A Beveridge–Nelson filters for the self normalization *

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Abstract. Let 
\[ X_t = \sum_{i=0}^{\infty} \psi_i \epsilon_{t-i} \]
be a linear process, where \( \epsilon_t, t \in \mathbb{Z} \), are i.i.d. r.v.’s in the domain of attraction of a normal law with zero mean and possibly infinite variance. Generalizing the class of Beveridge–Nelson filters this article proves a central limit theorem for the self-normalized sums
\[ U_n^{-1} \sum_{t=1}^{n} X_t, \]
where \( U_n^2 \) is a sum of squares of block-sums of size \( m \), as \( m \) and the number of blocks \( N = n/m \) tend to infinity.

Keywords: linear process, normal law.

1. Introduction

Since the work of Peter C.B. Phillips and V. Solo [3] a method of deriving asymptotics for weakly dependent linear processes has been used with an explicit algebraic decomposition of the linear filter. The method offers a simple unified approach to strong laws, central limit theorem and invariance principles for w. d. linear processes. In the article [2] Juodis and Račkauskas proves self-normalized central limit theorem for the Beveridge–Nelson [1] linear processes. In this paper we generalize this theorem allowing to widen the class of filters. We deal with the linear process of the following form
\[ X_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}, \quad \psi_j^+ = \sum_{i=j}^{\infty} \psi_i. \]  

In this section we consider filters satisfying the main summation condition
\[ \sum_{j=0}^{\infty} (\psi_j^+)^2 < \infty. \]  

This condition is important in the sense that the Beveridge–Nelson remains must be stationary. Now we are ready to define a new class of linear filters

\[ \Gamma_\psi := \bigcup_{p<2} \left\{ (\psi_k)_{k \geq 0} : \sum_{k=0}^{\infty} k^p |\psi_k|^p < \infty \right\}. \]  

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Observe that this class is smaller than

\[ \sum_{k=0}^{\infty} k^2 |\psi_k|^2 < \infty. \]  

(4)

But the difference between them is measured only by the slowly varying factor.

An important class of linear processes is the so-called Hall-Heyde condition

\[ \sum_{k=0}^{\infty} k^{1/2} |\psi_k| < \infty. \]  

(5)

Observe that both of them imply (2) but they differ from each other. Indeed filters \( \psi_j = 1/j^{3/2} \ln(j + 1) \) satisfies (4), but fails on (5). The essence that all classes mentioned above are of order 3/2, and they only can differ by slowly varying factor. And even the class (2) can’t afford the filters \( 1/j^{3/2} \).

The classical B-N case considers the filters

\[ \sum_{k=0}^{\infty} k |\psi_k| < \infty. \]  

(6)

This condition allows classical Beveridge–Nelson decomposition, but it is very strong (i.e., filters \( \psi_j = 1/j^{2} \) fails to satisfy it).

2. Results

Let \( X_t \) be a linear process of the form (1), where \( \varepsilon_t, t \in \mathbb{Z} \), are independent identically distributed random variables in the domain of attraction of a normal law with zero mean and possibly infinite variance (i.e., there exists constants \( b_n \) such that \( b_n^{-1}(X_1 + \ldots + X_n) \overset{D}{\longrightarrow} N(0, 1) \) denoted \( X_i \in DAN \), here and throughout \( \overset{D}{\longrightarrow} \) means weak convergence, and \( N(0, 1) \) standard normal law). Denote \( S_n = X_1 + \ldots + X_n \).

Set \( U_0 = 0 \) and define

\[ U_n := U_{m,k}^2 = \sum_{j=1}^{k} (S_{jm} - S_{(j-1)m})^2, \quad k = 1, \ldots, N, \quad 1 \leq m < n, \]  

(7)

where \( N = \lfloor n/m \rfloor \) and \( \lfloor a \rfloor \) denotes the integer part of \( a \). In the paper [2] the main result is Theorem 2 which states

\[ U_n^{-1} S_n \overset{D}{\longrightarrow} n \rightarrow \infty N(0, 1), \]  

(8)

for the class (6). The following theorem is the main result of this paper, and actually it generalizes Juodis Račkauskas Theorem 2 for the bigger class \( \Gamma_{\psi} \).

