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Multiplicative dependence of cubic algebraic numbers

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Abstract. We provide an infinite family of cubic algebraic integers α such that the set $\{\alpha + x \mid x \in \mathbb{Z}\}$ is multiplicatively dependent.

Keywords: algebraic numbers, multiplicative dependence.

Introduction and results

Throughout we denote by $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{C}$ the sets of integers, rational numbers and complex numbers respectively. Let α be an algebraic number and let $M \subset \mathbb{Q}$. We say that a complex number α is *M-dependent* if there are two distinct collections $x_1, \dots, x_n \in M$ and $y_1, \dots, y_m \in M$ such that

$$(\alpha + x_1) \cdots (\alpha + x_n) = (\alpha + y_1) \cdots (\alpha + y_m). \tag{1}$$

Here, $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$, where the right–hand side is assumed to be 1 for $m = 0$. Assume that α is M-dependent. We call its *length of multiplicative dependence* (and denote it by $\ell(\alpha, M)$) the smallest $n+m$ for which there are $x_1, \dots, x_n, y_1, \dots, y_m \in M$ satisfying (1).

Denote by $\mathbb{Z}_t$ the set of integers greater than or equal to t. $\mathbb{Z}_0$-dependence of algebraic numbers is important in the theory of Hurwitz zeta–function $\zeta(\alpha, s) = \sum_{j=0}^{\infty} (j+\alpha)^{-s}$ (see, e. g., [6]) as well as in the in the investigation of zero-distribution and the universality property of Lerch zeta-function (see, e.g., [4, 5]).

The first named author and Dubickas [1] raised the following question.

Question. Is every algebraic number $\mathbb{Z}_0$-dependent?

It was proved in [1] that every quadratic algebraic number α is $\mathbb{Z}_t$-dependent and $\ell(\alpha, \mathbb{Z}_t) \leqslant 8$ for any $t \in \mathbb{Z}$. On the other hand the equality $\alpha(\alpha+x) = \alpha+y$ shows that every quadratic algebraic integer is $\mathbb{Z}$-dependent and $\ell(\alpha, \mathbb{Z}) \leqslant 3$. Moreover, it was noted in [1] that every quadratic algebraic integer is $\mathbb{Z}_t$-dependent and $\ell(\alpha, \mathbb{Z}_t) \leqslant 5$ for every $t \in \mathbb{Z}$.

For quadratic algebraic numbers, which are not algebraic integers, the inequality $\ell(\alpha, \mathbb{Z}_t) \leqslant 8$ is not sharp. Indeed, it was stated in [1] that if α is a root of $2x^2 + 3$ and $\ell(\alpha, \mathbb{Z}_1) = 6$ then there exist two distinct collections $x_1, x_2, x_3 \in \mathbb{Z}_1$ and $y_1, y_2, y_3 \in \mathbb{Z}_1$ such that

$$(\alpha + x_1)(\alpha + x_2)(\alpha + x_3) = (\alpha + y_1)(\alpha + y_2)(\alpha + y_3)$$

and at least one of x_i, y_i is > 1000. However, we have found that

$$(\alpha+3)(\alpha+4)(\alpha+46)=(\alpha+2)(\alpha+11)(\alpha+24),$$

so that $\ell(\alpha,\mathbb{Z}_1)=6$.

Let α be an algebraic number whose minimal polynomial (the monic polynomial in $\mathbb{Q}[x]$ of smallest degree whose root is α) is

$$P(x)=x^d+a_{d-1}x^{d-1}+\cdots+a_0\in\mathbb{Q}[x].$$

Let $M\subset\mathbb{Q}$. Then M-dependence of α implies multiplicative dependence of the set $\{P(-m)\mid m\in M\}$ of values of the minimal polynomial of α. Indeed, let $K:=\mathbb{Q}(\alpha)$ be the number field generated by α and let $\mathrm{Nm}_{K/\mathbb{Q}}:K\to\mathbb{Q}$ be the norm in the extension $K/\mathbb{Q}$ (see, e.g., [3] or [7]). Assume that α is M-dependent. Since the norm map $\mathrm{Nm}_{K/\mathbb{Q}}$ is multiplicative, (1) implies

$$\mathrm{Nm}_{K/\mathbb{Q}}(\alpha+x_1)\cdots\mathrm{Nm}_{K/\mathbb{Q}}(\alpha+x_n)=\mathrm{Nm}_{K/\mathbb{Q}}(\alpha+y_1)\cdots\mathrm{Nm}_{K/\mathbb{Q}}(\alpha+y_m).$$

