Infinite point and Riemann–Stieltjes integral conditions for an integro-differential equation

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Abstract. In this paper, we study the existence of solutions for two nonlocal problems of integro-differential equation with nonlocal infinite-point and Riemann–Stieltjes integral boundary conditions. The continuous dependence of the solution will be studied.

Keywords: existence of solutions, continuous dependence, nonlocal condition, Riemann–Stieltjes condition, infinite point condition.

1 Introduction

In the last few years, some investigators have established a lot of useful and interesting functional differential equation with the nonlocal condition in order to achieve various goals; see [1–9, 11, 12, 14–21] and the references cited therein.

In this paper, we are concerned with the nonlocal problem for the integro-differential equation

\[
\frac{dx}{dt} = f\left(t, x(t), \int_0^t g(s, x(s)) \, ds\right), \quad \text{a.e. } t \in (0, 1),
\]

with the nonlocal condition

\[
\sum_{k=1}^{m} a_k x(\tau_k) = x_0, \quad a_k \geq 0, \quad \tau_k \in (0, 1).
\]
As applications, the nonlocal problem of equation (1) with the Riemann–Stieltjes integral condition

\[ \int_0^1 x(s) \, dg(s) = x_0 \]  

will be studied. Also, the nonlocal problem of equation (1) with infinite-point boundary condition

\[ \sum_{k=1}^{\infty} a_k x(\tau_k) = x_0 \]  

will be studied.

2 Main results

2.1 Integral representation

Lemma 1. Let \( B = \sum_{k=1}^{m} a_k \neq 0 \), the solution of the nonlocal problem (1)–(2), if it exist, then it can be represented by the integral equation

\[
x(t) = B^{-1} \left[ x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f \left( s, x(s), \int_0^s g(\theta, x(\theta)) \, d\theta \right) \, ds \right] \\
+ \int_0^t f \left( s, x(s), \int_0^s g(\theta, x(\theta)) \, d\theta \right) \, ds.
\]  

(5)

Proof. Let \( x \) be a solution of the nonlocal problem (1)–(2). Integrating both sides of (1), we get

\[
x(t) = x(0) + \int_0^t f \left( s, x(s), \int_0^s g(\theta, x(\theta)) \, d\theta \right) \, ds.
\]  

(6)

Using the nonlocal condition (2), we get

\[
\sum_{k=1}^{m} a_k x(\tau_k) = x(0) \sum_{k=1}^{m} a_k + \sum_{k=1}^{m} a_k \int_0^{\tau_k} f \left( s, x(s), \int_0^s g(\theta, x(\theta)) \, d\theta \right) \, ds,
\]

then

\[
x(0) = \frac{1}{\sum_{k=1}^{m} a_k} \left[ x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f \left( s, x(s), \int_0^s g(\theta, x(\theta)) \, d\theta \right) \, ds \right].
\]  

(7)
Using (6) and (7), we obtain
\[
x(t) = B^{-1}\left[ x_0 - \sum_{k=1}^{m} a_k \int_0^t f\left( s, x(s), \int_0^s g(\theta, x(\theta)) d\theta \right) ds \right] \\
+ \int_0^t f\left( s, x(s), \int_0^s g(\theta, x(\theta)) d\theta \right) ds.
\]

2.2 Existence of solution

2.2.1 Functional equation approach

Consider the nonlocal problem (1)–(2) with the assumptions:

(i) \( f : [0, T] \times \mathbb{R}^2 \to \mathbb{R} \) satisfies Caratheodory condition, i.e., \( f \) is measurable in \( t \) for any \( x, y \in \mathbb{R} \) and continuous in \( x, y \) for almost all \( t \in [0, 1] \). There exist a function \( c_1 \in L^1[0, 1] \) and a positive constant \( b_1 > 0 \) such that
\[
|f(t, x, y)| \leq c_1(t) + b_1|x| + b_1|y|.
\]

(ii) \( g : [0, 1] \times \mathbb{R} \to \mathbb{R} \) satisfies Caratheodory condition, i.e., \( g \) is measurable in \( t \) for any \( x \in \mathbb{R} \) and continuous in \( x \) for almost all \( t \in [0, 1] \). There exist a function \( c_2 \in L^1[0, 1] \) and a positive constant \( b_2 > 0 \) such that
\[
|g(t, x)| \leq c_2(t) + b_2|x|.
\]

(iii) \[
\sup_{\tau \in [0, 1]} \int_0^\tau c_1(s) ds \leq M_1, \quad \sup_{\tau \in [0, 1]} \int_0^\tau c_2(s) ds \leq M_2,
\]

(iv) \( 2b_1 + b_1b_2 < 1 \).

**Definition 1.** By a solution of the nonlocal problem (1)–(2) we mean a function \( x \in C[0, 1] \) that satisfies (1)–(2).

**Theorem 1.** Let assumptions (i)–(iv) be satisfied, then the nonlocal problem (1)–(2) has at least one solution.

**Proof.** Define the operator \( A \) associated with the integral equation (5) by
\[
Ax(t) = B^{-1}\left[ x_0 - \sum_{k=1}^{m} a_k \int_0^t f\left( s, x(s), \int_0^s g(\theta, x(\theta)) d\theta \right) ds \right] \\
+ \int_0^t f\left( s, x(s), \int_0^s g(\theta, x(\theta)) d\theta \right) ds.
\]

