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On sheaves on midsymmetrical quantaloids

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1. Introduction

In an earlier paper [2] we studied presheaves and sheaves on an arbitrary quantaloid (category enriched in the category of complete sup-lattices subject to certain laws). In [3] we began a study of presheaves on quantaloids from a special subclass of quantaloids satisfying "Midsymmetry". In this note we discuss sheaves which specialize in that "midsymmetrical" case.

2. Midsymmetrical quantaloids

DEFINITION 2.1. A *quantaloid* is a locally small category Q such that:

(i) for all u, v objects in Q, the hom-set $Q(u, v)$ is a complete lattice,
(ii) composition of morphisms of Q (in this paper denoted by $\&$) preserves arbitrary joins in both variables: $p\& \bigvee_i q_i = \bigvee_i p\&q_i$ and $(\bigvee_i p_i)\&q = \bigvee_i p_i\&q$ for all morphisms p,q of Q and for all families (p_i), (q_i) of morphisms of Q (forming respective composable pairs).

A quantaloid Q will be called midsymmetrical whenever it satisfies

Midsymmetry: $p\&(q\&r)\&p' = p\&(r\&q)\&p'$ for all $p \in Q(u, v)$, $q, r \in Q(v, v)$, $p' \in Q(v, v')$.

We focus on midsymmetrical quantaloids and the traditional requirement on the existence of their units is not imposed. Examples of the one-object quantaloids (which are called quantales) include frames (and thus complete Boolean algebras) and various ideal lattices of rings or C^*-algebras. Many other quantales and quantaloids can be found in [5]. Our basic example is the following

EXAMPLE 2.2. Let Q be a right-sided idempotent quantale (belonging to a class of quantales considered in [4]), i.e., an one-object quantaloid such that, for any $p, q \in Q$, $p\&p \leqslant p$ and $p\&p = p$. The routine check shows that it is midsymmetrical.

From now Q will be an arbitrary midsymmetrical quantaloid having a small set of objects. Let Q_0 denote this set and Q_1 the set of morphisms of Q.

3. Sheaves on a (midsymmetrical) quantaloid

The notion of a sheaf and a few facts (with omitted proofs) are taken from [2]. We will present the axioms of a sheaf (in Proposition 3.4) and the concept of the compatibility of sub-Q-sets (in Corollary 3.6) in a more close form to those of C.J. Mulvey and M. Nawaz. For the concepts not defined here see [2] (or [3]).

DEFINITION 3.1. Let $(X, A, \ulcorner\urcorner)$ be a separated presheaf on a quantaloid Q.

(i) We say that, for a singleton $\mathcal{S} = (S, S^{\#})$ of the underlying Q-set (X, A), there exist enough restrictable triplets $(s_x, x, s_x^{\#}) \in Q_1 \times X \times Q_1$ if the following conditions hold:

$$s_x = \bigvee\{s_{x'} \& a_{x',x} | x' \in X, (s_{x'}, x', s_{x'}^{\#}) \text{ restrictable}\} \tag{1}$$

and

$$s_x^{\#} = \bigvee\{a_{x,x'} \& s_{x'}^{\#} | x' \in X, (s_{x'}, x', s_{x'}^{\#}) \text{ restrictable}\} \tag{2}$$

for all $x \in X$. (Note that $(s_x, x, s_x^{\#})$ is restrictable if and only if $s_x = s_x \& s_x^{\#} \& s_x$ and $s_x^{\#} = s_x^{\#} \& s_x \& s_x^{\#}$.)

(ii) A sub-Q-set $(J, {}_J A) \subseteq (X, A)$ with $J \subseteq X_u$ (for some $u \in Q_0$) is said to be *compatible* if the pair $\mathcal{E} = (E, E^{\#})$ with $E = (e_x)_{x \in J}$, $E^{\#} = (e_x^{\#})^{x \in J}$, and $e_x = e_x^{\#} = a_x$ (representing a "diagonal" element of A) for $x \in J$) constitutes a singleton of $(J, {}_J A) \subseteq (X, A)$. The singleton $\mathcal{E}$ itself is called an *extent* of $(J, {}_J A) \subseteq (X, A)$.

Observe that the bottom extension ${}^X\mathcal{E} = ({}^X E, {}^X E^{\#})$ of the extent $\mathcal{E} = (E, E^{\#})$ of a compatible sub-Q-set $(J, {}_J A) \subseteq (X, A)$ to a singleton of (X, A) is obtained by the formulas obtained by:

$$^X e_x = \bigvee_{x' \in J} a_{x',x} \text{and} ^X e_x^{\#} = \bigvee_{x' \in J} a_{x,x'} \tag{3}$$

for all $x' \in X$.

