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Distance between the fractional Brownian motion and the space of adapted Gaussian martingales

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Abstract. We consider the distance between the fractional Brownian motion defined on the interval $[0,1]$ and the space of Gaussian martingales adapted to the same filtration. As the distance between stochastic processes, we take the maximum over $[0,1]$ of mean-square deviances between the values of the processes. The aim is to calculate the function a in the Gaussian martingale representation $\int_0^t a(s)\,\mathrm{d}W_s$ that minimizes this distance. So, we have the minimax problem that is solved by the methods of convex analysis. Since the minimizing function a cannot be either presented analytically or calculated explicitly, we perform discretization of the problem and evaluate the discretized version of the function a numerically.

Keywords: fractional Brownian motion, Gaussian martingales, convex programming, minimax approximation.

1 Introduction

Fractional Brownian motion on $[0,1]$ is a Gaussian process $B^H = \{B_t^H,\, t\in[0,1]\}$ with zero mean and covariance function

$$\mathsf{E}\, B_s^H B_t^H = \frac{1}{2}\bigl(t^{2H} + s^{2H} - |t-s|^{2H}\bigr),$$

where $H \in (0,1)$ is the Hurst index. This process admits the Molchan–Golosov representation, see [14–16]:

$$B_t^H = \int_0^t z(t,s)\,\mathrm{d}W_s, \tag{1}$$

where $W = \{W_t,\, t\in[0,1]\}$ is a standard Brownian motion,

$$z(t,s) = c_H s^{1/2-H} \times \left(\bigl(t(t-s)\bigr)^{H-1/2} - \left(H-\frac{1}{2}\right)\int_s^t u^{H-3/2}(u-s)^{H-1/2}\,\mathrm{d}u\right) 1_{0<s<t}, \tag{2}$$

and

$$c_H = \left(\frac{2H\Gamma(\frac{3}{2} - H)}{\Gamma(H + \frac{1}{2})\Gamma(2 - 2H)}\right)^{1/2}.$$

If $H > 1/2$, then the kernel z from (2) can be simplified to

$$z(t,s) = \left(H - \frac{1}{2}\right) c_H s^{1/2-H} \int_s^t u^{H-1/2}(u-s)^{H-3/2}\,\mathrm{d}u\, 1_{0<s<t}.$$

Obviously, $z(t,s) > 0$ for all $H \in (0,1)$ and $0 < s < t$. Note that B^H and W generate the same filtration, and denote it by $\mathcal{F}^W = \{\mathcal{F}_t,\, t \in [0,1]\}$.

It is well known that fBm is not a semimartingale and certainly is not a martingale unless $H = 1/2$. Therefore, a natural question of the approximation of fBm by martingales and semimartingales arises. On the one hand, as to semimartingales, the situation is more simple: fBm can be approximated by the Gaussian semimartingales in $\sup L_2$-norm (see, e.g., [9]). On the other hand, it is obvious that we cannot approximate the fBm by martingales with an arbitrary accuracy in any reasonable norm, see, e.g., [3]. Therefore, it is natural to put the question differently: what is the distance between the fBm and the martingale space in a reasonable metric, and on which element from the martingale space does this distance reach? This is a question of projection of fBm on the space of Gaussian martingales generated by the Wiener process. Solving this problem, we can, in the future, move forward a few steps, namely, we can consider the projection of the fBm on the space of repeated Wiener integrals, that is, to calculate the projections on the finite-dimensional spaces of Wiener chaos, so, to clarify the structure of fractional Brownian motion in comparison with the spaces of Wiener chaos. Being very simple in appearance in his statement, the question of the distance between the fBm and the space of adapted Gaussian martingales, nevertheless turned out to be very difficult. The existence and uniqueness of the minimizing function in the representation of Gaussian martingale, as well as a number of its properties were established in the paper [17] by the methods of convex analysis. However, it is still impossible to get the analytical representation of the minimizing function. Therefore, we decided to find this function approximately, for which we used the concept of the Chebyshev center and its properties. Now, let us state the problem and consider the methods of its solution in more detail.

For Gaussian stochastic process of the form $X_t = \int_0^t a(s)\,\mathrm{d}W_s$, we define the square distance between X and B^H as follows:

$$F(a) = \max_{t\in[0,1]} \mathsf{E}\big(X_t - B_t^H\big)^2 = \max_{t\in[0,1]} \int_0^t \big(a(s) - z(t,s)\big)^2\,\mathrm{d}s. \tag{3}$$

The maximum is attained because the random processes X and B^H are mean-square continuous and, as a consequence, $\mathsf{E}(X_t - B_t^H)^2$ is a continuous function in t, which attains its minimum and maximum on $[0,1]$.

