LMD mokslo darbai, 35–39
1998, Vilnius

A joint limit theorem for Lerch zeta-function

A. Laurinčikas* **(VU)**

1. Introduction

Let $s = \sigma + it$ be a complex variable. The Lerch zeta-function $L(\lambda, \alpha, s)$ is defined, for $\sigma > 1$, by

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s}.$$

Here α, λ, $0 < \alpha \leqslant 1$, are fixed parameters. When λ is an integer, the Lerch zeta-function reduces to the Hurwitz zeta-function. Therefore we consider the case $0 < \lambda < 1$. It is well known that in this case the Lerch zeta-function is analytically continuable to an entire function.

Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S, and let, for $T > 0$,

$$\nu_T(\ldots) = \frac{1}{T} \operatorname{meas}\{t \in [0, T], \ldots\},$$

where $\operatorname{meas}\{A\}$ stands for the Lebesgue measure of the set A, and in place of dots some condition satisfied by t is to be written.

Let $D = \{s \in \mathbb{C} : \sigma > 1/2\}$, and denote by $H(D)$ the space of analytic on D functions equipped with the topology of uniform convergence on compacta. Denote by γ the unit circle on $\mathbb{C}$, and let

$$\Omega = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m = 0, 1, 2, \ldots$. Let m_H be the probability Haar measure on $(\Omega, \mathcal{B}(\Omega))$. Define on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ an $H(D)$-valued random element

$$L(s, \omega) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m} \omega(m)}{(m+\alpha)^s},$$

where $\omega(m)$ is the projection of $\omega \in \Omega$ to the coordinate space γ_m.

Let n be a natural number. Define on $(\Omega, \mathcal{B}(\Omega), m_H)$ an $H^n(D)$-valued random element $L_n(s, \omega)$ by

$$L_n(s, \omega) = \big(L(s, \omega), L^2(s, \omega), \ldots, L^n(s, \omega)\big),$$

*Partially supported by Grant from Lithuanian Foundation of Studies and Science.

and let P_{L_n} denote the distribution of $L_n(s,\omega)$. Our purpose is to prove a limit theorem for the probability measure

$$P_T(A) = \nu_T\Big(\big(L(\lambda,\alpha,s+i\tau), L^2(\lambda,\alpha,s+i\tau), \dots, L^n(\lambda,\alpha,s+i\tau)\big) \in A\Big), \quad A \in \mathcal{B}(H^n(D)).$$

THEOREM. *Let α be a transcendental number. Then the probability measure P_T converges weakly to P_{L_n} as $T \to \infty$.*

The theorem remains valid in the case when the parameters α and λ in the definition of P_T are different.

2. Auxiliary results

For the proof of the theorem we will use an onedimensional limit theorem for the Lerch zeta-function.

LEMMA 1. *Let α be a transcendental number. Then the probability measure*

$$\nu_T\big(L(\lambda,\alpha,s+i\tau) \in A\big), \quad A \in \mathcal{B}(H(D)),$$

converges weakly to the distribution of the random element $L(s,\omega)$ as $T \to \infty$.

Proof of the lemma is given in [2], [3]. Let S and S_1 be two metric spaces, and let $h: S \to S_1$ be a measurable function. Then every probability measure P on $(S, \mathcal{B}(S))$ induces on $(S_1, \mathcal{B}(S_1))$ the unique probability measure Ph^{-1} defined by the equality $Ph^{-1}(A) = P(h^{-1}A)$, $A \in \mathcal{B}(S_1)$.

LEMMA 2. *Let h: $S \to S_1$ be a continuous function. If P_n converges weakly to P, then $P_n h^{-1}$ converges weakly to Ph^{-1} as $n \to \infty$.*

Proof can be found in [1].

LEMMA 3. *Let α be a transcendental number, and k be a natural number. Then the probability measure*

$$\nu_T\big(L^k(\lambda,\alpha,s+i\tau) \in A\big), \quad A \in \mathcal{B}(H(D)),$$

converges weakly, to the distribution of the random element $L^k(s,\omega)$.