DEFINITION 3.2. We say that a separated presheaf $(X, A, \ulcorner\urcorner)$ on Q is a *sheaf* on Q if it satisfies the following

Sheaf Conditions:

(i) for every singleton $\mathcal{S}$ of the underlying Q-set (X, A), there exist enough restrictable triplets $(s_x, x, s_x^{\#})$;

(ii) for every compatible sub-Q-set $(J, {}_J A) \subseteq (X, A)$, the bottom extension of its extent $\mathcal{E}$ to a singleton of (X, A) (given by (3)) is of the form

$$^X\mathcal{E} = \mathcal{A}_y \tag{4}$$

for some (unique) element $y \in X$ (where $\mathcal{A}_y = ((a_{y,x})_{x\in X}, (a_{x,y})^{x\in X})$).

The next fact is taken from [2].

PROPOSITION 3.3. Let $(X, A, \ulcorner\urcorner)$ be a separated presheaf and let $(J, {}_JA) \subseteq (X, A)$ be a sub-Q-set with $J \subseteq X_u$ (for some $u \in Q_0$). Then the following two conditions are equivalent:

(i) $(J, {}_JA) \subseteq (X, A)$ is compatible;
(ii) $a_x \& a_{x'} = a_{x,x'} \leqslant a_x \wedge a_{x'}$ for all $x, x' \in J$.

The following proposition shows that we can weaken the axiom (ii) of Definition 3.2.

PROPOSITION 3.4. Let $(X, A, \ulcorner\urcorner)$ be a separated presheaf and $(J, {}_JA) \subseteq (X, A)$ be a compatible sub-Q-set. Then the following assertions are equivalent:

(i) the bottom extension of the extent of $(J, {}_JA)$ is of the form

$$^X\mathcal{E} = \mathcal{A}_y$$

for some element $y \in X$, that is, there exists a unique (by Separation) element $y \in X$ for which

$$a_{y,x} = \bigvee_{x'\in J} a_{x',x} \quad \text{and } a_{x,y} = \bigvee_{x'\in J} a_{x,x'} \tag{5}$$

for all $x \in X$;

(ii) the extent of $(J, {}_JA)$ is of the form

$$\mathcal{E} = {}_J\mathcal{A}_y$$

for some unique element $y \in X$ for which

$$a_y = \bigvee_{x\in J} a_x, \tag{6}$$

where ${}_J\mathcal{A}_y = ((a_{y,x})_{x\in J}, (a_{x,y})^{x\in J})$, that is, there exists a unique element $y \in X$ satisfying (6) and the following relations

$$a_{y,x} = a_x = a_{x,y} \tag{7}$$

hold for all $x \in J$.

The implication from (i) to (ii) is trivial (taking $x = y$ in (5), one has (6)). To prove that (ii) implies (i), assume (ii). Then, given $x \in X$, on the one hand, we have that

$$\bigvee_{x'\in J} a_{x',x} = \bigvee_{x'\in J} a_{x'} \& a_{x',x} = \bigvee_{x'\in J} a_{y,x'} \& a_{x',x} \leqslant a_{y,x},$$

while, on the other,

$$a_{y,x} = a_y \& a_{y,x} = \bigvee_{x' \in J} a_{x'} \& a_{y,x} = \bigvee_{x' \in J} a_{x',y} \& a_{y,x} \leqslant \bigvee_{x' \in J} a_{x',x},$$

whence the first equality of (5). The second part follows similarly.

PROPOSITION 3.5. Let $(X, A, \Rsh\Lsh)$ be a separated presheaf and let $(J, {}_J A) \subseteq (X, A)$ be a sub-Q-set with $J \subseteq X_u$ (for some $u \in Q_0$) such that the inequality

$$a_x \& a_{x'} \leqslant a_x \wedge a_{x'} \tag{8}$$

holds for all $x, x' \in J$. Then the following conditions are equivalent:

(i) $a_x \& a_{x'} \Rsh x \Lsh a_{x'} = a_x \Rsh x' \Lsh a_x$ for all $x, x' \in J$,
(ii) $a_x \& a_{x'} = a_{x,x'}$ for all $x, x' \in J$.