We are searching for the function $a \in L_2[0,1]$ for which the square distance in minimal. The choice of the representation of the process X is justified as follows. Any square-integrable martingale with respect to $\mathcal{F}^W$ with right-continuous trajectories admits a representation $X_t = x_0 + \int_0^t a(s)\,\mathrm{d}W_s$, where a is a stochastic process adapted to the same filtration as W and satisfying $\mathsf{E}\int_0^1 a(s)^2\,\mathrm{d}s < \infty$ [11, Thm. 5.5]. In [17, p. 541] it is proved that the minimum of $\sup_{t\in[0,1]} \mathsf{E}(X_t - B_t^H)^2$ is attained for $x_0 = 0$ and for a being a nonrandom function. A problem of approximation of the fBm by a Gaussian martingale from some smaller classes of Gaussian martingales, e.g., from a finite-dimensional subsets, is much simpler than a general problem, and was solved in [1, 2, 8, 13]. The reciprocal problem of approximation of the Wiener process by an integral with respect to the fractional Brownian motion, also from some specific class of integrands, was solved in [4]. It is proved in [17] that the minimum of $F(a)$ from (3) for any $H \in (0,1)$ is attained at unique point, and recall again that even though we know some properties of a, we do not have for it an analytical expression.

In order to get function a at least numerically, in the present paper, we approximate a discrete-time slice $\{B_t^H,\ t = 1/N, 2/N, \dots, 1\}$ of the fBm instead of performing continuous-time approximation.

We find a "discrete analogue" for the function a numerically. In order to do this, we use the so-called alternating minimization [5], and search for the Chebyshev center of the finite set, see, e.g., [6]. The problem of locating the Chebyshev center is a linear minimization problem with quadratic constraints, while the dual problem is the quadratic minimization with linear constraints, see [7, 18]. Our goal is evaluation of $\min_{a\in L_2[0,1]} F(a)$. For that reason, we prove the following result:

$$\min_{\vec{a}\in\mathbb{R}^N} F\bigl(\vec{a}, \mathbf{K}^{(N)}\bigr) \leqslant \min_{a\in L_2[0,1]} F(a) = \lim_{N\to\infty} \min_{\vec{a}\in\mathbb{R}^N} F\bigl(\vec{a}, \mathbf{K}^{(N)}\bigr), \tag{4}$$

where $F(\vec{a}, \mathbf{K}^{(N)})$, $N \geqslant 1$, is the sequence of the convex functions of the vector $\vec{a} \in \mathbb{R}^N$. Vector $\vec{a}$ is related to function a, and matrices $\mathbf{K}^{(N)}$ are related to the kernel z in the representation (2). These functions are discrete analogue of the function $F(a)$. Two-sided relation (4) is the main mathematical contribution of the present paper. It allows to calculate the minimizing function a numerically.

The paper is organized as follows. In Section 2 we construct the sequence of the functions $F(\vec{a}, \mathbf{K}^{(N)})$ and prove relation (4). In Section 3, we propose and justify the algorithm for numerical evaluation of $F(\vec{a}, \mathbf{K}^{(N)})$. In Section 4, the minimizing function is plotted. Appendix contains some auxiliary results.

2 Two-sided bounds and convergence result

In what follows, we denote matrices (but not matrix entries) with boldface letters. For vectors, we use notation like $\vec{a}$. The elements of the vector are denoted with the same letter as the vector itself; that is if $\vec{a} \in \mathbb{R}^N$, then $\vec{a} = (a_1, \dots, a_N)^\top$.

Let $N \in \mathbb{N}$, and consider the random variables $B^H_{k/N}$, $0 \leqslant k \leqslant N$. The discrete-time process $\{B^H_{k/N},\ k = 0, 1, \ldots, N\}$ is called a discrete-time fBm. It generates a discrete-time filtration $\mathcal{F}^{(N)} = \{\mathcal{F}^{(N)}_k,\ k = 0, 1, \ldots, N\}$, where $\mathcal{F}^{(N)}_k = \sigma(B^H_0, \ldots, B^H_{k/N})$, $0 \leqslant k \leqslant N$.

Now our goal is to establish the lower and upper bounds for $\min_{a \in L_2[0,1]} F(a)$, and to prove the equality in formula (4), which means that both the lower and upper bounds tend to $\min_{a \in L_2[0,1]} F(a)$ as $N \to \infty$. To achieve this result, we discretize the function $z(t,s)$ in the following way. Denote by $k^{(N)}_{mn}$ the average value of $N^{-1/2} z(m/N, s)$ over $((n-1)/N, n/N)$:

$$k^{(N)}_{mn} = \sqrt{N} \int\limits_{(n-1)/N}^{n/N} z\left(\frac{m}{N}, s\right) \mathrm{d}s, \quad 1 \leqslant m,\ n \leqslant N. \tag{5}$$

Since $z(t,s) \geqslant 0$, all entries $k^{(N)}_{mn}$ are nonnegative. Moreover, due to the relations $z(t,s) > 0$ for $0 < s < t$ and $z(t,s) = 0$ for $s > t$, the entries $k^{(N)}_{mn}$ form a lower-triangular matrix, which we denote by $\mathbf{K}^{(N)}$. So, $k^{(N)}_{mn} = 0$ if $1 \leqslant m < n \leqslant N$, and $k^{(N)}_{mn} > 0$ if $1 \leqslant n \leqslant m \leqslant N$.

