On the Green’s formula for a Stokes type problem

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Abstract. A time-periodic Stokes problem is studied in the domain with cylindrical outlets to infinity. Using the Fourier series the problem is reduced to a sequence of elliptic problem. For each of these elliptic boundary value problems a generalized Green’s formula is constructed. The analogous Green’s formula for the steady Stokes problem was obtained in [1].

Keywords: cylindrical outlets to infinity, time-periodic Stokes problem, generalized Green’s formula.

1. Formulation of the problem

Let \( \Omega \subset \mathbb{R}^3 \) be a domain with cylindrical outlets to infinity, i.e., outside the ball \( B_R = \{ x \in \mathbb{R}^3 : |x| \leq R \} \) the domain \( \Omega \) coincides with a system of \( J \) semi-infinite cylinders \( \Pi^+_j \) of a constant cross section \( \omega_j \). Let \( \Pi^+_j \cap \Pi^+_k = \emptyset, j \neq k \) and let the boundary \( \partial\Omega \) be smooth. We consider in \( \Omega \) the time-periodic Stokes problem

\[
\begin{align*}
v_t - \nu \Delta v + \nabla p &= f, \quad (x, t) \in \Omega \times (0, 2\pi), \\
-\nabla \cdot v &= 0, \quad (x, t) \in \Omega \times (0, 2\pi), \\
v &= 0, \quad (x, t) \in \partial\Omega \times (0, 2\pi), \\
v(x, 0) &= v(x, 2\pi), \quad x \in \Omega.
\end{align*}
\]

We assume that the external force \( f = (f_1, f_2, f_3)^T \) is \( 2\pi \)-time-periodic function. Problem (1)–(4) could be decomposed into a sequence of elliptic problems. Indeed, we can look for the solution to problem (1)–(4) in the form

\[
\begin{align*}
v(x, t) &= v_0 + \frac{1}{2\pi} \sum_{k=1}^{\infty} \{v_{ck}(x) \cos kt + v_{sk}(x) \sin kt\}, \\
p(x, t) &= p_0 + \frac{1}{2\pi} \sum_{k=1}^{\infty} \{p_{ck}(x) \cos kt + p_{sk}(x) \sin kt\}.
\end{align*}
\]

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Inserting series (5), (6) into equations and boundary conditions we get for coefficients $v_{ck}, v_{sk}, p_{ck}, p_{sk}$ series of the systems of elliptic problems

$$
\begin{align*}
  &k v_{ck} - \nu \Delta v_{ck} + \nabla p_{ck} = f_{ck}, \quad x \in \Omega, \\
  &-k v_{sk} - \nu \Delta v_{sk} + \nabla p_{sk} = f_{sk}, \quad x \in \Omega, \\
  &-\nabla \cdot v_{ck} = 0, \quad -\nabla \cdot v_{sk} = 0, \quad x \in \Omega, \\
  &v_{ck} = 0, \quad v_{sk} = 0, \quad x \in \partial \Omega.
\end{align*}
$$

(7)

Here $f_{00}/(2\pi), f_{ck}/\pi, f_{sk}/\pi, k = 0, 2, \ldots$, are Fourier coefficients of the function $f = f(x, t)$.

In this paper we derive so-called generalized Green’s formula for problem (7). The analogous Green’s formula for the steady Stokes problem was obtained in [1]. The obtained below results are important for the construction of correct asymptotic conditions at infinity which describe real time-periodic physical processes (for example bloodstream).

2. The asymptotics of the solution to problem (7)

Let $x_j = (x_j^1, x_j^2, x_j^3)$ be the local coordinate system related to the cylinder $\Pi^j_+$ such that the axis $x_j^3$ is directed along cylinder axis. We consider problem (7) in a weighted Sobolev space $W^j_l(\Omega)$ which is a closure of $C^\infty_0(\Omega)$ (a class of infinitely differentiable functions with compact supports in $\Omega$) with respect to the norm

$$
\|u; W^j_l(\Omega)\|^2 = \sum_{|\alpha| \leq l} \int_{\Omega} \rho_\beta(x) |D^{\alpha} u(x)|^2 \, dx,
$$

where $\rho_\beta$ is a smooth positive function on $\Omega$ such that $\rho_\beta(x) = \exp(\beta x_j^3)$ on $\Omega \cap B_R, \ j = 1, \ldots, J$. If $\beta > 0$, elements of this space exponentially vanish as $x_j^3$ tends to infinity, and they may exponentially grow, if $\beta < 0$.

