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On simultaneous approximation of algebraic conjugates by roots of unity

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Let α be an algebraic number of degree $d \geqslant 2$, $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$ its conjugates and

$$P(x) = a(x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_d) \in \mathbb{Z}[x]$$

its minimal polynomial. We define the Mahler measure of α, $M(\alpha)$, by

$$M(\alpha) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|P(e^{it})\right|dt\right) = |a|\prod_{k=1}^{d}\max(1, |\alpha_k|).$$

In 1994, M. Mignotte and M. Waldschmidt [8] gave the following lower bound:

$$|\alpha - 1| > \exp\Big(-(1+\varepsilon)\sqrt{d\log d\log M(\alpha)}\Big) \tag{1}$$

provided that $\varepsilon > 0$, $d > d_0(\varepsilon)$ and α is an algebraic number of degree d which is not a root of unity. In 1995 [4], the constant $1+\varepsilon$ in this inequality was replaced by $\sqrt{2/3}+\varepsilon$. Finally, in [5] the author obtained the constant $\pi/4+\varepsilon < \sqrt{2/3}+\varepsilon$. On the other hand, F. Amoroso [1] proved that the estimate (1) is not far from being optimal. He gave an example showing that the inequality (1) can be strengthened only at the expense of $\log d$, but not $\sqrt{d}$. He also gave an example of the polynomial Φ for which „good" upper bound for the quantity

$$h(\Phi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \max\left(\log\left|\Phi(e^{it})\right|, 0\right)dt$$

is equivalent to the Riemann hypothesis. The measure $h(\Phi)$ was introduced by M. Mignotte [7].

Let $\mathbb{K}$ be a number field of degree d and let $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(n)}$ be multiplicatively independent elements of $\mathbb{K}$. Recently F. Amoroso [2] obtained a lower bound for $\max_{1\leqslant j\leqslant n}|\alpha^{(j)} - 1|$ in terms of $d, n, M(\alpha^{(1)}), M(\alpha^{(2)}), \ldots, M(\alpha^{(n)})$. On the other hand, the author [6] announced a theorem on simultaneous approximation where different conjugates of α were approximated by different roots of unity:

THEOREM 1. *Let $\{\delta_1, \delta_2, \dots \delta_m\}$ be the set of different roots of unity of degrees $n_1, n_2, \dots, n_m$ respectively. Then for every $\varepsilon > 0$ there exists an effective $d_0 = d_0(\varepsilon, m)$ such that for $d > d_0$*

$$|\alpha_1 - \delta_1|^{1/n_1} \dots |\alpha_m - \delta_m|^{1/n_m} > \exp\Big(- \Big(\frac{\pi}{4} + \varepsilon\Big) \sqrt{d \log d \log M(\alpha)} \Big), \tag{2}$$

where $\alpha_1, \alpha_2, \dots, \alpha_m$ are different conjugates of an algebraic number α of degree d which is not a root of unity.

The case, when α is a root of unity, is of no interest in the above theorem. After doing a little algebra in the case $\alpha^d = 1$, $\alpha \neq 1$, one can easily get

$$|\alpha - 1| > \big(1 + o_d(1)\big) 2\pi e^{-\gamma} (d \log\log d)^{-1}$$

where $o_d(1)$ is a function of d satisfying $\lim_{d\to+\infty} o_d(1) = 0$, γ is Euler's constant. The above inequality is much stronger than (1). Analogously, one can get a much stronger inequality than (2) if α is a root of unity. However, formally $\log M(\alpha) = 0$ whenever α is a root of unity, so we need this condition in the theorem.

In this paper we prove Theorem 1 and the following corollary:

COROLLARY. *Suppose that M is a subset of $\{1, 2, \dots, s\}$ and $\varepsilon > 0$. Then there exists an effective $d_1 = d_1(\varepsilon, s)$ such that for $d > d_1$*

$$\sum_{j \in M} \frac{1}{j} \log \big|\Phi_j(\alpha_j)\big| > -\Big(\frac{\pi}{4} + \varepsilon\Big) \sqrt{d \log d \log M(\alpha)}, \tag{3}$$

where $\{\alpha_j,\ j \in M\}$ are some conjugates of an algebraic number α of degree d which is not a root of unity, Φ_j is the j-th cyclotomic polynomial.