Denote by $z^{(N)}(t,s)$ the piecewise-constant approximation of $z(t,s)$ with step $1/N$:

$$z^{(N)}(t,s) = N \int\limits_{(n-1)/N}^{n/N} z(t,u)\, \mathrm{d}u \quad \text{for all } s \in \left[\frac{n-1}{N}, \frac{n}{N}\right), \tag{6}$$

and $z^{(N)}(t,1) = 0$. With this notation,

$$z^{(N)}\left(\frac{m}{N}, s\right) = \sqrt{N} k^{(N)}_{mn} \quad \text{if } s \in \left[\frac{n-1}{N}, \frac{n}{N}\right).$$

The discrete-time distance in $L_2[0,1]$ between the function z presented by its discrete counterpart $z(m/N, \cdot\,)$ and approximation $z^{(N)}$ is denoted by d_{mN}, so that

$$\begin{aligned} d_{mN} &= \int\limits_0^1 \left(z\left(\frac{m}{N}, s\right) - z^{(N)}\left(\frac{m}{N}, s\right)\right)^2 \mathrm{d}s \\ &= \sum_{n=1}^N \int\limits_{(n-1)/N}^{n/N} \left(z\left(\frac{m}{N}, s\right) - \sqrt{N} k^{(N)}_{mn}\right)^2 \mathrm{d}s. \end{aligned} \tag{7}$$

In this section, we compare minimal values of two functionals. The first functional is defined in (3). The second one has the following form:

$$F\big(\vec{a}, \mathbf{K}^{(N)}\big) = \max_{1 \leqslant m \leqslant N} \sum_{n=1}^m \big(a_n - k^{(N)}_{mn}\big)^2, \quad \vec{a} \in \mathbb{R}^N, \tag{8}$$

where a_n are the elements of the vector $\vec{a}$. The next result gives both nonasymptotic and asymptotic bounds for $(\min_{a\in L_2[0,1]} F(a))^{1/2}$.

Theorem 1.

(i) *The following two-sided inequality holds*:

$$\begin{aligned}\Big(\min_{\vec{a}\in\mathbb{R}^N} F\big(\vec{a},\mathbf{K}^{(N)}\big)\Big)^{1/2} &\leqslant \Big(\min_{a\in L_2[0,1]} F(a)\Big)^{1/2}\\ &\leqslant \Big(\min_{\vec{a}\in\mathbb{R}^N} F\big(\vec{a},\mathbf{K}^{(N)}\big)\Big)^{1/2} + \max_{1\leqslant m\leqslant N}\sqrt{x_{mN}}\\ &\quad + \frac{1}{\sqrt{2}}\max_{1\leqslant m,n\leqslant N} k_{mn}^{(N)} + \frac{1}{(2N)^H}.\end{aligned} \tag{9}$$

(ii) *Additionally, the following equality holds*:

$$\min_{a\in L_2[0,T]} F(a) = \lim_{N\to\infty}\min_{\vec{a}\in\mathbb{R}^N} F\big(\vec{a},\mathbf{K}^{(N)}\big).$$

Proof. (i) *Lower bound*. Prove the first inequality in (9). In order to do this, define a linear operator of the form

$$P^{(N)}f(t) = N\int\limits_{(n-1)/N}^{n/N} f(s)\,\mathrm{d}s,\quad t\in\left[\frac{n-1}{N},\frac{n}{N}\right);\qquad P^{(N)}f(1)=0.$$

Let us mention several properties of the operator $P^{(N)}$. First, the operator $P^{(N)}$ is the orthogonal projector in the Hilbert space $L_2[0,1]$ onto the subspace of functions that are constant on the intervals $[(n-1)/N, n/N)$. Hence, $\|P^{(N)}\| = 1$, where $\|\cdot\|$ is the operator norm. Second, for the entity

$$(f1_{[0,s)})(t) = \begin{cases} f(t) & \text{if } 0\leqslant t<s,\\ 0 & \text{if } s\leqslant t\leqslant 1,\end{cases}$$

we have an equality

$$P^{(N)}\big(f1_{[0,m/N)}\big) = \big(P^{(N)}f\big)1_{[0,m/N)}$$

for all integers m, $1\leqslant m\leqslant N$, and all $f\in L_2[0,1]$. Third, the functional equality $z^{(N)}(t,\cdot) = P^{(N)}z(t,\cdot)$ holds true for all $t\in[0,1]$.

