On the uniqueness of ARCH processes

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Abstract. In this note we prove the uniqueness of the solution to ARCH equations under conditions, which are weaker than in some earlier results.

Keywords: ARCH process, strictly stationary solution.

1. The main result

Let \((\epsilon_k | k \in \mathbb{Z})\) be a family of iid nonnegative random variables, \((a_i | i \geq 1)\) a sequence of nonnegative numbers and \(a_0 > 0\). Consider the following system of equations:

\[
x_k = (a_0 + \sum_{i=1}^{\infty} a_i x_{k-i}) \epsilon_k, \quad k \in \mathbb{Z}.
\]

(1.1)

Any strictly stationary nonnegative solution to (1.1), \((x_k)\), is called an ARCH process. A solution \((x_k)\) is called non-anticipative if, for all \(k\), \(x_k\) is independent of \(\epsilon_l\), \(l > k\).

The most known example of ARCH processes is a sequence \((r_k^2)\), where \((r_k)\) is a so-called GARCH\((p, q)\) process, a stationary solution to the equations

\[
r_k = \sigma_k \epsilon_k;
\]

\[
\sigma_k^2 = \delta + \sum_{i=1}^{p} \beta_i \sigma_{k-i}^2 + \sum_{j=1}^{q} \alpha_j r_{k-j}^2.
\]

(1.2)

where the \(\epsilon_k\) are iid with zero mean, \(\delta > 0\), \(\beta_i \geq 0\), \(\alpha_j \geq 0\) for all \(i, j\). [4] showed that \((r_k^2)\) satisfies the associated ARCH equations (1.1) with \(\epsilon_k = \epsilon_k^2\), \(a_0 = \delta/(1 - \beta(1))\) and with coefficients \(a_i\) defined by the equality \(a(t) = \alpha(t)/(1 - \beta(t))\); here \(a(t) = \sum_{i=1}^{\infty} a_i t^i\), \(\alpha(t) = \sum_{j=1}^{q} \alpha_j t^j\) and \(\beta(t) = \sum_{i=1}^{p} \beta_i t^i\).

This paper investigates the question, whether a solution to (1.1) is unique. The main result is the following.

THEOREM 1.1. Suppose the following conditions are satisfied with some \(q > 1\):

\[
E \log^{-} \epsilon_0 < \infty, \quad (1.3)
\]

\[
\sum_{i \geq 1} a_i q^i < \infty. \quad (1.4)
\]

Then system (1.1) can have only one strictly stationary solution.
In the literature, there exist few results concerning uniqueness of ARCH processes. [1] considered GARCH\((p, q)\) processes and proved the uniqueness of the integrable non-anticipative solution to (1.2). [3] generalized his results to the general ARCH processes. These results are not comparable with Theorem 1.1: the later does not cover all ARCH processes because of condition (1.4); on the other hand, Theorem 1.1 does not assume integrability of a solution.

[5] considered GARCH\((1, 1)\) processes and proved their uniqueness without integrability assumption. [2] generalized his results to the GARCH\((p, q)\) case. Theorem 1.1 generalizes the uniqueness part of Theorem 1.3 of [2], because the coefficients of ARCH equations, associated with (1.2), decay geometrically fast (see [1]).

Finally, [4] proved the uniqueness of an ARCH process under the following assumptions:
(i) \(a_i\) decrease, starting from some \(i_0\);
(ii) for some \(q > 1\),
\[
\sum_{n \geq 0} \eta_{kn} q^n < \infty, \tag{1.5}
\]
where the \(\eta_{kn}\) are defined by (2.6) below.

In [4], we showed that the convergence radius of the series \(\sum_n \eta_{kn} t^n\) does not exceed that of the series \(\sum_n a_n t^n\). Therefore condition (1.5) is stronger than (1.4). Moreover, Theorem 1.1 does not require monotonicity of coefficients. On the other hand, in [4] we didn’t impose any integrability condition on \(\epsilon_k\) such as (1.3).