Let \( Q_r = \{ x \in \mathbb{R} : ||x|| \leq r \} \), where \( r = B^{-1}(|x_0| + 2M_1 + 2b_2M_2)/(1 - (2b_1 + b_1b_2)) \). Then we have, for \( x \in Q_r \),

\[
|Ax(t)| \leq B^{-1} \left[ |x_0| + \sum_{k=1}^{m} a_k \int_0^{|x(s)|} f \left( s, x(s), \int_0^s g(\theta, x(\theta)) \, d\theta \right) \, ds \right] + \int_0^t \left| f \left( s, x(s), \int_0^s g(\theta, x(\theta)) \, d\theta \right) \right| \, ds
\]

\[
\leq B^{-1} \left[ |x_0| + \sum_{k=1}^{m} a_k \int_0^{|x(s)|} \left( c_1(s) + b_1|x(s)| + b_1 \int_0^s |g(\theta, x(\theta))| \, d\theta \right) \, ds \right] + \int_0^t \left( c_1(s) + b_1|x(s)| + b_1 \int_0^s |g(\theta, x(\theta))| \, d\theta \right) \, ds
\]

\[
\leq B^{-1} \left[ |x_0| + \sum_{k=1}^{m} a_k \left( M_1 + b_1r + b_1 \int_0^t \int_0^s c_2(\theta) + b_2|x(\theta)| \, d\theta \, ds \right) \right] + M_1 + b_1r + b_1 \int_0^t \int_0^s (c_2(\theta) + b_2|x(\theta)|) \, d\theta \, ds
\]

\[
\leq B^{-1}|x_0| + M_1 + b_1r + b_1M_2 + \frac{1}{2}b_1b_2r + M_1 + b_1r + b_1M_2 + \frac{1}{2}b_1b_2r
\]

\[
= B^{-1}|x_0| + 2M_1 + 2b_1r + 2b_1M_2 + b_1b_2r = r.
\]

This proves that \( A : Q_r \to Q_r \) and the class of functions \( \{Ax\} \) is uniformly bounded in \( Q_r \).

Now, let \( t_1, t_2 \in (0, 1) \) such that \(|t_2 - t_1| < \delta\), then

\[
|Ax(t_2) - Ax(t_1)|
\]

\[
\leq \int_{t_1}^{t_2} \left| f \left( s, x(s), \int_0^s g(\theta, x(\theta)) \, d\theta \right) \right| \, ds
\]

\[
\leq \int_{t_1}^{t_2} \left( c_1(s) + b_1|x(s)| + b_1 \int_0^s |g(\theta, x(\theta))| \, d\theta \right) \, ds
\]

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\[ \leq \int_{t_1}^{t_2} c_1(s) \, ds + (t_2 - t_1)b_1 r + b_1 \int_{t_1}^{t_2} c_2(\theta) \, d\theta \, ds \]

\[ + \frac{1}{2} b_1 b_2 \left( t_2^2 - t_1^2 \right). \]

This means that the class of functions \( \{ Ax \} \) is equicontinuous in \( Q_r \).

Let \( x_n \in Q_r, x_n \rightharpoonup x ( n \to \infty ) \), then from continuity of the functions \( f \) and \( g \) we obtain \( f(t, x_n(t), y_n(t)) \to f(t, x(t), y(t)) \) and \( g(t, x_n(t)) \to g(t, x(t)) \) as \( n \to \infty \). Also

\[
\lim_{n \to \infty} A x_n(t) = \lim_{n \to \infty} \left[ B^{-1} \left[ x_0 - \sum_{k=1}^{m} a_k \int_{0}^{\tau_k} f \left( s, x_n(s), \int_{0}^{s} g \left( \theta, x_n(\theta) \right) \, d\theta \right) \right] \right] \\
+ \int_{0}^{\tau} f \left( s, x_n(s), \int_{0}^{s} g \left( \theta, x_n(\theta) \right) \, d\theta \right) \, ds \]
\]

Using assumptions (i)–(ii) and Lebesgue dominated convergence theorem [13], from (8) we obtain

\[
\lim_{n \to \infty} A x_n(t) = \left[ B^{-1} \left[ x_0 - \sum_{k=1}^{m} a_k \int_{0}^{\tau_k} \lim_{n \to \infty} f \left( s, x_n(s), \int_{0}^{s} g \left( \theta, x_n(\theta) \right) \, d\theta \right) \right] \right] \\
+ \int_{0}^{\tau} \lim_{n \to \infty} f \left( s, x_n(s), \int_{0}^{s} g \left( \theta, x_n(\theta) \right) \, d\theta \right) \, ds = A x(t). \]

Then \( A x_n \to A x \) as \( n \to \infty \). This means that the operator \( A \) is continuous.

\[
\lim_{t \to 1} x(t) = \left[ B^{-1} \left[ x_0 - \sum_{k=1}^{m} a_k \int_{0}^{\tau_k} f \left( s, x(s), \int_{0}^{s} g \left( \theta, x(\theta) \right) \, d\theta \right) \right] \right] \\
+ \int_{0}^{1} f \left( s, x(s), \int_{0}^{s} g \left( \theta, x(\theta) \right) \, d\theta \right) \, ds \in C[0, 1],
\]

and

\[
\lim_{t \to 0} x(t) = B^{-1} \left[ x_0 - \sum_{k=1}^{m} a_k \int_{0}^{\tau_k} f \left( s, x(s), \int_{0}^{s} g \left( \theta, x(\theta) \right) \, d\theta \right) \right] \in C[0, 1].
\]

Then by Schauder fixed point theorem [10] there exist at least one solution \( x \in C[0, 1] \) of the integral equation (5).

To complete the proof, differentiating (5) we obtain
\[
\frac{dx}{dt} = \frac{d}{dt} \left\{ B^{-1} \left[ x_0 - \sum_{k=1}^{m} a_k \int_{0}^{\tau_k} \left( f \left( s, x(s), \int_{0}^{s} g(\theta, x(\theta)) \, d\theta \right) \right) \, ds \right] \right\}
+ \int_{0}^{t} f \left( s, x(s), \int_{0}^{s} g(\theta, x(\theta)) \, d\theta \right) \, ds
= 0 + \frac{d}{dt} \int_{0}^{t} f \left( s, x(s), \int_{0}^{s} g(\theta, x(\theta)) \, d\theta \right) \, ds
= f \left( s, x(s), \int_{0}^{s} g(\theta, x(\theta)) \, d\theta \right).
\]