In view of $\mathrm{Nm}_{K/\mathbb{Q}}(\alpha+x)=(-1)^dP(-x)$, which is valid for all $x\in\mathbb{Q}$, we obtain

$$(-1)^{dn}P(-x_1)\cdots P(-x_n)=(-1)^{dm}P(-y_1)\cdots P(-y_m),$$

and hence

$$P(-x_1)^2\cdots P(-x_n)^2=P(-y_1)^2\cdots P(-y_m)^2.$$

Therefore the set $\{P(-m)\mid m\in M\}$ is multiplicatively dependent. Dubickas [2] proved that for any quadratic polynomial $P(x)\in\mathbb{Q}[x]$ and any $t\in\mathbb{Z}$ the set $\{P(n)\mid n\in\mathbb{Z}_t\}$ is multiplicatively dependent.

It was proved in [1] that every cubic algebraic number α is $\mathbb{Q}$-dependent and $\ell(\alpha,\mathbb{Q})\leqslant 8$. We prove the following theorem.

Theorem 1. *Suppose that a, c and m are rational integers, $c\neq 0$, such that*

$$\left|2am-3m^2-1\right|>|c|\bigl(|a|+|c|+1\bigr).$$

Let α be a root of the polynomial $P(x)=x^3+ax^2+(2am-3m^2-1)x+c$. Then $P(x)$ is irreducible (in $\mathbb{Q}[x]$), α is $\mathbb{Z}$-dependent and $\ell(\alpha,\mathbb{Z})\leqslant 4$.

Proof. Since α is a root of $P(x)$, it follows that

$$\alpha^3+a\alpha^2+\bigl(2am-3m^2-1\bigr)\alpha+c=0. \tag{2}$$

On the other hand,

$$\begin{aligned}&\alpha^3+a\alpha^2+\bigl(2am-3m^2-1\bigr)\alpha+c\\&\quad=(\alpha+m)^3+(a-3m)(\alpha+m)^2-(\alpha+m)+c-am^2+2m^3+m.\end{aligned} \tag{3}$$

Let $r:=-2m^3+am^2-c$. Then (2) and (3) imply

$$(\alpha+m)^2(\alpha+a-2m)=\alpha+r,$$

so that α is $\mathbb{Z}$-dependent and $\ell(\alpha,\mathbb{Z})\leqslant 4$.

We are left to prove that the polynomial $P(x)$ is irreducible. Indeed, assume that it is reducible. Consequently, $P(x)$ is divisible by a linear polynomial from $\mathbb{Q}[x]$, and therefore it has a root $x_0 \in \mathbb{Q}$. Since $P(x)$ is monic with integer coefficients, $x_0 \in \mathbb{Z}$ and x_0 is a divisor of $P(0) = c \neq 0$. Hence $0 < |x_0| \leqslant |c|$ and

$$\begin{aligned} |2am - 3m^2 - 1| &= \left| \frac{P(x_0) - (x_0^3 + ax_0^2 + c)}{x_0} \right| = \\ &= \left| \frac{x_0^3 + ax_0^2 + c}{x_0} \right| = \left| x_0^2 + ax_0 + \frac{c}{x_0} \right| \leqslant \\ &\leqslant |x_0|^2 + |a| \cdot |x_0| + \left| \frac{c}{x_0} \right| \leqslant \\ &\leqslant |c|^2 + |a| \cdot |c| + |c| = |c|\big(|a| + |c| + 1\big), \end{aligned}$$

which is a contradiction. Therefore $P(x)$ is irreducible.

Corollary 1. *For any $a, c \in \mathbb{Z}$, $c \neq 0$, there exist infinitely many negative integers b such that the polynomial $P(x) := x^3 + ax^2 + bx + c$ is irreducible (over $\mathbb{Q}$) and if α is a root of $P(x)$ then it is $\mathbb{Z}$-dependent and $\ell(\alpha, \mathbb{Z}) \leqslant 4$.*

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REZIUMĖ

Kubinių algebrinių skaičių multiplikatyvusis priklausomumas

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Šiame darbe nagrinėjamas kubinių algebrinių skaičių multiplikatyviojo priklausomumo klausimas. Įrodoma, kad egzistuoja be galo daug algebrinių skaičių α, su kuriais aibė $\{\alpha + x \mid x \in \mathbb{Z}\}$ yra multiplikatyviai priklausoma.

Raktiniai žodžiai: algebrinis skaičius, multiplikatyvusis priklausomumas.