Proof. The lemma is a simple consequence of Lemmas 1 and 2.

3. Proof of the theorem

We will deduce the theorem from Lemma 3.

LEMMA 4. *The family of probability measures $\{P_T, T > 0\}$ is relatively compact.*

Proof. By Lemma 3 the probability measure

$$P_{kT}(A) = \nu_T\big(L^k(\lambda, \alpha, s + i\tau) \in A\big), \quad A \in \mathcal{B}(H(D)),$$

for any natural k converges weakly to the distribution of the random element $L^k(s, \omega)$ as $T \to \infty$. From this it follows that the family of probability measures $\{P_{kT}, T > 0\}$ is relatively compact. Since $H(D)$ is a complete separable space, hence we obtain by the second Prokhorov theorem [1] that the family $\{P_{kT}\}$ is tight, i.e. for an arbitrary $\varepsilon > 0$ there exists a compact set $K_k \subset H(D)$ such that

$$P_{kT}\big(H(D) \setminus K_k\big) < \frac{\varepsilon}{n} \tag{1}$$

for all $T > 0$. Define on a probability space $(\Omega_0, \mathcal{F}, \mathbb{P})$ a random variable η_T by

$$\mathbb{P}(\eta_T \in A) = \frac{1}{T}\int_0^T I_A\, dt, \quad A \in \mathcal{B}(\mathbb{R}),$$

where I_A is the indicator function of the set A. Consider the $H(D)$-valued random element $L_{kT}(s) = L^k(\lambda, \alpha, s + i\eta_T)$, and let

$$L_T(s) = \big(L_{1T}(s), \dots, L_{nT}(s)\big).$$

Then, by (1)

$$\mathbb{P}\big(L_{kT}(s) \in H(D) \setminus K_k\big) < \frac{\varepsilon}{n}.$$

Hence, putting $K = K_1 \times \dots \times K_n$, we obtain

$$P_T\big(H^n(D) \setminus K\big) = \mathbb{P}\big(L_T(s) \in H^n(D) \setminus K\big) = \mathbb{P}\bigg(\bigcup_{k=1}^{n}\big(L_{kT}(s) \in H(D) \setminus K_k\big)\bigg) < \varepsilon$$

for all $T > 0$. Consequently, the family $\{P_T\}$ is tight. Hence by the first Prokhorov theorem [1] it is relatively compact.

Let $s_1, \dots, s_r$ be arbitrary points on D, $\sigma_1 = \min_{1 \leqslant l \leqslant r} \Re s_l$, $\sigma_2 = 1/2 - \sigma_1 < 0$, and $D_1 = \{s \in \mathbb{C} : \sigma > \sigma_2\}$. We take arbitrary complex number u_{kl}, $1 \leqslant k \leqslant u$, $1 \leqslant l \leqslant r$, and let $h : H^n(D) \to H(D_1)$ be given by the formula

$$h(f_1, \dots, f_n) = \sum_{k=1}^{n}\sum_{l=1}^{r} u_{kl} f_k(s_l + s), \quad s \in D,\ f_j \in H(D).$$

Moreover, let

$$L_h(s) = h\big(L(\lambda, \alpha, s), L^2(\lambda, \alpha, s), \dots, L^n(\lambda, \alpha, s)\big).$$

Then precisely as in [4] we find that

$$L_h(s + i\eta_T) \xrightarrow[T\to\infty]{\mathcal{D}} h\big(L_n(s)\big), \tag{2}$$

where $L_n(s) = L_n(s, \omega)$.