Also, from the integral equation (5), we obtain
\[
x(\tau_k) = B^{-1} \left[ x_0 - \sum_{k=1}^{m} a_k \int_{0}^{\tau_k} \left( f \left( s, x(s), \int_{0}^{s} g(\theta, x(\theta)) \, d\theta \right) \right) \, ds \right]
+ \sum_{k=1}^{m} a_k \int_{0}^{\tau_k} f \left( s, x(s), \int_{0}^{s} g(\theta, x(\theta)) \, d\theta \right) \, ds
\]
and
\[
\sum_{k=1}^{m} a_k x(\tau_k) = \sum_{k=1}^{m} a_k B^{-1} \left[ x_0 - \sum_{k=1}^{m} a_k \int_{0}^{\tau_k} \left( f \left( s, x(s), \int_{0}^{s} g(\theta, x(\theta)) \, d\theta \right) \right) \, ds \right]
+ \sum_{k=1}^{m} a_k \int_{0}^{\tau_k} f \left( s, x(s), \int_{0}^{s} g(\theta, x(\theta)) \, d\theta \right) \, ds.
\]

Then
\[
\sum_{k=1}^{m} a_k x(\tau_k) = x_0.
\]

Then there exist at least one solution \( x \in C[0,1] \) of the nonlocal problem of functional differential equation (1)–(2).

2.2.2 Coupled system approach

Let the function \( f \) and \( g \) satisfies the conditions:

\( (i^*) \) \( f : [0,T] \times \mathbb{R}^2 \rightarrow \mathbb{R} \) satisfies Caratheodory condition, i.e., \( f \) is measurable in \( t \) for any \( x,y \in \mathbb{R} \) and continuous in \( x,y \) for almost all \( t \in [0,1] \). There exist a function \( m_1 \in L^1[0,1] \) such that
\[
|f(t,x,y)| \leq m_1(t).
\]

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(ii*) \( g : [0, 1] \times \mathbb{R} \to \mathbb{R} \) satisfies Caratheodory condition, i.e., \( g \) is measurable in \( t \) for any \( x \in \mathbb{R} \) and continuous in \( x \) for almost all \( t \in [0, 1] \). There exist a function \( m_2 \in L^1[0, 1] \) such that
\[
|g(t, x)| \leq m_2(t).
\]

(iii*)
\[
\sup_{t \in [0, 1]} \int_0^t m_1(s) \, ds \leq M_1, \quad \sup_{t \in [0, 1]} \int_0^t m_2(s) \, ds \leq M_2.
\]

Now, let
\[
y(t) = \int_0^t g(\theta, x(\theta)) \, d\theta,
\]
then
\[
x(t) = B^{-1} \left[ x_0 - \sum_{k=1}^m a_k \int_0^t f(s, x(s), y(s)) \, ds \right] + \int_0^t f(s, x(s), y(s)) \, ds. \tag{10}
\]

Let \( X \) be the Banach space of all order pairs \((x, y)\) with the norm
\[
\|(x, y)\|_X = \|x\|_C + \|y\|_C = \sup_{t \in [0, 1]} |x(t)| + \sup_{t \in [0, 1]} |y(t)|.
\]

**Definition 2.** By a solution of the nonlocal problem (1)–(2) we mean a function \( x \in C^1[0, 1] \) that satisfies \((1)–(2)\).

**Theorem 2.** Let assumptions (i*)–(iii*) be satisfied, then the nonlocal problem (1)–(2) has at least one solution.

**Proof.** Define the operator \( A \) associated with the integral equation (9)–(10) by
\[
A(x(t), y(t)) = \left( B^{-1} \left[ x_0 - \sum_{k=1}^m a_k \int_0^t f(s, x(s), y(s)) \, ds \right] + \int_0^t f(s, x(s), y(s)) \, ds, \right)
\]

Let \( Q_r = \{(x, y) \in \mathbb{R}^2 : \|x\| \leq r_1, \|y\| \leq r_2, \|(x, y)\| \leq r_1 + r_2 = r\} \), where \( r = M_1 + M_2 \).

Then we have, for \((x, y) \in Q_r\)
\[
A(x(t), y(t)) = \left( B^{-1} \left[ x_0 - \sum_{k=1}^m a_k \int_0^t f(s, x(s), y(s)) \, ds \right] + \int_0^t f(s, x(s), y(s)) \, ds, \right)
\]

but
\[
\left| B^{-1} \left[ x_0 - \sum_{k=1}^{m} a_k \int_0^\tau f(s, x(s), y(s)) \, ds \right] + \int_0^t f(s, x(s), y(s)) \, ds \right| \\
\leq B^{-1} \left[ |x_0| + \sum_{k=1}^{m} a_k \int_0^\tau m_1(s) \, ds \right] + \int_0^t m_1(s) \, ds \\
\leq B^{-1} |x_0| + 2M_1 \tag{11}
\]

and
\[
\left| \int_0^t g(\theta, x(\theta)) \, d\theta \right| \leq \int_0^t m_2(\theta) \, d\theta \leq M_2. \tag{12}
\]

From (11) and (12) we get
\[
\| A(x, y) \|_X \leq B^{-1} |x_0| + 2M_1 + M_2.
\]

This prove that \( A : Q_r \rightarrow Q_r \) and the class of functions \( \{ A(x, y) \} \) is uniformly bounded in \( Q_r \).