Consider problem (7) in the cylinder $\Pi^j_+$. Using the methods of the book [2] and arguing in the same way as in [1] we obtain four special solutions of the homogeneous problem (7):

$$
\begin{align*}
  &u_{0k}^0 = (0, 0, 0, 1, 0, 0, 0, 0)^t, \quad u_{0k}^1 = (0, 0, \psi_k^0, x_j^3, 0, 0, 0, 0)^t, \\
  &u_{sk}^0 = (0, 0, 0, 0, 0, 0, 1)^t, \quad u_{sk}^1 = (0, 0, \psi_k^0, 0, 0, 0, \psi_k^0, x_j^3)^t.
\end{align*}
$$

(8)

(9)

where the pair of functions $(\psi_k^0, \psi_k^1)$ is the unique solution of the problem

$$
\begin{align*}
  &k \psi_k^j + \nu \Delta \psi_k^j = 1, \quad x_j^j = (x_j^1, x_j^2) \in \omega^j, \\
  &k \psi_k^j - \nu \Delta \psi_k^j = 0, \quad x_j^j \in \partial \omega^j, \\
  &\psi_k^j = 0, \quad x_j^j \in \partial \omega^j.
\end{align*}
$$

(10)

According to Theorem 3.1.4 in [2] the sum of linear combinations of these solutions gives the main term (up to an exponentially vanishing term) of the asymptotic decom-
position of the "growing" at infinity solution. Let \( \chi_j(x) \) be a smooth cut-off function such that \( \text{supp}(\chi_j) \subseteq \Pi^j_+ \) and \( \chi_j(x) = 1 \) if \( x^j > L \) for \( j = 1, \ldots, J \).

**Theorem 1.** Let \( \beta > 0 \). If \( u_k = (v_{ck}, p_{ck}, v_{sk}, p_{sk}) \in D^{l}_{\beta W}(\Omega) \) is the solution to problem (7) with the right-hand side \( f_k = (f_{ck}, f_{sk}) \in W^{l-1}_{\beta}(\Omega)^6 \)

\[
\mathbf{u}_k(x) = \sum_{j=1}^{J} \chi_j(x) \left\{ a^{j}_{ck} u_k^{j0}_{ck} + a^{j}_{sk} u_k^{j0}_{sk} + b^{j}_{ck} u_k^{j1}_{ck} + b^{j}_{sk} u_k^{j1}_{sk} \right\} + \mathbf{\tilde{u}}_k, \tag{11}
\]

where \( \mathbf{\tilde{u}}_k \in D^{l}_{\beta W}(\Omega) \), \( a^{j}_{ck}, a^{j}_{sk}, b^{j}_{ck}, b^{j}_{sk} \in \mathbb{C} \). Here \( D^{l}_{\beta W}(\Omega) = W^{l+1}_{\beta}(\Omega)^6 \times W^{l}_{\beta}(\Omega)^2 \).