**Theorem 1.** If \( (\psi_n) \in \Gamma_{\psi}, \) and \( \varepsilon_i \in DAN, \) \( E\varepsilon_i = 0, \) then

\[ U_n^{-1} S_n \overset{D}{\longrightarrow} n \rightarrow \infty N(0, 1). \]
Proof. If we check the proof of the Theorem 2 [2] we see that we only need to re-estate the following statements

\[(V_{n}^{\varepsilon})^{-2} \sum_{j=1}^{N} \left( \sum_{k=0}^{j-1} \psi_{k}^{n} \varepsilon_{j-m-k} \right)^{2} \rightarrow P 0 \]  

(9)

and

\[(V_{n}^{\varepsilon})^{-2} \sum_{j=1}^{N} \left( \sum_{k=0}^{\infty} \left( \psi_{j+m+k}^{n} - \psi_{(j-1)m+k}^{n} \right) \varepsilon_{-k} \right)^{2} \rightarrow P 0. \]  

(10)

Observe that by stationarity of the process \((\varepsilon_{k}/V_{n}^{\varepsilon}, k = 1, \ldots, n)\) we have

\[E(V_{n}^{\varepsilon})^{-2} \sum_{j=1}^{N} \left( \sum_{k=0}^{j-1} \psi_{k}^{n} \varepsilon_{j-m-k} \right)^{2} \leq N \sum_{k=0}^{\infty} (\psi_{k}^{n})^{2} + CN^{2} \sum_{k=0}^{\infty} (\psi_{k}^{n})^{2}. \]  

Thus condition (2) is sufficient for the (9) convergence. Recall that

\[\Gamma_{N}^{2} = \sum_{j=1}^{N} \left( \sum_{k=0}^{\infty} \left( \psi_{j+m+k}^{n} - \psi_{(j-1)m+k}^{n} \right) \varepsilon_{-k} \right)^{2}. \]

Now convergence (10) reduces in showing that \(\Gamma_{N}\) is stochastically bounded. To this aim we need seven steps. First observe that

\[|\Gamma_{N}|^{p} \leq \sum_{j=1}^{N} \left| \sum_{k=0}^{\infty} \left( \psi_{j+m+k}^{n} - \psi_{(j-1)m+k}^{n} \right) \varepsilon_{-k} \right|^{p}. \]

Next we use the moment inequality

\[E \left| \sum_{j=1}^{N} \xi_{i} \right|^{p} \leq 2 \sum_{j=1}^{N} E|\xi_{i}|^{p}, \]  

which is true for any r.v.\(\xi_{i}\) if \(0 < p \leq 1\) and for any martingale differences \(\xi_{i}\)'s if \(1 < p \leq 2\). Hence

\[E|\Gamma_{N}|^{p} \leq 2 E|\xi_{1}|^{p} \sum_{j=1}^{N} \sum_{k=0}^{\infty} \sum_{m=1}^{j+m} \psi_{i}|^{p}. \]

Third we interchange the summation order
\[ = 2E|\varepsilon_1|^p \sum_{k=0}^{\infty} \sum_{j=1}^{N} \sum_{i=k+(j-1)m+1}^{k+jm} |\psi_i|^p. \]

Fourth step

\[ \leq 2E|\varepsilon_1|^p \sum_{k=0}^{\infty} \sum_{j=1}^{N} \left( \sum_{i=k+(j-1)m+1}^{k+jm} |\psi_i|^p \right). \]

Since \((a_1^p + \cdots + a_k^p)^{1/p}\) is monotonically decreasing, thus

\[ \leq 2E|\varepsilon_1|^p \sum_{k=0}^{\infty} \left( \sum_{i=k+1}^{\infty} |\psi_i|^p \right). \]

Next step

\[ \leq 2E|\varepsilon_1|^p \sum_{k=0}^{\infty} \left( \sum_{i=k+1}^{k+n} |\psi_i|^p \right). \]

And finally the seventh step we use [3] page 987 top inequality

\[ \leq 2E|\varepsilon_1|^p \left( \text{const} \cdot \sum_{k=0}^{\infty} k^p |\psi_k|^p \right). \]

Hence

\[ E|\Gamma_n|^p \leq E|\varepsilon_1|^p \sum_{j=1}^{N} \sum_{k=0}^{\infty} \sum_{i=k+(j-1)m+1}^{k+jm} |\psi_i|^p \leq cE|\varepsilon_1|^p \sum_{k=0}^{\infty} k^p |\psi_k|^p. \]

Thus the proof for generalized clt for \(\Gamma_{\psi}\) class is completed.

**References**


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*M. Juodis. Beveridge–Nelson filtru apibendrinimai autonomuotoms sumoms*