Now, let \( t_1, t_2 \in (0, 1) \) such that \( |t_2 - t_1| < \delta \), then
\[
|A(x(t_2), y(t_2)) - A(x(t_1), y(t_1))|
\]
\[
= \left| \left( B^{-1} \left[ x_0 - \sum_{k=1}^{m} a_k \int_0^\tau f(s, x(s), y(s)) \, ds \right] + \int_0^t f(s, x(s), y(s)) \, ds, \right. \right |
\]
\[
\left. \int_0^{t_2} g(\theta, x(\theta)) \, d\theta \right) \right| \\
- \left( B^{-1} \left[ x_0 - \sum_{k=1}^{m} a_k \int_0^\tau f(s, x(s), y(s)) \, ds \right] + \int_0^t f(s, x(s), y(s)) \, ds, \right. \right |
\]
\[
\left. \int_0^{t_1} g(\theta, x(\theta)) \, d\theta \right) \right|
\]
\[
= \left| \left( \int_{t_1}^{t_2} f(s, x(s), y(s)) \, ds, \int_{t_1}^{t_2} g(\theta, x(\theta)) \, d\theta \right) \right|.
\]

but
\[
\left| \int_{t_1}^{t_2} f(s, x(s), y(s)) \, ds \right| \leq \int_{t_1}^{t_2} m_1(s) \, ds, \quad \left| \int_{t_1}^{t_2} g(\theta, x(\theta)) \, d\theta \right| \leq \int_{t_1}^{t_2} m_2(s) \, ds. \tag{13}
\]
From (13) we get

\[ |A(x(t_2), y(t_2)) - A(x(t_1), y(t_1))| \leq \int_{t_1}^{t_2} (m_1(s) + m_2(s)) \, ds. \]

This means that the class of functions \( \{A(x, y)\} \) is equicontinuous in \( Q_r \).

Let \( x_n \in Q_r \), \( x_n \to x \) \( (n \to \infty) \), then from continuity of the functions \( f \) and \( g \) we obtain \( f(t, x_n(t), y_n(t)) \to f(t, x(t), y(t)) \) and \( g(t, x_n(t)) \to g(t, x(t)) \) as \( n \to \infty \).

Also

\[ \lim_{n \to \infty} A(x_n(t), y_n(t)) = \lim_{n \to \infty} \left( B^{-1} \left[ x_0 - \sum_{k=1}^{m} a_k \int_0^{T_k} f(s, x_n(s), y_n(s)) \, ds \right] + \int_0^t f(s, x_n(s), y_n(s)) \, ds, \right. \]

\[ \left. \quad \int_0^t g(s, x_n(\theta)) \, d\theta \right) \]

\[ = \left( B^{-1} \left[ x_0 - \sum_{k=1}^{m} a_k \int_0^{T_k} f(s, x(s), y(s)) \, ds \right] + \int_0^t f(s, x(s), y(s)) \, ds, \right. \]

\[ \left. \quad \int_0^t g(s, x(\theta)) \, d\theta \right) \]

\[ = A(x(t), y(t)). \]

Using assumptions (i)–(ii) and Lebesgue dominated convergence theorem [13], from (14) we obtain

\[ \lim_{n \to \infty} A(x_n(t), y_n(t)) = A(x(t), y(t)). \]

Then \( Ax_n \to Ax \) as \( n \to \infty \). This means that the operator \( A \) is continuous.

\[ \lim_{t \to 1} x(t) = \left\{ B^{-1} \left[ x_0 - \sum_{k=1}^{m} a_k \int_0^{T_k} f(s, x(s), y(s)) \, ds \right] + \int_0^1 f(s, x(s), y(s)) \, ds \right\} \]

\[ \in C[0, 1], \]

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and
\[
\lim_{t\to 1} y(t) = \int_0^1 g(s, x(\theta)) \, d\theta \in C[0, 1],
\]
\[
\lim_{t\to 0} x(t) = B^{-1} \left[ x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} f(s, x(s), y(s)) \, ds \right] \in C[0, 1],
\]
\[
\lim_{t\to 0} y(t) = 0 \in C[0, 1],
\]

Then by Schauder fixed point theorem [10] there exist at least one solution
\( x \in C[0, 1] \) of the integral equation (9)–(10).

To complete the proof, differentiating (10), we obtain
\[
\frac{dx}{dt} = \frac{d}{dt} \left\{ B^{-1} \left[ x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} f(s, x(s), y(s)) \, ds \right] + \int_0^t f(s, x(s), y(s)) \, ds \right\}
\]
\[
= 0 + \frac{d}{dt} \int_0^t f(s, x(s), y(s)) \, ds = f(s, x(s), y(s)),
\]
\[
y(t) = \int_0^t g(s, x(\theta)) \, d\theta.
\]

Also, from the integral equation (9)–(10) we obtain
\[
x(\tau_k) = B^{-1} \left[ x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} f(s, x(s), y(s)) \, ds \right] + \int_0^{\tau_k} f(s, x(s), y(s)) \, ds,
\]
\[
y(t) = \int_0^t g(s, x(\theta)) \, d\theta,
\]

and
\[
\sum_{k=1}^m a_k x(\tau_k) = \sum_{k=1}^m a_k B^{-1} \left[ x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} f(s, x(s), y(s)) \, ds \right]
\]
\[
+ \sum_{k=1}^m a_k \int_0^{\tau_k} f(s, x(s), y(s)) \, ds
\]
\[
y(t) = \int_0^t g(s, x(\theta)) \, d\theta.
\]
Then
\[
\sum_{k=1}^{m} a_k x(\tau_k) = x_0.
\]

Hence, the nonlocal problem (1)–(2) is equivalent to integral equation (9)–(10).

2.3 Uniqueness of the solution

Let \( f \) and \( g \) satisfy the following assumptions:

\( (v) \) \( f : [0, T] \times \mathbb{R}^2 \to \mathbb{R} \) is measurable in \( t \) for any \( x, y \in \mathbb{R} \) and satisfies the Lipschitz condition
\[
|f(t, x, y) - f(t, u, v)| \leq b_1 |x - u| + b_1 |y - v|.
\]

\( (vi) \) \( g : [0, T] \times \mathbb{R} \to \mathbb{R} \) is measurable in \( t \) for any \( x \in \mathbb{R} \) and satisfies the Lipschitz condition
\[
|g(t, x) - g(t, u)| \leq b_2 |x - u|.
\]

\( (vii) \) \( \sup_{t \in [0,1]} \int_0^t |f(s, 0, 0)| \, ds \leq L_1 \), \( \sup_{t \in [0,1]} \int_0^t \int_0^s |g(\theta, 0)| \, d\theta \, ds \leq L_2 \).