3. Generalized Green’s formula

Let \( u_k = (v_{ck}, p_{ck}, v_{sk}, p_{sk}) \), \( U_k = (V_{ck}, P_{ck}, V_{sk}, P_{sk}) \in C_{\infty}^{\infty}(\overline{\Omega}) \). Integrating twice by parts in \( \Omega \) one gets the standard Green’s formula (see [3])

\[
\begin{align*}
\nonumber &(-\Delta v_{ck} + \nabla p_{ck} + k v_{sk}, V_{ck})_\Omega + (-\Delta v_{sk} + \nabla p_{sk} - k v_{ck}, V_{sk})_\Omega + (-\nabla \cdot v_{sk}, P_{sk})_\Omega \\
&+ (v_{ck}, n P_{ck} - \nu \partial_n v_{ck})_{\partial \Omega} + (v_{sk}, n P_{sk} - \nu \partial_n v_{sk})_{\partial \Omega} \\
&-(v_{ck}, -\Delta V_{ck} + \nabla P_{ck} - k V_{sk})_\Omega - (p_{ck}, -\nabla \cdot V_{ck})_\Omega \\
&-(v_{sk}, -\Delta V_{sk} + \nabla P_{sk} + k V_{ck})_\Omega - (p_{sk}, -\nabla \cdot V_{sk})_\Omega \\
&-(\mathbf{n} P_{ck} - \nu \partial_n V_{ck}, V_{ck})_{\partial \Omega} - (\mathbf{n} P_{sk} - \nu \partial_n V_{sk}, V_{sk})_{\partial \Omega} = 0,
\end{align*}
\]

here \((,)_\Omega\) stands for a scalar product in \( L_2(\Omega) \). Denoting by \( q(u_k, U_k) \) the left-hand side of the above formula we get

\[
q(u, U) = q(U, u) = 0
\]

for any \( u \in D^{l}_{\beta W}(\Omega) \) and \( U \in D^{l}_{-\beta W}(\Omega) \). Let \( S \) be an operator of problem (7) and \( S^* \) be an operator of the problem

\[
\begin{align*}
-k V_{sk} - \nu \Delta V_{ck} + \nabla P_{ck} = F_{ck}, & \quad x \in \Omega, \\
-k V_{ck} - \nu \Delta V_{sk} + \nabla P_{sk} = F_{sk}, & \quad x \in \Omega, \\
-\nabla \cdot V_{ck} = 0, & \quad -\nabla \cdot V_{sk} = 0, \quad x \in \Omega, \\
V_{ck} = 0, V_{sk} = 0, & \quad x \in \partial \Omega.
\end{align*}
\]

(13)

It is clear that \( S^* \) is an adjoint operator to \( S \) with respect to the Green’s formula (12). Note that \( S \) is not self-adjoint operator. Homogeneous problem (13) in the cylinder \( \Pi^j_+ \) has four special solutions

\[
U^{j0}_{ck} = (0, 0, 0, 1, 0, 0, 0, 0, 0)^t, \quad U^{j1}_{ck} = (0, 0, \psi^j_k, x^j_k, 0, 0, \psi^j_k, 0)^t, \tag{14}
\]

\[
U^{j0}_{sk} = (0, 0, 0, 0, 0, 0, 0, 0, 1)^t, \quad U^{j1}_{sk} = (0, 0, -\psi^j_k, 0, 0, \psi^j_k, x^j_k)^t, \tag{15}
\]

where functions \( \psi^j_k \) and \( \psi^j_k \) are defined by formula (10). We denote by \( D^{l}_{\pm \beta W}(\Omega) \) the subset of functions \( u_k \in D^{l}_{-\beta W}(\Omega) \) having expansion (11) and by \( D^{l}_{\pm \beta W}(\Omega)^* \) the
subset of $\mathcal{D}^l_{-\beta} W(\Omega)$ consisting of functions having an expansions

$$U_k = \sum_{j=1}^{J} \chi_j \left[ A^j_{ck} U^{0j}_{ck} + A^j_{sk} U^{0j}_{sk} + B^j_{ck} U^{1j}_{ck} + B^j_{sk} U^{1j}_{sk} \right] + \tilde{U}_k,$$  \hspace{1cm} (16)

where $U^{jh}_{o,k}, h \in \{0, 1\}, o \in \{c, s\}$, are defined by (14) and (15), $\tilde{U}_k \in \mathcal{D}^l_{-\beta} W(\Omega)$, $A^j_{ck}, A^j_{sk}, B^j_{ck}, B^j_{sk} \in \mathbb{C}$.

