Multiconditional probabilities

Remigijus Petras GYLYS (MII)
e-mail: gyliene@ktl.mii.lt

Abstract. We introduce a generalization of (Kolmogorovian) conditional probabilities.

Keywords: multiconditional event, expected values, multiconditional probability.

1. Introduction

In this note we introduce multiconditional events and describe their expected values (which we call multiconditional probabilities) generalizing some results of H.T. Nguyen, I.R. Goodman and E.A. Walker [6] and partially of U. Höhle and S. Weber [1–4,7,8].

2. Multiconditional events

Let $B$ be a Boolean algebra, and let $a$ and $f_0$ be elements of $B$. In [5] (further developed in op.cit.), the “conditional event” “$a$ given $f_0$”, written $(a \parallel f_0)$, is defined as the order interval

$$(a \parallel f_0) = [a \wedge f_0, a \vee \neg f_0],$$

i.e., the set of all elements of $B$ between $a \wedge f_0$ and $a \vee \neg f_0$ (which is the same thing as the pair $(a \wedge f_0, a \vee \neg f_0)$). In this note we propose the notion of “multiconditional events” in a Boolean algebra.

Let $B$ be a Boolean algebra and let $(f_0, \ldots, f_{n-2})$ (with $n \geq 2$) be an event of elements of $B$ such that

$$f_0 \leq \ldots \leq f_{n-2} \text{ and } a \wedge f_0 = \ldots = a \wedge f_{n-2}.$$

Then we define the multiconditional event $(a \parallel f_0, \ldots, f_{n-2})$ of “$a$ given conditions $f_0, \ldots, f_{n-2}$” as the following isotonic chain:

$$(a \parallel f_0, \ldots, f_{n-2}) = (a \wedge f_0, a \vee \neg f_{n-2}, \ldots, a \vee \neg f_0).$$

We denote by $B^n$ the set of all isotonic chains $f: \mathbf{n} \to B$, where $\mathbf{n}$ denotes the sequence of integers: $\mathbf{n} = \{0, 1, \ldots, n - 1\}$. We provide this set with the point-wise partial ordering: $f \leq g \iff f_0 \leq g_0, \ldots, f_{n-1} \leq g_{n-1}$. Obviously these chains form a bounded lattice, and the lattice-theoretic operations and universal bounds are given by:

$$(f_0, \ldots, f_{n-1}) \wedge (g_0, \ldots, g_{n-1}) = (f_0 \wedge g_0, \ldots, f_{n-1} \wedge g_{n-1}).$$
\((f_0, \ldots, f_{n-1}) \vee (g_0, \ldots, g_{n-1}) = (f_0 \vee g_0, \ldots, f_{n-1} \vee g_{n-1})\),
\((\bot, \ldots, \bot) = \bot\) and \((\top, \ldots, \top) = \top\),

where \(\bot\) and \(\top\) (in brackets) denote the least element and the largest element in \(B\), respectively. If we identify \(B\) with the sublattice \(B^n_c\) of \(B^n\) of constant sequences then \(B^n \) becomes a sublattice of \(B^n\).

We denote by \(\tilde{B}^n\) the set of all multiconditional events. The following equalities are worth pointing out
\[(a \land f_0 \parallel f_0, \ldots, f_{n-2}) = \ldots = (a \land f_{n-2} \parallel f_0, \ldots, f_{n-2}) = (a \parallel f_0, \ldots, f_{n-2}).\]

It can be easily checked that chains of \(B^n\) are in an one-to-one correspondence to multiconditional events in \(\tilde{B}^n\) via
\[(g_0, g_1, \ldots, g_{n-1}) = (g_0 \parallel g_0 \lor \neg g_{n-1}, \ldots, g_0 \lor \neg g_1).\]

Further, it is easy to see that the set \(\tilde{B}^n\) of multiconditional events extends the set \(B^n\) in the following sense:
\[(a \parallel \top, \ldots, \top) = (a, \ldots, a),\]
\[(a \parallel f_0, \ldots, f_{n-2}) = (b \parallel g_0, \ldots, g_{n-2})\]
\[\iff a \land f_0 = b \land g_0, f_0 = g_0, \ldots, f_{n-2} = g_{n-2}.\]