**Theorem 3.** Let assumptions (v)–(vii) be satisfied, then the solution of the nonlocal problem (1)–(2) is unique.

**Proof.** From assumption (v) we have that \( f \) is measurable in \( t \) for any \( x, y \in \mathbb{R} \) and satisfies the Lipschitz condition, then it is continuous in \( x, y \in \mathbb{R} \) for all \( t \in [0, 1] \), and
\[
|f(t, x, y)| \leq b_1 |x| + b_1 |y| + |f(t, 0, 0)|.
\]

Condition (i) is satisfied. Also by the same way we can show that assumption (ii) satisfied by assumption (vi). Now, from Theorem 1 the solution of the nonlocal problem (1)–(2) exists.

Let \( x, y \) be two the solution of (1)–(2), then
\[
|x(t) - y(t)| = \left| B^{-1} \left[ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f \left( s, x(s), \int_0^s g(\theta, x(\theta)) \, d\theta \right) \, ds \right] 
\]
\[
+ \int_0^t f \left( s, x(s), \int_0^s g(\theta, x(\theta)) \, d\theta \right) \, ds 
\]
\[
- B^{-1} \left[ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f \left( s, y(s), \int_0^s g(\theta, y(\theta)) \, d\theta \right) \, ds \right] 
\]
\[
- \int_0^t f \left( s, y(s), \int_0^s g(\theta, y(\theta)) \, d\theta \right) \, ds \right|.
\]
\[ \leq B^{-1} \sum_{k=1}^{m} a_k \int_0^{\tau_k} \left| f \left( s, x(s), \int_0^{s} g(\theta, x(\theta)) \, d\theta \right) \right| \, ds \\
- \left| f \left( s, y(s), \int_0^{s} g(\theta, y(\theta)) \, d\theta \right) \right| \, ds \\
+ \int_0^{t} \left| f \left( s, x(s), \int_0^{s} g(\theta, x(\theta)) \, d\theta \right) \right| \, ds - f \left( s, y(s), \int_0^{s} g(\theta, y(\theta)) \, d\theta \right) \, ds, \]

\[ \leq B^{-1} \sum_{k=1}^{m} a_k \int_0^{\tau_k} \left( b_1 \| x - y \| + b_1 \int_0^{s} \left| g(\theta, x(\theta)) - g(\theta, y(\theta)) \right| \, d\theta \right) \, ds \\
+ \int_0^{t} \left( b_1 \| x - y \| + b_1 \int_0^{s} \left| g(\theta, x(\theta)) - g(\theta, y(\theta)) \right| \, d\theta \right) \, ds \\
\leq b_1 \| x - y \| + \frac{1}{2} b_1 b_2 \| x - y \| + b_1 \| x - y \| + \frac{1}{2} b_1 b_2 \| x - y \| \\
= (2b_1 + b_1 b_2) \| x - y \|. \]

Hence,

\[ (1 - 2b_1 + b_1 b_2) \| x - y \| \leq 0. \]

Since \((2b_1 + b_1 b_2) < 1\), then \(x(t) = y(t)\), and the solution of the nonlocal problem (1)–(2) is unique. \(\square\)

### 2.4 Continuous dependence

#### 2.4.1 Continuous dependence on \(x_0\)

**Definition 3.** The solution \(x \in C[0,1]\) of the nonlocal problem (1)–(2) depends continuously on \(x_0\) if

\[ \forall \epsilon > 0, \ \exists \delta(\epsilon): \ |x_0 - x_0^*| < \delta \implies \|x - x^*\| < \epsilon, \]

where \(x^*\) is the solution of the nonlocal problem

\[ \frac{dx^*}{dt} = f \left( t, x^*(t), \int_0^{t} g(s, x^*(s)) \, ds \right), \ \text{a.e. } t \in (0,1), \quad (15) \]

with the nonlocal condition

\[ \sum_{k=1}^{n} a_k x^*(\tau_k) = x_0^*, \ \ a_k \geq 0, \ \tau_k \in (0,1). \quad (16) \]

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Theorem 4. Let the assumptions of Theorem 3 be satisfied, then the solution of the nonlocal problem (1)–(2) depends continuously on $x_0$.

Proof. Let $x, x^*$ be two solutions of the nonlocal problems (1)–(2) and (15)–(16), respectively. Then

$$
|x(t) - x^*(t)|
$$

$$
= \left| B^{-1} \left[ x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} f \left( s, x(s), \int_0^s g(\theta, x(\theta)) \, d\theta \right) \, ds \right] 
$$

$$
+ \int_0^t f \left( s, x(s), \int_0^s g(\theta, x(\theta)) \, d\theta \right) \, ds 
$$

$$
- B^{-1} \left[ x_0^* - \sum_{k=1}^m a_k \int_0^{\tau_k} f \left( s, x^*(s), \int_0^s g(\theta, x^*(\theta)) \, d\theta \right) \, ds \right] 
$$

$$
+ \int_0^t f \left( s, x^*(s), \int_0^s g(\theta, x^*(\theta)) \, d\theta \right) \, ds 
$$

$$
\leq B^{-1} |x_0 - x_0^*| 
$$

$$
+ B^{-1} \sum_{k=1}^m a_k \int_0^{\tau_k} \left| f \left( s, x^*(s), \int_0^s g(\theta, x^*(\theta)) \, d\theta \right) - f \left( s, x(s), \int_0^s g(\theta, x(\theta)) \, d\theta \right) \right| \, ds 
$$

$$
+ \int_0^t \left| f \left( s, x(s), \int_0^s g(\theta, x(\theta)) \, d\theta \right) - f \left( s, x^*(s), \int_0^s g(\theta, x^*(\theta)) \, d\theta \right) \right| \, ds 
$$

$$
\leq B^{-1} |x_0 - x_0^*| 
$$

$$
+ B^{-1} \sum_{k=1}^m a_k \int_0^{\tau_k} \left( b_1 \|x - x^*\| + b_1 \int_0^s \left| g(\theta, x^*(\theta)) - g(\theta, x(\theta)) \right| \, d\theta \right) \, ds 
$$

$$
+ \int_0^t \left( b_1 \|x - x^*\| + b_1 \int_0^s \left| g(\theta, x(\theta)) - g(\theta, x^*(\theta)) \right| \, d\theta \right) \, ds 
$$

$$
\leq B^{-1} |x_0 - x_0^*| + b_1 \|x - y\| + \frac{1}{2} b_1 b_2 \|x - x^*\| + b_1 \|x - x^*\| 
$$

$$
+ \frac{1}{2} b_1 b_2 \|x - x^*\| 
$$

$$
\leq B^{-1} \delta + (2b_1 + b_1 b_2) \|x - x^*\|.
$$
Hence,

\[ \| x - x^* \| \leq \frac{B^{-1}\delta}{[1 - (2b_1 + b_1b_2)]} = \epsilon. \]

This means that the solution of the nonlocal problem (1)–(2) depends continuously on \( x_0 \).