Since $\text{supp}(\chi_j) \cap \text{supp}(\chi_l) = \emptyset, j \neq l$, we have

$$q(\chi_j u^{jh}_{o,k}, \chi_j U^{jh}_{\infty, k}) = 0, \quad h, m \in \{0, 1\}, \quad o, oo \in \{c, s\}.$$  

Using the fact that functions (8), (9) and (14), (15) are exact solutions to homogeneous problems (7) and (13), respectively, we get, after cumbersome computation, that

$$q(\chi_j u^{jh}_{o,k}, \chi_j U^{jh}_{\infty, k}) = 0, \quad h = 0, 1, \quad o, oo \in \{c, s\}.$$  

Inserting representations (11) and (16) into $q(u_k, U_k)$ we get that a number of terms in $q(u_k, U_k)$ vanishes and, finally, we find

$$q(u_k, U_k) = \sum_{j=1}^{J} \left[ a^j_{ck} \bar{B}^j_{ck} q(\chi_j u^{0j}_{ck}, \chi_j U^{1j}_{ck}) + a^j_{sk} \bar{B}^j_{sk} q(\chi_j u^{0j}_{sk}, \chi_j U^{1j}_{sk}) 
+ a^j_{ck} \bar{B}^j_{sk} q(\chi_j u^{0j}_{ck}, \chi_j U^{1j}_{sk}) + a^j_{sk} \bar{B}^j_{ck} q(\chi_j u^{0j}_{sk}, \chi_j U^{1j}_{ck}) 
+ b^j_{ck} \bar{A}^j_{ck} q(\chi_j u^{1j}_{ck}, \chi_j U^{0j}_{ck}) + b^j_{sk} \bar{A}^j_{sk} q(\chi_j u^{1j}_{sk}, \chi_j U^{0j}_{sk}) 
+ b^j_{sk} \bar{A}^j_{ck} q(\chi_j u^{1j}_{sk}, \chi_j U^{0j}_{sk}) + b^j_{ck} \bar{A}^j_{sk} q(\chi_j u^{1j}_{ck}, \chi_j U^{0j}_{sk}) \right].$$

Let us calculate the term $q(\chi_j u^{0j}_{ck}, \chi_j U^{1j}_{ck})$. We note, firstly, that the cut-off function $\chi_j$ restricts all considerations to the cylinder $\Pi^j_L$, secondly, that $S(\chi_j u^{jh}_{o,k})$ and $S^{*}(\chi_j u^{jh}_{o,k})$ have compact supports. Applying the Green’s formula (12) in the domain $\Omega_L = \{x \in \Omega: \text{if } x \in \Pi^j_L \text{ then } \chi_j^j < L, \ j = 1, \ldots, J\}$ we get an additional integral over the cross-section $\omega^j$. Let $n = (0, 0, 1)^t$ be the outward normal to $\partial \Omega_L$ on $\omega^j$ and \(\partial_3 = \partial / \partial x_3^j\). Taking into account (8), (9) and (14), (15) we get

$$q(\chi_j u^{0j}_{ck}, \chi_j U^{1j}_{ck}) = (v_{\partial 3} u^{1j}_{ck} - v_{\partial 3} u^{0j}_{ck}, v_{\partial 3} U^{1j}_{ck} - v_{\partial 3} U^{0j}_{ck})_\omega
- (n_p^{0j} - v_{\partial 3} v^{0j}_{sk}, v^{1j}_{sk} U^{0j}_{sk})_\omega
+ (n_p^{1j} - v_{\partial 3} v^{1j}_{sk}, v^{0j}_{sk} U^{1j}_{sk})_\omega
= -1, \phi_k^{(j)}_\omega. $$
The rest terms in the Green’s formula could be computed in the same way. Finally, we arrive at

\[ q(\mathbf{u}_k, \mathbf{U}_k) = \sum_{j=1}^{J} \left\{ \left( b_{ck}^j \overline{A}_{ck}^j + b_{sk}^j \overline{A}_{sk}^j - a_{ck}^j \overline{B}_{ck}^j - a_{sk}^j \overline{B}_{sk}^j \right) (\varphi_{k}^j, 1)_{\omega_j} \right. \]

\[ + \left. \left( a_{ck}^j \overline{B}_{sk}^j + b_{sk}^j \overline{A}_{ck}^j - b_{ck}^j \overline{A}_{sk}^j - a_{sk}^j \overline{B}_{ck}^j \right) (\psi_{k}^j, 1)_{\omega_j} \right\}. \]