First of all, we note that the assumption (8) ensures the restrictability of the triplets $(a_x \& a_{x'}, x, a_{x'})$ and (a_x, x', a_x) for all $x, x' \in J$ (and thus legitimates the expression in (i)). To prove the implication (i)⇒(ii), assume (i), Then, given $x, x' \in J$, we have the following chain of implications:

$$a_{a_x \& a_{x'} \Rsh x \Lsh a_{X'}, a_x \Rsh x' \Lsh a_x} = a_{a_x \Rsh x' \Lsh a_x}$$

$$\Rightarrow a_x \& a_{x'} \& a_{x,x'} \& a_x = a_x \& a_{x'} \& a_x$$

$$\Rightarrow a_x \& a_{x'} \& a_{x,x'} \& a_x \& a_{x'} = a_x \& a_{x'} \& a_x \& a_{x'}$$

$$\Rightarrow a_x \& a_x \& a_{x,x'} \& a_{x'} \& a_{x'} = a_x \& a_x \& a_{x'} \& a_{x'} (\text{by Midsymmetry})$$

$$\Rightarrow a_{x,x'} = a_x \& a_{x'} (\text{by Strictness}),$$

whence (ii). To prove that (ii) implies (i), assume (ii). Then, given $x, x' \in J$, we have that $a_x \& a_{x'} \& a_x \& a_{x'} = a_x \& a_{x'} \& a_x$, $a_x \& a_{x'} \& a_x = a_x \& a_{x'} \& a_{x,x'} \& a_x$, and that $a_x \& a_{x'} \& a_x \& a_{x'} = a_x \& a_{x',x} \& a_{x'}$, which may be written

$$a_{a_x \& a_{x'} \Rsh x \Lsh a_{x'}} = a_{a_x \Rsh x' \Lsh a_x} = a_{a_x \& a_{x'} \Rsh x \Lsh a_{x'}, a_x \Rsh x' \Lsh a_x} = a_{a_x \Rsh x' \Lsh a_x, a_x \& a_{x'} \Rsh x \Lsh a_{x'}},$$

whence (i) (by Separation).

In view of Proposition 3.3, we have

COROLLARY 3.6. In the setting of the preceding proposition, the following two conditions are equivalent:

(i) $)(J, {}_J A) \subseteq (X, A)$ is compatible;
(ii) $a_x \& a_{x'} \leqslant a_x \wedge a_{x'}$ and $a_x \& a_{x'} \Rsh x \Lsh a_{x'} = a_x \Rsh x' \Lsh a_x$ for all $x, x' \in J$.

Now consider the case of a right-sided idempotent quantale in order to compare our concept of sheaf with that of C.J. Mulvey and M. Nawaz.

DEFINITION 3.7. (Definition 27 [4] and Definition 28 [4]). Let Q be a right-sided idempotent quantale and let $(X, E, \downharpoonleft, \upharpoonleft)$ be a presheaf in the sense of C.J. Mulvey and M. Nawaz.

(i) A subset J of X is said to be compatible if

$$Ex \downharpoonleft x' = x \upharpoonleft Ex'$$

for all $x, x' \in J$.

(ii) A quadruple $(X, E, \downharpoonleft, \upharpoonleft)$ is called a sheaf on Q if, for any compatible subset $J \subseteq X$, there exists a "join" of J, i.e., a unique element $y \in X$ such that

$$Ey = \bigvee_{x \in J} Ex$$

and $Ex \downharpoonleft y = x$ for all $x \in J$.

PROPOSITION 3.8. (Proposition 7.6 [2]). Let (X, A) be a separated quantal set on a right-sided idempotent quantale Q such that its every singleton $\mathcal{S} = (S, S^{\#})$ satisfies the condition that $s_x^{\#} = a_x \& s_x$ for all $x \in X$ (observing that every singleton in the sense of C.J. Mulvey and M. Nawaz is so). Let $(X, A, \Rsh\Lsh)$ be a presheaf on Q and let $(X, E, \downharpoonleft, \upharpoonleft)$ be the presheaf associated to $(X, A, \Rsh\Lsh)$ in the setting of Proposition 6.5 [2]. If $(X, E, \downharpoonleft, \upharpoonleft)$ is a sheaf in the sense of Definition 3.7, then the underlying presheaf $(X, A, \Rsh\Lsh)$ is a sheaf in our sense.

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Apie pluoštus virš midsimetriškų kvantaloidų

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Nagrinėjami priešpluoščiai ir pluoštai virš midsimetriškų kvantaloidų. Nustatytos sąlygos, kuriomis priešpluoštis tampa pluoštu.