According to [17], the function $F(a)$ attains its minimum at the unique point. Let this point be $\tilde{a}\in L_2[0,1]$. Denote

$$\tilde{a}_n = \sqrt{N}\int\limits_{(n-1)/N}^{n/N}\tilde{a}(s)\,\mathrm{d}s,\quad \vec{\tilde{a}}_n = (\tilde{a}_1,\dots,\tilde{a}_N)^\top.$$

With this notation,

$$P^{(N)}\tilde{a}(s) = \sqrt{N}\tilde{a}_n \quad \text{if } s \in \left[\frac{n-1}{N}, \frac{n}{N}\right).$$

For every integer m, $1 \leqslant m \leqslant N$, we have that

$$\begin{aligned}
\sum_{n=1}^{m} \bigl(\tilde{a}_n - k_{mn}^{(N)}\bigr)^2 &= \int_0^{m/N} \left(P^{(N)}\tilde{a}(s) - z^{(N)}\left(\frac{m}{N}, s\right)\right)^2 \mathrm{d}s \\
&= \left\| \left(P^{(N)}\left(\tilde{a} - z\left(\frac{m}{N}, \cdot\right)\right)\right) 1_{[0,m/N]} \right\|^2 \\
&= \left\| P^{(N)}\left(\left(\tilde{a} - z\left(\frac{m}{N}, \cdot\right)\right) 1_{[0,m/N]}\right) \right\|^2 \\
&\leqslant \bigl\|P^{(N)}\bigr\|^2 \left\| \left(\tilde{a} - z\left(\frac{m}{N}, \cdot\right)\right) 1_{[0,m/N]} \right\|^2 \\
&= \left\| \left(\tilde{a} - z\left(\frac{m}{N}, \cdot\right)\right) 1_{[0,m/N]} \right\|^2 \\
&= \int_0^{m/N} \left(\tilde{a}(s) - z\left(\frac{m}{N}, s\right)\right)^2 \mathrm{d}s.
\end{aligned}$$

Maximizing over m, we obtain

$$\max_{1\leqslant m\leqslant N} \sum_{n=1}^{m} \bigl(\tilde{a}_n - k_{mn}^{(N)}\bigr)^2 \leqslant \max_{1\leqslant m\leqslant N} \int_0^{m/N} \left(\tilde{a}(s) - z\left(\frac{m}{N}, s\right)\right)^2.$$

Now we are ready to obtain the first inequality in (9):

$$\begin{aligned}
\min_{\vec{a}\in\mathbb{R}^N} F\bigl(\vec{a}, \mathbf{K}^{(N)}\bigr) &\leqslant F\bigl(\vec{\tilde{a}}, \mathbf{K}^{(N)}\bigr) = \max_{1\leqslant m\leqslant N} \sum_{n=1}^{m} \bigl(\tilde{a}_n - k_{mn}^{(N)}\bigr)^2 \\
&\leqslant \max_{1\leqslant m\leqslant N} \int_0^{m/N} \left(\tilde{a}(s) - z\left(\frac{m}{N}, s\right)\right)^2 \mathrm{d}s \\
&\leqslant \max_{t\in[0,1]} \int_0^{t} \bigl(\tilde{a}(s) - z(t,s)\bigr)^2 \mathrm{d}s \\
&= F(\tilde{a}) = \min_{a\in L_2[0,1]} F(a).
\end{aligned}$$

So, the first inequality in (9) is proved.

(i) *Upper bound*. Prove the second inequality in (9). Let N be fixed. Let $\vec{a}_* = (a_{*1}, \dots, a_{*N})^\top$ be the point where the functional $F^{(N)}$ attains its minimum. Without loss of generality, we can assume that

$$0 \leqslant \min_{n \leqslant m \leqslant N} k_{mn}^{(N)} \leqslant a_{*n} \leqslant \max_{n \leqslant m \leqslant N} k_{mn}^{(N)} \quad \text{for all } 1 \leqslant n \leqslant N. \tag{10}$$

It is a consequence of the following fact: if some element a_n of the vector $\vec{a}$ lies outside interval $[\min_{n \leqslant m \leqslant N} k_{mn}^{(N)}, \max_{n \leqslant m \leqslant N} k_{mn}^{(N)}]$, then, moving a_n into the nearest endpoint of the interval, we do not increase the value of $F(\vec{a}, \mathbf{K}^{(N)})$. If the function $F(\cdot, \mathbf{K}^{(N)})$ attains its minimum at the unique point, this point of minimum must satisfy inequality (10). Define the function $a(s)$ as follows:

$$a(s) = \sqrt{N} a_{*n} \quad \text{for all } s \in \left[\frac{n-1}{N}, \frac{n}{N}\right), \qquad a(1) = 0,$$

and consider a Gaussian process $\{X_t,\, t \in [0,1]\}$ of the form $X_t = \int_0^t a(s)\,\mathrm{d}W_s$. According to isometry property for stochastic integrals, we have that for every integer $1 \leqslant m \leqslant N$,

$$\begin{aligned} d_{mN}^1 &:= \mathsf{E}\left(B_{m/N}^H - \int_0^{m/N} z^{(N)}\left(\frac{m}{N}, s\right) \mathrm{d}W_s\right)^2 \\ &= \int_0^{m/N} \left(z\left(\frac{m}{N}, s\right) - z^{(N)}\left(\frac{m}{N}, s\right)\right)^2 \mathrm{d}s, \end{aligned}$$

and

$$\begin{aligned} d_{mN}^2 &:= \mathsf{E}\left(\int_0^{m/N} z^{(N)}\left(\frac{m}{N}, s\right) \mathrm{d}W_s - X_{m/N}\right)^2 \\ &= \int_0^{m/N} \left(z^{(N)}\left(\frac{m}{N}, s\right) - a(s)\right)^2 \mathrm{d}s. \end{aligned}$$