The proof is completed. \( \square \)

2.4.2 Continuous dependence on \( a_k \)

**Definition 4.** The solution \( x \in C[0, 1] \) of the nonlocal problem (1)–(2) depends continuously on \( a_k \) if

\[ \forall \epsilon > 0, \exists \delta(\epsilon): |a_k - a_k^*| < \delta \implies \| x - x^* \| < \epsilon, \]

where \( x^* \) is the solution of the nonlocal problem

\[ \frac{dx^*}{dt} = f(t, x^*(t), \int_0^t g(s, x^*(s)) \, ds), \quad \text{a.e. } t \in (0, 1), \quad (17) \]

with the nonlocal condition

\[ \sum_{k=1}^n a_k^* x^*(\tau_k) = x_0, \quad a_k \geq 0, \quad \tau_k \in (0, 1). \quad (18) \]

**Theorem 5.** Let the assumptions of Theorem 3 be satisfied, then the solution of the nonlocal problem (1)–(2) depends continuously on \( a_k \).

**Proof.** Let \( B^* = \sum_{k=1}^n a_k^* \neq 0 \), and let \( x, x^* \) be two solutions of the nonlocal problems (1)–(2) and (17)–(18), respectively. Then

\[
\begin{align*}
| x(t) - x^*(t) | &= B^{-1} \left[ x_0 - \sum_{k=1}^n a_k \int_0^{\tau_k} f(s, x(s), \int_0^s g(\theta, x(\theta)) \, d\theta) \, ds \right] \\
&\quad + \int_0^t f(s, x(s), \int_0^s g(\theta, x(\theta)) \, d\theta) \, ds \\
&\quad - B^{*-1} \left[ x_0 - \sum_{k=1}^n a_k^* \int_0^{\tau_k} f(s, x^*(s), \int_0^s g(\theta, x^*(\theta)) \, d\theta) \, ds \right] \\
&\quad - \int_0^t f(s, x^*(s), \int_0^s g(\theta, x^*(\theta)) \, d\theta) \, ds \\
\end{align*}
\]
Infinite point and Riemann–Stieltjes integral conditions for an integro-differential equation

\[ \begin{align*}
\leq B^{-1}B^{-1}m\delta x_0 \\
&+ B^{-1}\sum_{k=1}^{m} a_k^* \int_0^\tau_k \left| f \left( s, x^* (s), \int_0^s g (\theta, x^* (s)) \, d\theta \right) \right| \\
&+ B^{-1}\sum_{k=1}^{m} |a_k^* - a_k| \int_0^\tau_k \left| f \left( s, x(s), \int_0^s g (\theta, x(s)) \, d\theta \right) \right| \\
&+ B^{-1}B^{-1}m\delta x_0 + (2b_1 + b_1 b_2)\|x - x^*\| \\
&+ B^{-1}m\delta (2b_1 \|x\| + b_1 b_2 \|x\| + 2L_1 + 2b_1 L_2). 
\end{align*} \]

Hence,

\[ \|x - x^*\| \leq \frac{m\delta x_0 + m\delta B((2b_1 + b_1 b_2)\|x\| + 2L_1 + 2b_1 L_2)}{1 - (2b_1 + b_1 b_2)BB^*} = \epsilon. \]

This means that the solution of the nonlocal problem (1)–(2) depends continuously on \(a_k\).

The proof is completed. \(\square\)

2.4.3 Continuous dependence on the function \(g\)

**Definition 5.** The solution \(x \in C[0, 1]\) of the nonlocal problem (1)–(2) depends continuously on the function \(g\) if

\[ \forall \epsilon > 0, \exists \delta (\epsilon) : \|g - g^\ast\| < \delta \implies \|x - x^\ast\| < \epsilon, \]

where \(x^\ast\) is the solution of the nonlocal problem

\[ \frac{dx^\ast}{dt} = f \left( t, x^\ast (t), \int_0^t g^\ast (s, x^\ast (s)) \, ds (s, x^\ast (s)) \right), \text{ a.e. } t \in (0, 1), \quad (19) \]

with the nonlocal condition

\[ \sum_{k=1}^{n} a_k x^\ast (\tau_k) = x_0, \quad a_k \geq 0, \quad \tau_k \in (0, 1). \quad (20) \]