We recall that

$$\Phi_j(x) = \prod_{\zeta \in \mu_j^*} (x - \zeta),$$

where μ_j^* is the set of primitive j-th roots of unity.

Proof of Theorem 1. In our paper [5] we have considered the following function:

$$R(z) = \prod_{1 \leqslant v < u \leqslant K} |z^{u-v} - 1|^{J_u J_v},$$

where $J_u = [K \sin(\pi u / K)]$. Let $l = [(K-1)/n]$ where n is a fixed natural number. The key point of our argument is to write the function $R(z)$ in the following form:

$$R(z) = \prod\nolimits^{(1)} |z^{jn} - 1|^{J_u J_v} \prod\nolimits^{(2)} |z^{u-v} - 1|^{J_u J_v}.$$

Here the product $\prod^{(1)}$ is for u, v such that $u - v = jn$, $1 \leqslant j \leqslant l$, and the product $\prod^{(2)}$ is for u, v such that $u - v$ is not divisible by n.

Since

$$|z^{jn} - 1| = |z^n - 1|\,|1 + z^n + z^{2n} + \ldots + z^{(j-1)n}| \leqslant |z^n - 1| \max\left(1, |z|\right)^{(j-1)n} j,$$

we bound

$$\prod^{(1)} \leqslant |z^n - 1|^{\sum_1 J_u J_v} \max\left(1, |z|\right)^{\sum_1 (u-v) J_u J_v} \cdot \exp\left(\sum\nolimits_1 J_u J_v \log\left((u-v)/n\right)\right).$$

Here $\sum_1$ denotes the sum over u and v such that $u - v$ is divisible by n. As for the product $\prod^{(2)}$, we have

$$\prod^{(2)} \leqslant \max\left(1, |z|\right)^{\sum_2 (u-v) J_u J_v} \cdot \exp\left(\sum\nolimits_2 J_u J_v \log 2\right),$$

where $\sum_2$ denotes the sum over u and v such that $u - v$ is not divisible by n. Combining the estimates for $\prod^{(1)}$ and $\prod^{(2)}$, we have

$$\begin{aligned} R(z) \leqslant |z^n - 1|^{\sum_1 J_u J_v} \max\left(1, |z|\right)^{\sum J_u J_v (u-v)} \\ \times \exp\left(\sum\nolimits_1 J_u J_v \log(u-v) + \sum\nolimits_2 J_u J_v\right). \end{aligned} \tag{4}$$

For K tending to infinity we get (see [5])

$$B = \sum J_u J_v (u - v) \sim K^5 / 2\pi^2. \tag{5}$$

Analogously,

$$L(n) = \sum\nolimits_1 J_u J_v \sim 2K^4 / n\pi^2, \tag{6}$$

$$H = \sum\nolimits_1 J_u J_v \log(u - v) + \sum\nolimits_2 J_u J_v \sim c_1 K^4 \log K, \tag{7}$$

where c_1 is some absolute constant.

On the other hand (see (7) and (12) in [5]),

$$R(z) < \max\left(1, |z|\right)^{\sum J_u J_v (u-v)} e^C, \tag{8}$$

where

$$C \sim \frac{1}{4} K^3 \log K. \tag{9}$$

Since α is not a root of unity, the product

$$|a|^B \prod_{j=1}^{d} R(\alpha_j)$$

is a non-zero integer. For $1 \leqslant j \leqslant m$, we estimate $R(\alpha_j)$ by (4) taking $n = n_j$. For $j > m$, we estimate $R(\alpha_j)$ by (8). Thus, by (4)–(9)

$$1 < \prod_{j=1}^{m} \left|\alpha_j^{n_j} - 1\right|^{L(n_j)} M(\alpha)^B e^{mH+dC},$$

$$1 < e^T \prod_{j=1}^{m} \left|\alpha_j^{n_j} - 1\right|^{1/n_j},$$

where

$$T \sim \frac{\pi^2 B \log M(\alpha)}{2K^4} + \frac{\pi^2(mH + dC)}{2K^4}$$
$$\sim \frac{K \log M(\alpha)}{4} + \frac{\pi^2 d \log K}{8K} + c_2 \log K.$$