Furthermore,

$$\left(\mathsf{E}\left(B_{m/N}^H - X_{m/N}\right)^2\right)^{1/2} \leqslant \left(d_{mN}^1\right)^{1/2} + \left(d_{mN}^2\right)^{1/2}.$$

Taking maximum over $1 \leqslant m \leqslant N$, we obtain

$$\left(\max_{1 \leqslant m \leqslant N} \mathsf{E}\left(X_{m/N} - B_{m/N}^H\right)^2\right)^{1/2} \leqslant \max_{1 \leqslant m \leqslant N} \sqrt{d_{mN}^1} + \sqrt{F(\vec{a}_*, \mathbf{K}^{(N)})}.$$

Now apply Lemma A.5. Because of (10),

$$\max_{s \in [0,1]} \left|a(s)\right| = \max_{n=1,\dots,N} \left|\sqrt{N} a_{*n}\right| \leqslant \sqrt{N} \max_{1 \leqslant m,n \leqslant N} k_{mn}^{(N)}.$$

Thus, we have

$$\begin{aligned}\big(F(a)\big)^{1/2} &\leqslant \Big(\max_{1\leqslant n\leqslant N} \mathsf{E}\big(X_{n/N} - B^H_{n/N}\big)^2\Big)^{1/2} + \frac{1}{(2N)^H} + \frac{1}{\sqrt{2N}} \max_{1\leqslant n\leqslant N} |\sqrt{N} a_n| \\ &\leqslant \max_{1\leqslant m\leqslant N} \sqrt{x_{mN}} + \sqrt{F(\vec{a}_*, \mathbf{K}^{(N)})} + \frac{1}{(2N)^H} + \frac{1}{\sqrt{2}} \max_{1\leqslant m,n\leqslant N} k^{(N)}_{mn}.\end{aligned}$$

Now we recall that the minimum of the functional (3) is attained, $\vec{a}_*$ is the minimum point of $F(\cdot, \mathbf{K}^{(N)})$, hence $F(\vec{a}_*, \mathbf{K}^{(N)}) = \min_{\vec{a}\in\mathbb{R}^N} F(\vec{a}, \mathbf{K}^{(N)})$. This completes the proof of the first assertion.

(ii) Due to Lemma A.3, $\max_{1\leqslant m\leqslant N} d_{mN} \to 0$ as $N \to \infty$, and due to Lemma A.4 $\max_{1\leqslant m,n\leqslant N} k^{(N)}_{mn} \to 0$ as $N \to \infty$. Hence, the second assertion of Theorem 1 follows from the first one. More precisely, we rewrite inequality (9) as

$$\begin{aligned}&\Big(\min_{a\in L_2[0,1]} F(a)\Big)^{1/2} - \max_{1\leqslant m\leqslant N} \sqrt{x_{mN}} - \frac{1}{\sqrt{2}} \max_{1\leqslant m,n\leqslant N} k^{(N)}_{mn} - \frac{1}{(2N)^H} \\ &\quad\leqslant \Big(\min_{\vec{a}\in\mathbb{R}^N} F\big(\vec{a}, \mathbf{K}^{(N)}\big)\Big)^{1/2} \leqslant \Big(\min_{a\in L_2[0,1]} F(a)\Big)^{1/2}\end{aligned}$$

and apply squeeze theorem. □

Thus, instead of minimizing (3), we are searching for the minimum of the functional $F(\vec{a}, \mathbf{K}^{(N)})$, defined in (8), over $\vec{a} \in \mathbb{R}^N$. In the next section, we propose the numerical algorithm of finding $\min_{\vec{a}\in\mathbb{R}^N} F(\vec{a}, \mathbf{K}^{(N)})$.