Proof of theorem. By Lemma 4 there exists a sequence $T_1 \to \infty$ such that P_{T_1} converges weakly to some probability measure P. Let P is the distribution of an $H^n(D)$-valued random element

$$\tilde{L}(s) = \big(\tilde{L}_1(s), \dots, \tilde{L}_n(s)\big),$$

i.e.

$$L_{T_1} \xrightarrow[T_1 \to \infty]{\mathcal{D}} \tilde{L}.$$

Hence and from Lemma 2 we have that

$$h(L_{T_1}) \xrightarrow[T_1 \to \infty]{\mathcal{D}} h(\tilde{L}),$$

or

$$L_h(s + i\eta_{T_1}) \xrightarrow[T_1 \to \infty]{\mathcal{D}} h(\tilde{L}). \tag{3}$$

By (2)

$$L_h(s + i\eta_{T_1}) \xrightarrow[T \to \infty]{\mathcal{D}} h(L_n).$$

Hence and from (3) it follows that

$$h(L_n) \stackrel{\mathcal{D}}{=} h(\tilde{L}). \tag{4}$$

Let a function $h_1 : H(D_1) \to \mathbb{C}$ be given by the formula

$$h_1(f) = f(0), \quad f \in H(D_1).$$

Then the relation (4) implies

$$h_1\big(h(L)\big) \stackrel{\mathcal{D}}{=} h_1\big(h(\tilde{L})\big).$$

This yields

$$\sum_{k=1}^{n} \sum_{l=1}^{r} u_{kl} L^k(\lambda, \alpha, s_l, \omega) \stackrel{\mathcal{D}}{=} \sum_{k=1}^{n} \sum_{l=1}^{r} u_{kl} \tilde{L}_k(s_l)$$

for arbitrary complex number u_{kl}. Hence, using properties of hyperplanes in $\mathbb{C}^n$, we deduce that $\big(L^k(\lambda, \alpha, s_l, \omega)\big)$ and $\big(\tilde{L}_k(s_l)\big)$, $1 \leqslant k \leqslant n$, $1 \leqslant l \leqslant r$, have the same distribution.

Let K be an arbitrary compact subset of D, and let the sequence $\{s_l\}$ is dense in K. Moreover, we set

$$G = \left\{(g_1, \dots, g_n) \in H^n(D) : \sup_{s \in K} \big|g_j(s) - f_j(s)\big| \leqslant \varepsilon,\ j = 1, \dots, n\right\},$$

$$G_r = \left\{(g_1, \dots, g_n) \in H^n(D) : \big|g_k(s_l) - f_k(s_l)\big| \leqslant \varepsilon,\ k = 1, \dots, n,\ l = 1, \dots, r\right\}$$

Then we obtain that

$$m_H\big(\omega \in \Omega:\ L_n(s,\omega) \in G_r\big) = P\big(\tilde{L}(s) \in G_r\big).$$

Since $G_r \to G$ as $r \to \infty$, letting $r \to \infty$, hence we find

$$m_H\big(\omega \in \Omega:\ L_n(s,\omega) \in G\big) = P\big(\tilde{L}(s) \in G\big).$$

This gives

$$L_n \overset{\mathcal{D}}{=} \tilde{L}.$$

Thus

$$L_{T1} \xrightarrow[T_1\to\infty]{\mathcal{D}} L_n.$$

From this the theorem easily follows.

REFERENCES

[1] P. Billingsley, *Convergence of Probability Measures*, John Wiley, New York, 1968.

[2] A. Laurinčikas, *On limit distribution of the Lerch zeta-function*, New Trends in Probability and Statistics, Vol. 4, VSP/TEV, Utrecht/Vilnius, 1997, 135–148.

[3] A. Laurinčikas, A limit theorem for the Lerch zeta-function in the space of analytic functions, *Lith. Math. J.*, **37(2)** (1997), 146–155.

[4] A. Laurinčikas and K. Matsumoto, Joint value-distribution theorems on Lerch zeta-functions, *Liet. Matem. Rink.*, (to appear).

Daugiamatė ribinė teorema Lercho dzeta funkcijai

A. Laurinčikas (*VU*)

Straipsnyje įrodoma ribinė teorema Lercho dzeta funkcijos laipsniams analizinių funkcijų erdvėje.