Testing AR(1) model*

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Abstract. In this paper we investigate a simple AR(1) model by testing a presence of changed segment in a data. We suggest test statistics based on a behavior of partial sums of residuals.

Keywords: AR(1) model, changed segment, partial sums of residuals.

1. Introduction

Structural stability of a time series is very important in applied econometrics. Estimates derived from unstable processes can be biased and forecasts lose accuracy. A considerable attention of testing the parameter constancy of time series have given Pickard [4], Lee and Park [3] and many others.

The CUSUM method has been utilized for testing a change of a mean, a variance and other parameters of regression type models, see, e.g., Kulperger [2], Bai [1] and references therein. Shin [7] established the weak limit of partial sums of residuals of AR models and investigated various tests for one change alternatives. We investigate in this paper a simple AR(1) model under changed segment type alternatives. The paper is organized as follows. Section 2 presents a model under consideration and test statistics. In Section 3 we study a behavior of test statistics under some alternatives. In section 4 some simulation results are presented.

2. Model and test statistic

In this paper we consider a simple AR(1) model:

\[ y_k = \rho y_{k-1} + a_k + e_k, \quad k = 1, 2, ..., n, \quad y_0 = 0, \]

where \( e_1, ..., e_n \) are i.i.d. with mean zero and finite variance \( \sigma^2 < \infty \), a sequence \( (a_k) \) will be specified later. We want to test the null hypothesis

\[ H_0: a_k = 0 \quad \text{for all} \quad k = 1, ..., n \]

against various type alternatives.

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Let \((\hat{e}_k, k = 1, \ldots, n)\) denote the residuals of model (1) under the null hypothesis. Set \(\hat{S}_0 = 0, \hat{S}_k = \hat{e}_1 + \cdots + \hat{e}_k, k = 1, \ldots, n\). Define the test statistics

\[
T(n; \alpha) = \max_{1 < l < n} \max_{\rho \in \mathbb{R}} \left| \frac{1}{1 - \rho} \sum_{i=k}^{k+l} (a_i - \hat{a}) \right|
\]

where \(0 \leq \alpha < 1/2\). In Račkauskas and Rastené [5] limiting distributions for normalized statistics \(n^{-1/2+\alpha} \sigma^{-1} T(n; \alpha)\) are established under the null hypothesis.

3. Behavior of test statistics under alternatives

Consider model (1) where \(|\rho| \neq 1\). Then \(T(n; \alpha)\) can be estimated \(T(n; \alpha) \geq T_1(n; \alpha) - T_2(n; \alpha)\), where

\[
T_1(n; \alpha) = \max_{1 < l < n} \max_{\rho \in \mathbb{R}} \left| \frac{1}{1 - \rho} \sum_{i=k}^{k+l} (a_i - \hat{a}) \right|
\]

\[
T_2(n; \alpha) = \max_{1 < l < n} \max_{\rho \in \mathbb{R}} \left| \frac{1}{1 - \rho} \sum_{i=k}^{k+l} (\hat{e}_i - \hat{\epsilon}) - \frac{\rho - \hat{\rho}}{1 - \rho} \left( y_{k+l} - y_k - \frac{l}{n} y_n \right) \right|
\]

\(\hat{\rho}\) denotes an estimate of \(\rho\) under null, \(\hat{a} = n^{-1} \sum_{k=1}^{n} a_k, \hat{\epsilon} = n^{-1} \sum_{k=1}^{n} \hat{e}_k\).

By Račkauskas and Rastené [5], Račkauskas and Suquet [6] assuming the conditions

\[
\frac{1}{n} \sum_{i=1}^{n} a_i = O_P(1), \tag{2}
\]

\[
\lim_{l \to \infty} P(\{|e_1| \geq t^{1/2-\alpha}\}) = 0 \tag{3}
\]

it follows that \(n^{-1/2+\alpha} \sigma^{-1} T_2(n; \alpha) = O_P(1)\). Hence, if under an alternative hypothesis we have \(n^{-1/2+\alpha} \sigma^{-1} T_1(n; \alpha) \xrightarrow{p} \infty\), then statistics \(T(n; \alpha)\) are proper for testing.

Next we consider two examples of changed segment alternatives.

**Example 1.** There exist \(l^*, k^*\), \(1 < l^* < n\), \(k^* < n\), such that

\[
a_k = a \mathbb{1}_{k^* - k < k^* + l^*}, \quad k = 1, \ldots, n.
\]

where \(a \in R, a \neq 0\). Moreover, we assume that \(l^* \to \infty\) and \(l^*/n \to 0\) as \(n \to \infty\). In this case

\[
T_1(n; \alpha) \geq \left| \frac{1 - \hat{\rho}}{1 - \rho} \left( 1 - \frac{l^*}{n} \right) \right|.
\]

Hence, under conditions (2) and (3), we have \(n^{-1/2+\alpha} \sigma^{-1} T_1(n; \alpha) \xrightarrow{p} \infty\) provided

\[
\sigma^{-1} |a| l^*(1-\alpha) n^{-1/2+\alpha} \to \infty \quad \text{as} \; n \to \infty.
\]

**Example 2.** There exists \(l^*, k^*\), \(1 < l^*, k^* < n\), such that

\[
a_k = (1 - \rho) y_{k-1} \mathbb{1}_{k^* - k < k^* + l^*}.
\]
Under this alternative, model (1) takes the form

\[
y_k = \begin{cases} 
0, & \text{if } k = 0 \\
\rho y_{k-1} + e_k, & \text{if } 1 \leq k \leq k^*, \ k^* + l^* < k \leq n, \\
y_{k-1} + e_k, & \text{if } k^* < k \leq k^* + l^*, 
\end{cases}
\]

i.e., there exists a segment where AR(1) process is non-stationary. This example is investigated by simulations.

Fig. 1. Example 1.
4. Simulations

For different values of $l^*$, $\alpha$, $\alpha$, $\rho$ we have computed 300 realizations of the test statistics, where $n$ is equal to 1000. Residuals were generated from the standard normal distribution. For the $p$-values analysis we use $p$-values discrepancy plots. We compare the empirical distribution function for $p$-values with the distribution function of the true $p$-values. A difference between the empirical and the true distribution functions is set on $y$-axis and an argument of the distribution function on $x$-axis. For a power analysis we have presented size-power curves. On the $x$-axis we have set values of empirical $p$-values distribution function under the null hypothesis whereas on the $y$-

Fig. 2. Example 2.
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axis values of empirical $p$-value distribution function under the alternative (empirical power function).

Example 1. From the Fig. 3 we see that almost in all cases the test is a bit conservative (in average accept the null hypothesis too often) except when $\alpha = 2/32$, $\alpha = 0.5$ and a change segment is equal 30%. The right column of plots shows that the power increases when a changed segment and constant $a$ increases and $\alpha$ is closer to 0 rather than to 1/2.

Example 2. From the Fig. 3 we see that the test accept the null hypothesis too often. The size-power curves show that the test power decreases when $\rho$ tends to 1 and change segment decreases. Best results gives $\alpha = 1/4$.

References


REZIUMĖ

I. Rastenė. AR(1) modelio testavimas

Darbe nagrinėjamas AR(1) modelio galimąsų segmento pasikeitimus. Pasiūlyta testinė statistika, paremta modelio liekanų dalinių sumų elgesiu.