3 Iterative minimization of the squared distance using alternating minimization method

In this section, the dimension $N \in \mathbb{N}$ is fixed. Let $\mathbf{K}$ be a fixed lower-triangle $N \times N$ matrix, whose elements are denoted as k_{mn} and rows are denoted as $\vec{k}_{m\bullet}$. The function $F(\vec{a}, \mathbf{K})$, $\vec{a} \in \mathbb{R}^N$, is defined by the following relation:

$$F(\vec{a}, \mathbf{K}) = \max_{1\leqslant m\leqslant N} \sum_{n=1}^{m} (a_n - k_{mn})^2, \quad \vec{a} \in \mathbb{R}^N.$$

Reduce the problem of finding the minimum of the convex function $F(\cdot, \mathbf{K})$ to the problem of finding the minimum of biconvex functional of two vectors. Denote

$$F(\vec{a}, \vec{b}, \mathbf{K}) = \max_{1\leqslant m\leqslant N} \Bigg(\sum_{n=1}^{m} (a_n - k_{mn})^2 + \sum_{n=m+1}^{N} (a_n - b_n)^2\Bigg).$$

Here, by convention, $\sum_{n=N+1}^{N} (a_n - b_n)^2 = 0$. For fixed $\vec{a}$, $F(\vec{a}, \vec{b}, \mathbf{K})$ is attained for $\vec{b} = \vec{a}$, i.e.,

$$F(\vec{a}, \mathbf{K}) = F(\vec{a}, \vec{a}, \mathbf{K}) = \min_{\vec{b}\in\mathbb{R}^N} F(\vec{a}, \vec{b}, \mathbf{K}). \tag{11}$$

Definition 1. Let Z be a nonempty bounded set in $\mathbb{R}^N$. Chebyshev center of the set Z is the point $\vec{a} \in \mathbb{R}^N$ where the minimum of $\sup_{\vec{z}\in Z} \|\vec{a} - \vec{z}\|$ is attained:

$$\operatorname{ChebyCenter}(Z) = \arg\min_{\vec{a}\in\mathbb{R}^N} \sup_{\vec{z}\in Z} \|\vec{a} - \vec{z}\|. \tag{12}$$

Note that the minimum in (12) is attained because the criterion function

$$\sqrt{Q(\vec{a})} = \sup_{\vec{z}\in Z} \|\vec{a} - \vec{z}\|$$

is continuous in $\mathbb{R}^N$ and tends to $+\infty$ as $\|a\| \to \infty$. It is attained at a unique point since the square criterion function

$$Q(\vec{a}) = \sup_{\vec{z}\in Z} \|\vec{a} - \vec{z}\|^2$$

is strongly convex,

$$\begin{aligned} Q\bigl(t\vec{a} + (1-t)\vec{b}\bigr) &\leqslant tQ(\vec{a}) + (1-t)Q(\vec{b}) - t(1-t)\|\vec{a} - \vec{b}\|^2 \\ &< tQ(\vec{a}) + (1-t)Q(\vec{b}) \quad \text{if } 0 < t < 1 \text{ and } \vec{a} \neq \vec{b}. \end{aligned}$$

For fixed $\vec{b}$, $\min F(\vec{a}, \vec{b}, \mathbf{K})$ is attained for $\vec{a}$ being the Chebyshev center of the N-point set $\{\vec{k}_1(\vec{b}), \vec{k}_2(\vec{b}), \dots, \vec{k}_N(\vec{b})\}$, where

$$\begin{aligned} &\vec{k}_m(\vec{b}) = (k_{m1}, \dots, k_{mm}, b_{m+1}, \dots, b_N)^\top, \quad 1 \leqslant m \leqslant N-1, \\ &\vec{k}_N(\vec{b}) = (k_{N1}, \dots, k_{NN})^\top = \vec{k}_{N\bullet}. \end{aligned} \tag{13}$$

Due to (11), the problem of minimization of $F(\vec{a}, \mathbf{K})$ is equivalent to the problem of minimization of $F(\vec{a}, \vec{b}, \mathbf{K})$:

$$\min_{\vec{a}\in\mathbb{R}^N} F(\vec{a}, \mathbf{K}) = \min_{\vec{a}\in\mathbb{R}^N} \min_{\vec{b}\in\mathbb{R}^N} F(\vec{a}, \vec{b}, \mathbf{K}).$$

If the minimum of $F(\vec{a}, \mathbf{K})$ is attained for $\vec{a} = \vec{a}_*$, then the minimum of $F(\vec{a}, \vec{b}, \mathbf{K})$ is attained for $\vec{a} = \vec{b} = a_*$. If the minimum of $F(\vec{a}, \vec{b}, \mathbf{K})$ is attained for $\vec{a} = \vec{a}_*$, $\vec{b} = \vec{b}_*$, then the minimum of $F(\vec{a}, \mathbf{K})$ is attained for $\vec{a} = \vec{a}_*$. In what follows, $\min F$ will mean a common minimum of functions $F(\vec{a}, \mathbf{K})$ and $F(\vec{a}, \vec{b}, \mathbf{K})$.

Let us summarize the properties of the functions $F(\vec{a}, \mathbf{K})$ and $F(\vec{a}, \vec{b}, \mathbf{K})$ in the following proposition.