**Theorem 6.** Let the assumptions of Theorem 3 be satisfied, then the solution of the nonlocal problem (1)–(2) depends continuously on the function \(g\).
Proof. Let \( x, x^* \) be two solutions of the nonlocal problem (1)–(2) and (19)–(20), respectively. Then

\[
|x(t) - x^*(t)| = | B^{-1} \left[ x_0 - \sum_{k=1}^{m} a_k \int_0^t f(s, x(s), \int_0^s g(\theta, x(\theta)) d\theta) \right] ds \\
+ \int_0^t f(s, x(s), \int_0^s g(\theta, x(\theta)) d\theta) ds \\
- B^{-1} \left[ x_0 - \sum_{k=1}^{m} a_k \int_0^t f(s, x^*(s), \int_0^s g^*(\theta, x^*(\theta)) d\theta) \right] ds \\
- \int_0^t f(s, x^*(s), \int_0^s g^*(\theta, x^*(\theta)) d\theta) ds \\
\leq B^{-1} \sum_{k=1}^{m} a_k \int_0^t \left| f(s, x^*(s), \int_0^s g^*(\theta, x^*(\theta)) d\theta) - f(s, x(s), \int_0^s g(\theta, x(\theta)) d\theta) \right| ds \\
+ \int_0^t \left| f(s, x(s), \int_0^s g(\theta, x(\theta)) d\theta) - f(s, x^*(s), \int_0^s g^*(\theta, x^*(\theta)) d\theta) \right| ds,
\]

\[
\leq B^{-1} \sum_{k=1}^{m} a_k \int_0^t \left( b_1 \| x - x^* \| + b_1 \int_0^s \left| g^*(\theta, x^*(\theta)) - g(\theta, x(\theta)) \right| d\theta \right) ds \\
+ \int_0^t \left( b_1 \| x - x^* \| + b_1 \int_0^s \left| g^*(\theta, x^*(\theta)) - g^*(\theta, x^*(\theta)) \right| d\theta \right) ds \\
\leq b_1 \| x - x^* \| + b_1 \| x - x^* \| + b_1 \| x - x^* \| + \frac{1}{2} b_1 \| x - x^* \| + b_1 \| x - x^* \| \\
+ \frac{1}{2} b_1 b_2 \| x - x^* \| \\
\leq b_1 \delta + (2b_1 + b_1 b_2) \| x - x^* \|.
\]

Hence,

\[
\| x - x^* \| \leq \frac{b_1 \delta}{1 - (2b_1 + b_1 b_2) \sum_{k=1}^{m} a_k} = \epsilon.
\]

This means that the solution of the nonlocal problem (1)–(2) depends continuously on the function \( g \). The proof is completed.

\[ \text{http://www.journals.vu.lt/nonlinear-analysis} \]
2.5 Nonlocal Riemann–Stieltjes integral condition

Let \( x \in C[0, 1] \) be the solution of the nonlocal problem (1)–(2). Let \( a_k = g(t_k) - g(t_{k-1}) \), \( g \) is increasing function, \( \tau_k \in (t_{k-1}, t_k) \), \( 0 = t_0 < t_1 < t_2 \cdots < t_m = 1 \), then, as \( m \to \infty \), the nonlocal condition (2) will be

\[
\sum_{k=1}^{m} g(t_k) - g(t_{k-1}) x(\tau_k) = x_0
\]

and

\[
\lim_{m \to \infty} \sum_{k=1}^{m} g(t_k) - g(t_{k-1}) x(\tau_k) = \int_{0}^{1} x(s) \, dg(s) = x_0.
\]

**Theorem 7.** Let assumptions (i)–(iv) be satisfied, then the nonlocal problem of (1)–(3) has at least one solution.

**Proof.** As \( m \to \infty \), the solution of the nonlocal problem (1)–(2) will be

\[
x(t) = \lim_{m \to \infty} \frac{1}{\sum_{k=1}^{m} a_k} \left[ x_0 - \sum_{k=1}^{m} a_k \int_{0}^{\tau_k} f \left( s, x(s), \int_{0}^{s} g(\theta, x(\theta)) \, d\theta \right) \, ds \right]
\]

\[
+ \int_{0}^{t} f \left( s, x(s), \int_{0}^{s} g(\theta, x(\theta)) \, d\theta \right) \, ds
\]

\[
= \frac{1}{g(1) - g(0)} \left[ x_0 - \lim_{m \to \infty} \sum_{k=1}^{m} a_k \int_{0}^{\tau_k} f \left( s, x(s), \int_{0}^{s} g(\theta, x(\theta)) \, d\theta \right) \, ds \left( g(t_k) - g(t_{k-1}) \right) \right]
\]

\[
+ \int_{0}^{t} f \left( s, x(s), \int_{0}^{s} g(\theta, x(\theta)) \, d\theta \right) \, ds
\]

\[
= \frac{1}{g(1) - g(0)} \left[ x_0 - \frac{1}{t} \int_{0}^{t} f \left( s, x(s), \int_{0}^{s} g(\theta, x(\theta)) \, d\theta \right) \, ds \, dt \right]
\]

\[
+ \int_{0}^{t} f \left( s, x(s), \int_{0}^{s} g(\theta, x(\theta)) \, d\theta \right) \, ds.
\]

2.6 Infinite-point boundary condition

**Theorem 8.** Let assumptions (i)–(iv) be satisfied, then the nonlocal problem of (1)–(4) has at least one solution.


Proof. Let the assumptions of Theorem 1 be satisfied, and let \( \sum_{k=1}^{m} a_k \) be convergent, then

\[
x_m(t) = \frac{1}{\sum_{k=1}^{m} a_k} \left[ x_0 - \sum_{k=1}^{m} a_k \int_{0}^{t} f\left(s, x(s), \int_{0}^{s} g(\theta, x(\theta)) d\theta\right) ds \right] + \int_{0}^{t} f\left(s, x_m(s), \int_{0}^{s} g(\theta, x_m(\theta)) d\theta\right) ds.
\]

Taking the limit to (21) as \( m \to \infty \), we have

\[
\lim_{m \to \infty} x_m(t) = \lim_{n \to \infty} \left[ x_0 - \sum_{k=1}^{m} a_k \int_{0}^{t} f\left(s, x(s), \int_{0}^{s} g(\theta, x(\theta)) d\theta\right) ds \right] + \int_{0}^{t} f\left(s, x_m(s), \int_{0}^{s} g(\theta, x_m(\theta)) d\theta\right) ds.
\]

(21)

Now, \(|a_k\tau_k| \leq |a_k||x|\), then by comparison test \( \sum_{k=1}^{\infty} a_k x(\tau_k) \) is convergent.