2. The proof

To prove Theorem 1.1, we need two lemmas.

**Lemma 2.1.** Let \((\rho_n)\) be a stationary sequence of quasi-integrable random variables. Then there exists a random variable \(\xi\) with values in the extended real line, such that almost surely
\[
\frac{\rho_1 + \cdots + \rho_n}{n} \to \xi. \tag{2.1}
\]

**Proof.** If \(\mathbb{E}|\rho_1| < \infty\), the lemma follows from the ergodic theorem, see, for example, Shiryaev [7, Chapter V, Theorem 3]. If \(\mathbb{E}\rho_1^+ < \infty\), \(\mathbb{E}\rho_1^- = \infty\), it follows from the subadditive ergodic theorem, applied to the process
\[
X_{st} = \rho_{s+1} + \cdots + \rho_t,
\]
see [6], Theorem 2. If \(\mathbb{E}\rho_1^+ = \infty\), \(\mathbb{E}\rho_1^- < \infty\), the subadditive ergodic theorem should be applied to the process \((-X_{st})\).

**Lemma 2.2.** Suppose, condition (1.3) is satisfied and let \((x_k)\) be a stationary solution to (1.1). If \(a_{i_0} > 0\) for some \(i_0 \geq 1\), then almost surely
\[
\frac{\log x_n - n}{n} \to 0. \tag{2.2}
\]
Proof. Let \((x_k)\) be a stationary solution to (1.1). By (1.3), \(\epsilon_0 > 0\) almost surely. The inequality \(x_0 \geq a_0 \epsilon_0\) then implies that \(x_0 > 0\) almost surely. By stationarity, all \(x_k\) are positive with probability 1.

For \(j \geq 1\) define
\[
\rho_j = \log \frac{x_{j+i_0}}{x_{(j-1)i_0}}.
\] (2.3)
Clearly, \((\rho_j)\) is a stationary sequence. Moreover, from
\[
x_0 \geq \epsilon_0 a_0 x_{-i_0}
\]
and (2.3) we get
\[
\rho_1 \leq \log a_0^{-1} - \log \epsilon_0;
\]
therefore, by (1.3),
\[
E\rho_1^+ < \infty.
\]
Lemma 2.1 now yields the existence of a random variable \(\xi\), such that almost surely
\[
\frac{\rho_1 + \cdots + \rho_j}{j} \to \xi.
\]
But \(\rho_1 + \cdots + \rho_j = \log x_{-j+i_0} - \log x_0\), therefore almost surely
\[
\frac{\log x_{-j+i_0}}{j} \to \xi.\] (2.4)
On the other hand, \(j^{-1} \log x_{-j+i_0}\) is distributed identically with \(j^{-1} \log x_0\), which tends to 0 almost surely. Therefore
\[
\frac{\log x_{-j+i_0}}{j} \xrightarrow{p} 0.\] (2.5)
By (2.4)–(2.5), \(\xi = 0\), i.e., almost surely
\[
\frac{\log x_{-j+i_0}}{j} \to 0.\]
By stationarity, for each \(d = 0, \ldots, i_0 - 1\),
\[
\frac{\log x_{-j+i_0-d}}{j} \to 0,
\]
which implies that almost surely
\[
\frac{\log x_{-j+i_0-d}}{ji_0+d} \to 0.\]
We see that \(n^{-1} \log x_{-n}\) tends to 0, as \(n\) tends to \(\infty\) along each of the subsequence \(n = ji_0 + d\). Therefore, (2.2) holds.
Proof of Theorem 1.1. Denote
\[
y_{kn} = a_0 \varepsilon_k (\eta_{k0} + \eta_{k1} + \cdots + \eta_{kn}), \quad y_k = a_0 \varepsilon_k \sum_{n \geq 0} \eta_{kn},
\]
\[
z_{kn} = \sum_{i \geq n+1} (\eta_{i0} a_i + \eta_{i1} a_{i-1} + \cdots + \eta_{in} a_{i-n}) x_{k-i},
\]
where
\[
\eta_{kn} = \sum_{i_1 + \cdots + i_l = n} a_{i_1} \cdots a_{i_l} \varepsilon_{k-i_1} \cdots \varepsilon_{k-i_l-i_1}.
\] (2.6)