The following natural partial ordering can be defined on \(\tilde{B}^n\):
\[(a \parallel f_0, \ldots, f_{n-2}) \leq (b \parallel g_0, \ldots, g_{n-2})\]
\[\iff a \land f_0 \leq b \land g_0, a \lor \neg f_0 \leq b \lor \neg g_0, \ldots, a \lor \neg f_{n-2} \leq b \lor \neg g_{n-2}.\]

which extends the usual entailment relation in \(B\). The following monotonicity properties hold:
\[a \leq b \Rightarrow (a \parallel f_0, \ldots, f_{n-2}) \leq (b \parallel f_0, \ldots, f_{n-2}),\]
\[f_0 \leq g_0, \ldots, f_{n-2} \leq g_{n-2}, a \land g_0 \leq a \land f_0\]
\[\Rightarrow (a \parallel g_0, \ldots, g_{n-2}) \leq (a \parallel f_0, \ldots, f_{n-2}).\]

Furthermore, \(\tilde{B}^n\) is a bounded lattice. The lattice operations and universal bounds are given by
\[(a \parallel f_0, \ldots, f_{n-2}) \land (b \parallel g_0, \ldots, g_{n-2})\]
\[= (a \land f_0 \land g_0 \parallel (a \land f_0 \land b \land g_0) \lor (\neg a \land f_0) \lor (\neg b \land g_0),\]
\[\ldots, (a \land f_0 \land b \land g_0) \lor (\neg a \land f_{n-2}) \lor (\neg b \land g_{n-2})\],
\[(a \parallel f_0, \ldots, f_{n-2}) \lor (b \parallel g_0, \ldots, g_{n-2})\]
If we identify $B$ with the sublattice $\tilde{B}^n$ of $\hat{B}^n$ of multiconditional events
$$\tilde{B}^n = \{(a \parallel \top \ldots \top) \mid a \in B\}$$
then $B$ becomes a sublattice of $\hat{B}^n$.

### 3. Multiconditional probabilities

Let $P$ be a probability measure on a Boolean algebra $B$. If $\langle g_0, \ldots, g_{n-1} \rangle \subseteq B$ is an isotonic chain, then from isotonicity of $P$ it follows that the image $P\langle g_0, \ldots, g_{n-1} \rangle$ of $\langle g_0, \ldots, g_{n-1} \rangle$ under $P$

$$P\langle g_0, \ldots, g_{n-1} \rangle = \langle P(g_0), \ldots, P(g_{n-1}) \rangle$$

is an isotonic chain in the real unit interval $[0, 1]$ (with the ‘natural’ lattice structure given by max and min). For $(a \parallel f_0, \ldots, f_{n-2}) \in \hat{B}^n$, we have that

\begin{align*}
P(a \parallel f_0, \ldots, f_{n-2}) &= \langle P(a \land f_0), P(a \lor \lnot f_{n-2}), \ldots, P(a \lor \lnot f_0) \rangle \\
&= \langle P(a \land f_0), P(a \lor f_0) + 1 - P(f_{n-2}), \ldots, P(a \land f_0) + 1 - P(f_0) \rangle. \quad (1)
\end{align*}

Consider an “expected value” function $E$ on $[0, 1]$, an $n$-ary function $E \colon [0, 1]^n \to [0, 1]$ satisfying the following axioms: for $r \in [0, 1]$, $(r_0, \ldots, r_{n-1})$, $(g_0, \ldots, g_{n-1}) \in [0, 1]^n$

(with $r_0 \leq \ldots \leq r_{n-1}$ and $g_0 \leq \ldots \leq g_{n-1}$),

(i) $E(r, \ldots, r) = r$ (idempotency),
(ii) $r_0 \leq g_0, \ldots, r_{n-1} \leq g_{n-1} \Rightarrow E(r_0, \ldots, r_{n-1}) \leq E(g_0, \ldots, g_{n-1})$ (isotonicity).