Also

\[
\left| \int_{0}^{\tau_k} f\left(s, x(s), \int_{0}^{s} g(\theta, x(\theta)) d\theta\right) ds \right|
\]

\[
\leq \int_{0}^{\tau_k} \left(c_1(s) + b_1 |x(s)| + b_1 \int_{0}^{s} g(\theta, x(\theta)) d\theta\right) ds
\]

\[
\leq \int_{0}^{\tau_k} \left(c_1(s) + b_1 |x(s)| + b_1 \left(c_2(s) + b_2 |x(s)|\right) d\theta\right) ds
\]

\[
\leq M_1 + b_1 ||x|| + b_1 M_2 + \frac{1}{2} b_1 b_2 ||x|| \leq M,
\]

then

\[
|a_k| \int_{0}^{\tau_k} f\left(s, x(s), \int_{0}^{s} g(\theta, x(\theta)) d\theta\right) ds \leq |a_k|M,
\]

and by the comparison test \( \sum_{k=1}^{\infty} a_k \int_{0}^{\tau_k} f(s, x(s), \int_{0}^{s} g(\theta, x(\theta)) d\theta) ds \) is convergent.

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Now, $|f| \leq |c_1(s) + b_1\|x\| + b_1 M_2 + b_1 b_2\|x\|$, using assumptions (i)–(ii) and Lebesgue dominated convergence theorem [13], from (22) we obtain

$$x(t) = \frac{1}{\sum_{k=1}^{\infty} a_k} \left[ x_0 - \sum_{k=1}^{\infty} a_k \int_{0}^{t} f \left( s, x(s), \int_{0}^{s} g(\theta, x(\theta)) \, d\theta \right) \, ds \right]$$

$$+ \int_{0}^{t} f \left( s, x(s), \int_{0}^{s} g(\theta, x(\theta)) \, d\theta \right) \, ds.$$ 

The theorem proved.

### 3 Examples

In this section, we offer some examples to illustrate our results.

**Example 1.** Consider the following nonlinear integro-differential equation:

$$\frac{dx}{dt} = t^3 e^{-t} + \frac{\ln(1 + |x(t)|)}{3 + t^2}$$

$$+ \int_{0}^{t} \frac{1}{9} \left( \cos(3s + 3) + s^5 \cos x(s) + e^{-s} x(s) \right) \, dt,$$

a.e. $t \in (0, 1), \quad (23)$

with infinite point boundary condition

$$\sum_{k=1}^{\infty} \frac{1}{k^5} x \left( \frac{k-1}{k} \right) = x_0. \quad (24)$$

Set

$$f \left( t, x(t), \int_{0}^{t} g(s, x(s)) \, ds \right)$$

$$= t^3 e^{-t} + \frac{\ln(1 + |x(t)|)}{3 + t^2} + \frac{1}{9} \int_{0}^{t} \left( \cos(3s + 3) + s^5 \cos x(s) + e^{-s} x(s) \right) \, dt.$$

Then

$$\left| f \left( t, x(t), \int_{0}^{t} g(s, x(s)) \, ds \right) \right|$$

$$\leq t^3 e^{-t} + \frac{1}{3} \left( |x| + \frac{1}{3} \int_{0}^{t} \left( \cos(3s + 3) + s^5 \cos x(s) + e^{-s} x(s) \right) \, dt \right),$$

and also

$$|g(s, x(s))| \leq \frac{1}{3} |\cos(3s + 3)| + \frac{2}{3} |x(s)|.$$
It is clear that assumptions (i)–(iv) of Theorem 1 are satisfied with \( c_1(t) = t^3 e^{-t} \in L^1[0, 1] \), 
\( c_2(t) = \frac{|\cos(3t + 3)|}{2} \in L^1[0, 1] \), \( b_1 = 1/3 \), \( b_2 = 2/3 \), \( b_1 b_2 = 2/3 + 2/9 = 8/9 < 1 \), and the series \( \sum_{k=1}^{\infty} 1/k^3 \), is convergent. Therefore, by applying to Theorem 1 the given nonlocal problem (23)–(24) has a continuous solution.

**Example 2.** Consider the following nonlinear integro-differential equation:

\[
\frac{dx}{dt} = t^3 + t + 1 + \frac{x(t)}{\sqrt{t + 3}} + \int_0^t \frac{1}{4} \left( \sin^2(3s + 3) + \frac{sx(s)}{2s(1 + x(s))} \right) dt, \quad \text{a.e. } t \in (0, 1),
\]

(25)

with infinite point boundary condition

\[
\sum_{k=1}^{\infty} \frac{1}{k^3} x \left( \frac{k^2 + k - 1}{k^2 + k} \right) = x_0.
\]

(26)

Set

\[
f \left( t, x(t), \int_0^t g(s, x(s)) \, ds \right)
\]

\[
= t^3 + t + 1 + \frac{x(t)}{\sqrt{2t + 4}} + \frac{1}{4} \int_0^t \left( \sin^2(3s + 3) + \frac{sx(s)}{2s(1 + x(s))} \right) \, dt.
\]

Then

\[
\left| f \left( t, x(t), \int_0^t g(s, x(s)) \, ds \right) \right| \leq t^3 + t + 1 + \frac{3}{3} |x| + \frac{1}{3} \int_0^t \left| \sin^2(3s + 3) + \frac{sx(s)}{2s(1 + x(s))} \right| \, dt,
\]

and also

\[
|g(s, x(s))| \leq \frac{3}{4} \left| \sin^2(3s + 3) \right| + \frac{3}{8} |x(s)|.
\]

It is clear that the assumptions (i)–(iv) of Theorem 1 are satisfied with \( c_1(t) = t^3 + t + 1 \in L^1[0, 1] \), 
\( c_2(t) = (3/4)(\sin^2(3s + 3)) \in L^1[0, 1] \), \( b_1 = 1/3 \), \( b_2 = 3/8 \), \( 2b_1 + b_1 b_2 = 2/3 + 1/8 = 19/24 < 1 \), and the series \( \sum_{k=1}^{\infty} 1/k^3 \), is convergent. Therefore, by applying to Theorem 1 the given nonlocal problem (25)–(26) has a continuous solution.
References