In [4] we showed that, for all \(k\) and \(n\),
\[
x_k = y_{kn} + \varepsilon_k z_{kn}.
\] (2.7)
Moreover, \(y_{kn} \to y_k\) as \(n \to \infty\).

All random variables in (2.7) are nonnegative; therefore \(x_k \geq y_{kn}\) for all \(n\) and hence \(y_k \leq x_k < \infty\), i.e., almost surely
\[
\sum_{n \geq 0} \eta_{kn} < \infty.
\] (2.8)

It is easy to check that the sequence \((y_k)\) is a stationary solution to (1.1). Therefore it remains to prove that \(x_k = y_k\) almost surely. To do this, it suffices to show that \(z_{kn} \to 0\), as \(n \to \infty\) (here \(\to\) stands for the convergence in probability).

If all \(a_i\) equal 0, then \(z_{kn} = 0\) for all \(n\) and there is nothing to prove. Therefore suppose that \(a_{i_0} > 0\) for some \(i_0 \geq 1\). Let \(q > 1\) be any number, for which condition (1.4) is satisfied. By Lemma 2.2, almost surely
\[
\frac{\log x_{-n}}{n} \to 0 < \log q,
\]
hence there exists a random \(n_0\), such that, for all \(n \geq n_0\), \(x_{-n} \leq q^n\). Hence, a random variable \(C\), defined by
\[
C = \sup_{j \geq 1} \frac{q^{-j} x_{-j}}{j},
\]
is almost surely finite.

For all \(k \in \mathbb{Z}\), denote
\[
C_k = \sup_{j \geq 1} \frac{q^{-j} x_{k-j}}{j}.
\]
By stationarity, all \(C_k\) are distributed identically with \(C\). Furthermore, for all \(k\) and \(j \geq 1\),
\[
x_{k-j} \leq C_k q^j.
\] (2.9)
Now, by definition of $z_{kn}$ and (2.9),

$$
z_{kn} = \sum_{i \geq n+1}^{n} \sum_{j=0}^{n} \eta_{kj} a_{i-j} x_{k-i} = \sum_{j=0}^{n} \sum_{i \geq n+1}^{n} a_{i-j} x_{k-i}
\leq C_{k-n} \sum_{j=0}^{n} \sum_{i \geq n+1}^{n} a_{i-j} q^{j-n} = C_{k-n} \sum_{j=0}^{n} \eta_{kj} q^{j-n} \sum_{i \geq n}^{n} a_{i-j} q^{i-j}
\leq C_{k-n} \sum_{i \geq 1}^{n} a_{i} q^{j} \sum_{j=0}^{n} \eta_{kj} q^{j-n}.
$$

It is well known, that if $b_n$ and $c_n$ are nonnegative numbers, $\sum_{n}^{n} b_{n} < \infty$ and $c_{n} \to 0$, then $\sum_{j=0}^{n} c_{j-n} \to 0$. Therefore, by (2.8), almost surely

$$
\sum_{j=0}^{n} \eta_{kj} q^{j-n} \to 0, \quad n \to \infty.
$$

This yields $z_{kn} \to 0$, because the sequence $C_{k-n}$ is bounded in probability, as $n \to \infty$.

References


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V. Kazakevičius, R. Leipus. Apie ARCH procesų vienatį

Šiame darbe irodome ARCH lygčių sprendinio vienatį esant išpildytoms sąlygoms, kurios yra silpnesnės negu kai kuriuose ankstesniaose darbuose.