Now we are going to “extend” the probability measure $P$ to the lattice $\tilde{B}^n$ of multiconditional events in the following way:

$$(a \parallel f_0, \ldots, f_{n-2}) \rightarrow E(P(a \parallel f_0, \ldots, f_{n-2}))$$

$$= E(P(a \land f_0), P(a \lor f_0) + 1 - P(f_{n-2}), \ldots, P(a \land f_0) + 1 - P(f_0)).$$

We denote the values of this extension of $P$ as $P_E(a \mid f_0, \ldots, f_{n-2})$ and call it multiconditional probability (of “$a$ given conditions $f_0, \ldots, f_{n-2}$”). Obviously this quantity satisfies the following conditions:

(i) $P_E(\perp \mid \top, \ldots, \top) = 0$ and $P_E(\top \mid \top, \ldots, \top) = 1$,

(ii) $(a \parallel f_0, \ldots, f_{n-2}) \leq (b \parallel g_0, \ldots, g_{n-2})$ \Rightarrow $P_E(a \mid f_0, \ldots, f_{n-2}) \leq P_E(b \mid g_0, \ldots, g_{n-2})$.

To motivate the choice of the name “multiconditional probability”, consider an expected value function defined by

$$E_{2,k}(r_0, r_1) = \begin{cases} \frac{r_1^k}{r_0^k + r_1^k} & \text{if } \langle r_0, r_1 \rangle \neq (0, 1), \\
\frac{1}{2} & \text{if } \langle r_0, r_1 \rangle = (0, 1), \end{cases}$$
where \( k \) is an arbitrary number in \([0, 1]\). From (1) (with \( n = 2 \)) it follows that

\[
P_{E_{2,k}}(a \mid f_0) = \begin{cases} 
\frac{P(a \land f_0)}{P(f_0) + 1 - P(a \land f_0)} & \text{if } P(f_0) \neq 0, \\
\frac{P(a \land f_0)}{P(f_0)} & \text{if } P(f_0) = 0,
\end{cases}
\]

which (in the case when \( k = 1 \)) is the usual definition of (Kolmogorovian) conditional probability.

For \((r_0, r_1, r_2) \in [0, 1]^3\) (with \( r_0 \leq r_1 \leq r_2 \)) and \( k \in [0, 1] \), consider

\[
E_{3,k}(r_0, r_1, r_2) = \begin{cases} 
\frac{r_0 + 1}{r_2 + 1} & \text{if } (r_0, r_2) \neq (0, 1), \\
k & \text{if } (r_0, r_2) = (0, 1),
\end{cases}
\]

which defines an expected value function from \([0, 1]^3\) to \([0, 1]\). From (1) (with \( n = 3 \)) it follows that

\[
P_{E_{3,k}}(a \mid f_0, f_1) = \begin{cases} 
1 - \frac{P_3(a \land f_0) - P_3(a \land f_0)}{P_3(f_0) - P_3(a \land f_0)} & \text{if } P(f_0) \neq 0, \\
k & \text{if } P(f_0) = 0
\end{cases}
\]

(not forgetting the conditions: \( f_0 \leq f_1 \) and \( a \land f_0 = a \land f_1 \)). This formula can be considered as a generalization of the usual conditional probability.

Next, for \((r_0, r_1, r_2, r_3) \in [0, 1]^4\) (with \( r_0 \leq \ldots \leq r_3 \)) and \( k \in [0, 1] \), consider

\[
E_{4,k}(r_0, r_1, r_2, r_3) = \begin{cases} 
\frac{r_0 + 1}{r_2 + 1} & \text{if } (r_0, r_2) \neq (0, 1), \\
k & \text{if } (r_0, r_2) = (0, 1).
\end{cases}
\]

It is evident that this quantity defines an expected value function from \([0, 1]^4\) to \([0, 1]\). From this we obtain that

\[
P_{E_{4,k}}(a \mid f_0, f_1, f_2) = \begin{cases} 
1 - \frac{P_4(a \land f_0) - P_4(a \land f_0)}{P_4(f_0) - P_4(a \land f_0)} & \text{if } P(f_0) \neq 0, \\
k & \text{if } P(f_0) = 0
\end{cases}
\]

(with the conditions that \( f_0 \leq f_1 \leq f_2 \) and \( a \land f_0 = a \land f_1 = a \land f_2 \)), which can be considered as an another (more high level) generalization of the traditional conditional probability.

Similarly, one can consider the case \( n = 5 \) etc.

References


REZIUME

*R.P. Gyls. Daugiasalyginės tikimybės*

Pristatoma ir pailustruojama pavyzdžiais nauja daugiasalyginės tikimybės savoka.