Proposition 1. *Let $\mathbf{K}$ be a fixed $N \times N$ lower-triangular matrix.*

(i) *For a fixed $\vec{a} \in \mathbb{R}^N$, the minimum $\min_{\vec{b}\in\mathbb{R}^N} F(\vec{a}, \vec{b}, \mathbf{K}) = F(\vec{a}, \mathbf{K})$ is attained for $\vec{b} = \vec{a}$.*

(ii) *For a fixed $\vec{b} \in \mathbb{R}^N$, the minimum $\min_{\vec{a}\in\mathbb{R}^N} F(\vec{a}, \vec{b}, \mathbf{K})$ is attained for $\vec{a}$ being the Chebyshev center of the points $\vec{k}_1(\vec{b}), \vec{k}_2(\vec{b}), \dots, \vec{k}_N(\vec{b})$.*

(iii) *The minimal values of the functions* $F(\vec{a},\mathbf{K})$ *and* $F(\vec{a},\vec{b},\mathbf{K})$ *coincide. We denote them* $\min F$:

$$\min F := \min_{\vec{a}\in\mathbb{R}^N} F(\vec{a},\mathbf{K}) = \min_{\vec{a},\vec{b}\in\mathbb{R}^N} F(\vec{a},\vec{b},\mathbf{K}).$$

We find the minimum of $F(\vec{a},\vec{b},\mathbf{K})$ by alternating minimization. Let $\vec{a}^{(0)} \in \mathbb{R}^N$ be the initial approximation. The minimum of $F(\vec{a}^{(0)},\vec{b},\mathbf{K})$ is attained for $\vec{b} = \vec{a}^{(0)}$. Then minimize $F(\vec{a},\vec{a}^{(0)},\mathbf{K})$ with respect to $\vec{a}$:

$$\vec{a}^{(1)} = \arg\min_{\vec{a}\in\mathbb{R}^N} F\bigl(\vec{a},\vec{a}^{(0)},\mathbf{K}\bigr).$$

Again, the minimum of $F(\vec{a}^{(1)},\vec{b},\mathbf{K})$ is attained for $\vec{b} = \vec{a}^{(1)}$. Define the sequence $\{\vec{a}^{(i)}, i \geqslant 1\}$ iteratively

$$\vec{a}^{(i)} = \arg\min_{\vec{a}\in\mathbb{R}^N} F\bigl(\vec{a},\vec{a}^{(i-1)},\mathbf{K}\bigr), \quad i \geqslant 1. \tag{14}$$

Then we have the descend of the criterion function:

$$F\bigl(\vec{a}^{(0)},\vec{a}^{(0)},\mathbf{K}\bigr) \geqslant F\bigl(\vec{a}^{(1)},\vec{a}^{(0)},\mathbf{K}\bigr) \geqslant F\bigl(\vec{a}^{(1)},\vec{a}^{(1)},\mathbf{K}\bigr) \geqslant F\bigl(\vec{a}^{(2)},\vec{a}^{(0)},\mathbf{K}\bigr) \geqslant \cdots,$$

and since $F(\vec{a}^{(i)},\vec{a}^{(i)},\mathbf{K}) = F(\vec{a}^{(i)},\mathbf{K})$, we get that

$$F\bigl(\vec{a}^{(0)},\mathbf{K}\bigr) \geqslant F\bigl(\vec{a}^{(1)},\mathbf{K}\bigr) \geqslant F\bigl(\vec{a}^{(2)},\mathbf{K}\bigr) \geqslant \cdots.$$

The following theorem shows that the sequence $\{F(\vec{a}^{(i)},\mathbf{K}),\ i \geqslant 0\}$ converges to $\min F$.

Theorem 2. *Let*

$$\min_{n\leqslant m\leqslant N} k_{mn} \leqslant a_n^{(0)} \leqslant \max_{n\leqslant m\leqslant N} k_{mn} \quad \text{for all } n,\ 1 \leqslant n \leqslant N. \tag{15}$$

Then the sequence $\{\vec{a}^{(i)},\ i \geqslant 1\}$ *defined by* (14) *has the following properties*:

(i) $\lim_{i\to\infty} F(\vec{a}^{(i)},\mathbf{K}) = \min F$;
(ii) *If the minimal value of* $F(\cdot,\mathbf{K})$ *is attained at unique point* $\vec{a}_*$, *then* $\lim_{i\to\infty} \vec{a}^{(i)} = \vec{a}_*$.