Now taking $K = [\pi\sqrt{d \log d / 4 \log M(\alpha)}]$, for d tending to infinity, we obtain

$$T \sim \frac{\pi}{4}\sqrt{d \log d \log M(\alpha)}.$$

Hence, for $d > d_0$

$$\prod_{j=1}^{m} \left|\alpha_j^{n_j} - 1\right|^{1/n_j} > \exp\left(- \left(\frac{\pi}{4} + \varepsilon\right)\sqrt{d \log d \log M(\alpha)}\right). \tag{10}$$

Without loss of generality we can assume that $|\alpha_j| \leqslant 2$, $1 \leqslant j \leqslant m$. Then

$$\prod_{j=1}^{m} \left|\left(\alpha_j^{n_j} - 1\right)/\left(\alpha_j - \delta_j\right)\right|^{1/n_j} \leqslant \prod_{j=1}^{m} 3^{(n_j-1)/n_j} < 3^m,$$

and the inequality (2) follows.

Proof of the Corollary. Since

$$x^j - 1 = \prod_{r|j} \Phi_r(x),$$

we have $|\alpha_j^j - 1| \leqslant c(s)\Phi_j(\alpha_j)$. Utilizing (10), where $n_j = j$ and each α is with the required index, we get (3).

Alternatively, we can use the following statement (proposition 3.3 in [9]): for all $\alpha \in \mathbb{C}$ not a root of unity and satisfying $|\alpha| \leqslant 1$, and for all integers $m \geqslant 1$,

$$\min_{\zeta \in \mu_m^*} |\alpha - \zeta| \leqslant |\Phi_m(\alpha)|\,(118m)^{3\sigma(m)/2}.$$

Here $\sigma(m)$ is the number of divisors of m. Now utilizing the well known estimate (see e.g. [3])

$$\sigma(m) < c_3 m^{(1+\varepsilon)\log 2/\log\log m},$$

we can get the explicit expression for $c(s)$.

We proved the inequality (3) when the degree d is "large" compared to the maximal j. On the other hand, J. Silverman [9] gave a lower bound for $\log|\Phi_j(\alpha)|$ in the case when j is "large" compared to d. He proved the following result, which is essentially due to C. Stewart [10]: let α be an algebraic integer of degree $d \geqslant 2$ that is not a root of unity. If $j \geqslant (1000d)^{265}$, then

$$\log|\Phi_j(\alpha)| > (1000d)^{50} j^{3/5}.$$

In the following theorem we give the lower bound for $\log|\Phi_j(\alpha)|$. This strengthens the inequality (3) when $\operatorname{card} M = 1$.

THEOREM 2. *Let α be an algebraic number which is not a root of unity, d its degree and $M(\alpha)$ its Mahler's measure. Then for any $\varepsilon > 0$ and $j > 2$ there exists an effective constant $d_2 = d_2(\varepsilon, j)$ such that for $d > d_2$*

$$\log|\Phi_j(\alpha)| > -\left(\frac{\pi\sqrt{j}}{8} + \varepsilon\right)\sqrt{d\log d\log M(\alpha)},$$

where Φ_j is the j-th cyclotomic polynomial.

Proof of Theorem 2. In section 3 of our paper [5] we obtained the following lower bound:

$$\left|\alpha - \exp(2\pi ir/j)\right| > \exp\left(-\left(\frac{\pi\sqrt{j}}{8} + \varepsilon\right)\sqrt{d\log d\log M(\alpha)}\right) \qquad (11)$$

whenever $(r, j) = 1$ and $\exp(2\pi ir/j)$ is not equal to ± 1. If $j > 2$, then the polynomial $\Phi_j(x)$ has no real roots. Thus, the inequality (11) holds.

Let $\exp(2\pi ir/j)$ be the nearest primitive j-th root of unity to α. Without loss of generality we may assume that, e.g.,

$$\left|\alpha - \exp(2\pi ir/j)\right| < j^{-2}.$$

Then obviously α is well separated from the other primitive j-th roots of unity, and we have

$$\left|\alpha - \exp(2\pi ir/j)\right| < |\Phi_j(\alpha)| c(j).$$

This combined with the inequality (11) implies Theorem 2.

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Apie vienalaikę jungtinių algebrinių skaičių aproksimaciją šaknimis iš vieneto

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Darbe nagrinėjama jungtinių algebrinių skaičių aproksimacija šaknimis iš vieneto. Gautas atitinkamas apatinis įvertis nenaudojant algebrinio skaičiaus laipsnį ir jo Malerio matą.