, x, a)}{\partial x_1^2} + \ldots + \frac{\partial^2 F^m(t, x, a)}{\partial x_m^2}, \]  

(8)

under the boundary condition \( F^m(t, x, a)|_{x=a} = 0 \) and the initial condition \( F^m(t, x, a)|_{t=0} = 1 \).

Passing to spherical coordinates, we shall transform equation (8) into the following shape:

\[ \frac{2 \partial v^m(t, r, a)}{\partial t} = \frac{\partial^2 v^m(t, r, a)}{\partial r^2} + \frac{m-1}{r} \frac{\partial v^m(t, r, a)}{\partial r}, \]  

(9)

under the boundary condition

\[ v^m(t, r, a)|_{r=a} = 0 \]  

(10)

and the initial condition

\[ v^m(t, r, a)|_{t=0} = 1. \]  

(11)

This paper is meant for studying the properties of distribution functions \( F^m(t, x, a) = v^m(t, r, a) \), where \( a > r = |x| > 0 \).
2 Statement of the basic results

We consider the Brownian motion process \( B^m(t) \) in an \( m \)-space starting from the point \( x \) on the sphere \( S^m \) with the radius \( r = |x| < a \). We shall prove the following theorem.

**Theorem 1.** Let \( B^m(s), 0 \leq s \leq t, \) be an \( m \)-dimensional Brownian motion, starting from the point \( x \) on the sphere \( S^m \) with the radius \( r = |x| < a \). Then

\[
v^m(t, r, a) = \sum_{n=1}^{\infty} 2n^\nu J_\nu(\mu_n r/a) \exp\left( -\frac{\mu_n^2 t}{2a^2} \right),
\]

where \( \mu_n, n = 1, 2, \ldots \) are the positive roots of the Bessel function \( J_\nu(z) \) with \( \nu = m/2 - 1 \).

**Proof.** We find the solution to this differential diffusion equation (9) by the standard Fourier method. We try to find a solution of the form

\[
v^m(t, r, a) = T(t)R(r),
\]

where \( T(t) \) is a function only of the variable \( t \) and \( R(r) \) is a function only of the variable \( r \).

Substituting the proposed form of solution (13) into equation (9) and dividing both sides of the equality by \( T(t)R(r) \), we obtain

\[
2 \frac{T'(t)}{T(t)} = \frac{R''(r) + \frac{m-1}{r} R'(r) + \lambda^2 R(r)}{R(r)} = -\lambda^2.
\]

Then, from equality (14) we obtain two ordinary equations

\[
2T'(t) + \lambda^2 T(t) = 0,
\]
\[
R''(r) + \frac{m-1}{r} R'(r) + \lambda^2 R(r) = 0.
\]

Boundary condition (10) yields \( R(a) = 0 \). Thus, in view of the found function \( R(r) \), we derive the simplest problem on eigenvalues: find the values of the parameter \( \lambda \) at which there exist nontrivial solutions of equation (16) and the boundary condition \( R(a) = 0 \).

Set

\[
R(r) = \frac{u(r)}{r^\nu}
\]

in equation (16). Then \( u(r) \) satisfies the Bessel equation

\[
u^2 u''(r) + ru'(r) + (\lambda^2 r^2 - \nu^2) u(r) = 0, \quad \text{where} \quad \nu = \frac{m}{2} - 1.
\]

The general solution of equation (18) is of the shape:

\[
u(r) = c_1 J_\nu(\lambda r) + c_2 Y_\nu(\lambda r),
\]
where $J_\nu(\lambda r)$ is the Bessel function of the first kind of order $\nu$ and $Y_\nu(\lambda r)$ is the Bessel function of the second kind. It follows from (17) and (19) that

$$R(r) = \frac{c_1 J_\nu(\lambda r) + c_2 Y_\nu(\lambda r)}{r^\nu}.$$  

(20)

Since $Y_\nu(\lambda r) \to \infty$ as $r \to 0$, most probably $c_2 = 0$. Under the boundary condition (8) we get the following equation

$$J_\nu(\lambda a) = 0,$$  

(21)

that has infinitely many positive zeros $\mu_1, \mu_2, \mu_3, \ldots$ (see [14]). Hence we derive that $\lambda_k$ is defined by the formulas

$$\lambda_k = \frac{\mu_k}{a},$$

and

$$R_k(r) = \frac{J_\nu(\frac{\mu_k a}{r})}{r^\nu},$$

(22)

Now, in view of equations (13), (15) and (22), we find that the functions

$$v^m(t, r, a) = c_n \exp \left( -\frac{\mu_n^2 t}{2a^2} \right) \frac{J_\nu(\frac{\mu_n r}{a})}{r^\nu} J_\nu(\frac{\mu_n a}{r})$$  

(23)

satisfy equation (9) and the boundary condition (10) for any $c_n$. Let us compose a series

$$v^m(t, r, a) = \sum_{n=1}^{\infty} c_n \frac{J_\nu(\frac{\mu_n r}{a})}{r^\nu} \exp \left( -\frac{\mu_n^2 t}{2a^2} \right).$$  

(24)

To satisfy the initial condition (11), we need to fulfill the equality

$$\sum_{n=1}^{\infty} c_n J_\nu\left(\frac{\mu_n r}{a}\right) = r^\nu.$$  

(25)

The written series represents an expansion of the function $r^\nu$ in Bessel functions in the interval $(0, a)$. The coefficients of expansions are defined by the formula

$$c_n = \frac{2}{a^2 J_{\nu+1}^2(\mu_n)} \int_0^a r^{\nu+1} J_\nu\left(\frac{\mu_n r}{a}\right) \, dr.$$  

(26)