Now we define operators \( \pi_{0}^c, \pi_{0}^s, \pi_{1}^c, \pi_{1}^s : \mathbb{D}_{\pm \beta}^l W(\Omega) \to \mathbb{C}^J \) (operators \( \pi_{0}^c, \pi_{0}^s, \pi_{1}^c, \pi_{1}^s : \mathbb{D}_{\pm \beta}^l W(\Omega)^* \to \mathbb{C}^J \) are defined in the same way) as follows:

\[ \pi_{0}^c \mathbf{u} = (a_{1}^c, a_{2}^c, \ldots, a_{J}^c), \quad \pi_{0}^s \mathbf{u} = (a_{1}^s, a_{2}^s, \ldots, a_{J}^s), \]

\[ \pi_{1}^c \mathbf{u} = (b_{1}^c, b_{2}^c, \ldots, b_{J}^c), \quad \pi_{1}^s \mathbf{u} = (b_{1}^s, b_{2}^s, \ldots, b_{J}^s), \]

where the numbers \( a_{j}^c, a_{j}^s, b_{j}^c, b_{j}^s \) are the coefficients in expansion (11) of the function \( \mathbf{u} \in \mathbb{D}_{\pm \beta}^l W(\Omega) \) (in expansion (16) for \( \mathbf{U} \in \mathbb{D}_{\pm \beta}^l W(\Omega)^* \)). Let

\[ c_{k}^j = \int_{\omega_j} \varphi_{k}^j \, dx \quad d_{k}^j = -\int_{\omega_j} \psi_{k}^j \, dx \quad x_j' = (x_{1,j}', x_{2,j}'), \]

and

\[ C_k = \text{diag}\{c_{1}^k, c_{2}^k, \ldots, c_{J}^k\}, \quad D_k = \text{diag}\{d_{1}^k, d_{2}^k, \ldots, d_{J}^k\} \]

be the \( J \times J \) matrices. Taking into account previous results and notations we get the following formula

\[ q(\mathbf{u}_k, \mathbf{U}_k) = \langle C_k \pi_{0}^c \mathbf{u}_k - D_k \pi_{0}^s \mathbf{u}_k, \pi_{0}^c \mathbf{U}_k \rangle_J + \langle C_k \pi_{1}^c \mathbf{u}_k + D_k \pi_{1}^s \mathbf{u}_k, \pi_{0}^s \mathbf{U}_k \rangle_J \]

\[ - \langle \pi_{0}^c \mathbf{u}_k, C_k \pi_{1}^c \mathbf{U}_k + D_k \pi_{1}^s \mathbf{U}_k \rangle_J - \langle \pi_{0}^s \mathbf{u}_k, C_k \pi_{1}^s \mathbf{U}_k - D_k \pi_{1}^c \mathbf{U}_k \rangle_J, \]

(17)

where \( \langle , \rangle_J \) stands for a scalar product in \( \mathbb{C}^J \). We call (17) the generalized Green’s formula.

References


REZIUME

M. Skujus. Apie Gryno formulę vienam Stokso tipo uždavinui

Laiko atžvilgiu periodinis Stokso uždavinys begalinių cilindrų sistemoje Furjė eilučių pagalba suvedamas į elipsinių uždavinii seka. Šiems Stokso tipo kraštiniam uždaviniam suvedama apibendrintoji Gryno formulė.

Raktiniai žodžiai: begalinių cilindrų sistema, laiko atžvilgiu periodinis Stokso uždavinys, apibendrintoji Gryno formulė.