Let $y = \frac{\mu_n r}{a}$, then

$$c_n = \frac{2}{a^2 J_{\nu+1}^2(\mu_n)} \left(\frac{\mu_n}{a}\right)^{\nu+2} \int_0^\infty y^{\nu+1} J_\nu(y) \, dy.$$  

(27)
Making use of the recurrence relation
\[
\frac{d}{dy} y_{\nu+1}(y) = y_{\nu}(y) J_{\nu+1}(y),
\]
it is easy to find that
\[
\int_{0}^{\mu_n} y_{\nu+1} J_{\nu+1}(y) \, dy = \int_{0}^{\mu_n} d(y_{\nu+1} J_{\nu+1}(y)) = \mu_n^{\nu+1} J_{\nu+1}(\mu_n).
\]
(28)

It follows from (27) and (28) that
\[
c_n = 2 \frac{a^{\nu}}{\mu_n J_{\nu+1}(\mu_n)}. \tag{29}
\]
Formulae (24) and (29) complete the proof of Theorem 1.

Let us mention some corollaries.

**Corollary 1.** Let \( B^m(s) \) be an \( m \)-dimensional Brownian motion, starting from the origin. Then, passing to the limit from Theorem 1 as \( r \to 0 \), we obtain
\[
P\{ \sup_{0 \leq s \leq t} |B^m(s)| < a \} = \sum_{n=1}^{\infty} \frac{1}{2^{\nu-1} \Gamma(\nu + 1)} \frac{\mu_n^{\nu-1}}{J_{\nu+1}(\mu_n)} \exp \left( - \frac{\mu_n^2 t}{2a^2} \right), \tag{30}
\]
where \( a > 0 \).

**Proof.** We obtain the limit from formula (4.14.4) in [15]
\[
\lim_{r \to 0} \frac{J_{\nu}(\mu_n r/a)}{(\mu_n r/a)^{\nu}} = \frac{1}{2^{\nu} \Gamma(\nu + 1)}
\]
and
\[
\lim_{r \to 0} \frac{2a^{\nu} J_{\nu}(\mu_n r/a)}{\mu_n J_{\nu+1}(\mu_n)} = \frac{1}{2^{\nu-1} \Gamma(\nu + 1)} \frac{\mu_n^{\nu-1}}{J_{\nu+1}(\mu_n)}.
\]
Hence we derive the result [12]. The proof is complete.

We can easily find positive roots of the Bessel functions \( J_{\nu}(z) \) in formula (12) only for one-dimensional and three-dimensional cases. Therefore, only for that cases we present the following corollaries:

**Corollary 2.** Let \( B(s) \) be a one-dimensional Brownian motion, starting from the point \( x \in [-a, a] \). Then
\[
F^1(t, x, a) = P\{ \sup_{0 \leq s \leq t} |B(s) + x| < a \} = 4 \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp \left( - \frac{(2k+1)^2 \pi^2 t}{8a^2} \right) \cos \left( \frac{(2k+1)\pi x}{2a} \right), \tag{31}
\]
where \(-a < x < a\).
Proof. It is easy to see, that if \( m = 1 \), then \( \nu = -\frac{1}{2} \), \( J_\nu(x) = J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos(x) \), \( J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin(x) \). The positive roots of the Bessel function \( J_{-\frac{1}{2}}(x) \) are \( \mu_n = \frac{\pi}{2}(1 + 2n), \ n = 0, 1, 2, \ldots \).

Thus, we have
\[
2a^\nu J_\nu(\mu_n r/a) r^\nu \mu_n J_{\nu+1}(\mu_n) = 2 a \cos\left(\frac{\mu_n r}{a}\right) \mu_n \sin(\mu_n) = \frac{4}{\pi(2n+1)} \cos\left(\frac{(2k+1)\pi r}{2a}\right)(-1)^n.
\]

Applying this formula and (12), we get the proof of Corollary 2. The proof is complete.

This formula gives a probability that the one-dimensional Brownian motion leaving the point \( x \), will not be absorbed till the moment \( t \). Hence we derive the result [11].

**Corollary 3.** Let \( B^3(s), \ 0 \leq s \leq t \), be a three-dimensional Brownian motion, starting from the point \( x \) on the sphere \( S^3_r \) with the radius \( r = |x| < a \). Then
\[
v_3(t, r, a) = -2 \sum_{n=1}^{\infty} (-1)^n \frac{a}{\pi r n} \sin\left(\frac{\pi r n}{a}\right) \exp\left(-\frac{n^2\pi^2 t}{2a^2}\right).
\]  
(32)

**Proof.** If \( m = 3 \), then \( \nu = \frac{m}{2} - 1 = \frac{1}{2} \) and \( J_\nu(x) = J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin(x) \), \( J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}} (\sin(x) - \cos(x)) \). The positive roots of the Bessel function \( J_{\frac{1}{2}}(x) \) are \( \mu_n = \pi n, \ n = 1, 2, \ldots \).

Consequently
\[
2a^\nu J_\nu(\mu_n r/a) r^\nu \mu_n J_{\nu+1}(\mu_n) = -2 \frac{a}{\pi r n} \sin\left(\frac{\pi r n}{a}\right)(-1)^n.
\]

The proof is complete.

**Corollary 4.** Let \( B^3(s) \) be a three-dimensional Brownian movement, starting from the beginning of coordinates, then passing to the limit as \( r \to 0 \), we obtain.
\[
v_3(t, 0, a) = -2 \sum_{n=1}^{\infty} (-1)^n \exp\left(-\frac{n^2\pi^2 t}{2a^2}\right).
\]  
(33)

**Proof.** It is obvious, that the limit:
\[
\lim_{r \to 0} \frac{a}{\pi r n} \sin\left(\frac{\pi r n}{a}\right) = 1
\]

It proves (33). The proof is complete.
References