Proof. (i) Denote a rectangle $\mathbb{H} \subset \mathbb{R}^N$

$$\begin{aligned}\mathbb{H} &= \Bigl\{\vec{a} \in \mathbb{R}^N \Bigm| \forall 1 \leqslant n \leqslant N\colon \min_{n\leqslant m\leqslant N} k_{mn} \leqslant a_n \leqslant \max_{n\leqslant m\leqslant N} k_{mn}\Bigr\} \\ &= \Bigl[\min_{1\leqslant m\leqslant N} k_{m1}, \max_{1\leqslant m\leqslant N} k_{m1}\Bigr] \times \Bigl[\min_{2\leqslant m\leqslant N} k_{m2}, \max_{2\leqslant m\leqslant N} k_{m2}\Bigr] \\ &\quad \times \cdots \times \Bigl[\min(k_{N-1,N-1}, k_{N,N-1}),\ \max(k_{N-1,N-1}, k_{N,N-1})\Bigr] \times \{k_{NN}\}.\end{aligned}$$

Denote the diagonal of the rectangle $\mathbb{H}$ by d, so that

$$d^2 = \sum_{n=1}^{N} \Bigl(\max_{n\leqslant m\leqslant N} k_{mn} - \min_{n\leqslant m\leqslant N} k_{mn} \Bigr)^2.$$

Condition (15) means that $\vec{a}^{(0)} \in \mathbb{H}$. By induction, $\vec{a}^{(i)} \in \mathbb{H}$ for all i. Indeed, if $a^{(i-1)} \in \mathbb{H}$, then $\vec{k}_m(a^{(i-1)}) \in \mathbb{H}$ for all $1 \leqslant m \leqslant N$, where $\vec{k}_m(\vec{b})$ is defined in (13). The Chebyshev center of a set lies in the convex hull of the set. Thus, $\vec{a}^{(i)}$ lies in the convex hull of $\vec{k}_m(\vec{a}^{(i-1)})$, $1 \leqslant m \leqslant N$, whence $\vec{a}^{(i)} \in \mathbb{H}$.

The minimal value $\min_{\vec{a}\in\mathbb{H}} F(\vec{a}, \mathbf{K})$ is attained because $F(\cdot, \mathbf{K})$ is a continuous function and $\mathbb{H}$ is nonempty closed bounded set. Denote

$$\vec{a}_* = \operatorname*{arg\,min}_{\vec{a}\in\mathbb{H}} F(\vec{a}, \mathbf{K}).$$

The minimum of $F(\cdot, \mathbf{K})$ on entire space $\mathbb{R}^N$ is attained at $\vec{a}_*$, see explanation in the proof of Theorem 1 (and relation $\vec{a}_* \in \mathbb{H}$ is analogous to inequality (10)):

$$\min F = F(\vec{a}_*, \mathbf{K}) = \min_{\vec{a}\in\mathbb{R}^N} F(\vec{a}, \mathbf{K}) = \min_{\vec{a},\vec{b}\in\mathbb{R}^N} F(\vec{a}, \vec{b}, \mathbf{K}).$$

Since $\vec{a}^{(i)} \in \mathbb{H}$ and $\vec{a}_* \in \mathbb{H}$, we have $\|\vec{a}^{(i)} - \vec{a}_*\| \leqslant d$ for all $i \geqslant 0$. For all $\vec{a} \in \mathbb{H}$ and $\vec{b} \in \mathbb{H}$, the inequality $F(\vec{a}, \mathbf{K}) \leqslant F(\vec{a}, \vec{b}, \mathbf{K}) \leqslant d^2$ holds true.

The case $d = 0$ is trivial: in this case the set $\mathbb{H}$ consists of the only one point, $\vec{a}^{(i)} = \vec{a}^{(0)}$ and $F(a^{(i)}, \mathbf{K}) = 0$ for all i; therefore, the statement of this theorem holds true. So, in the rest of the proof assume that $d > 0$.

Denote pseudo-norms on $\mathbb{R}^N$:

$$\|\vec{x}\|_m = \left(\sum_{n=1}^{m} x_n^2 \right)^{1/2}, \qquad \|\vec{x}\|_{\perp m} = \left(\sum_{n=m+1}^{N} x_n^2 \right)^{1/2}$$

(for $m = N$, denote $\|\vec{x}\|_N = \|\vec{x}\|$ and $\|\vec{x}\|_{\perp N} = 0$). With this notation,

$$\|\vec{x}\|_m^2 + \|\vec{x}\|_{\perp m}^2 = \|\vec{x}\|^2 \quad \text{for all } 1 \leqslant m \leqslant N \text{ and } \vec{x} \in \mathbb{R}^N,$$
$$F(\vec{a}, \mathbf{K}) = \max_{1\leqslant m\leqslant N} \|\vec{a} - \vec{k}_{m\bullet}\|_m^2,$$
$$F(\vec{a}, \vec{b}, \mathbf{K}) = \max_{1\leqslant m\leqslant N} \bigl(\|\vec{a} - \vec{k}_{m\bullet}\|_m^2 + \|\vec{a} - \vec{b}\|_{\perp m}^2 \bigr).$$

In what follows, we are going to use inequalities

$$\|\vec{a} - \vec{k}_{m\bullet}\|_m^2 \leqslant F(\vec{a}, \mathbf{K}) \quad \text{for all } \vec{a} \in \mathbb{R}^N,$$
$$\|\vec{a} - \vec{b}\|_{\perp m}^2 \leqslant F(\vec{a}, \vec{b}, \mathbf{K}) \leqslant d^2 \quad \text{for all } \vec{a} \in \mathbb{H} \text{ and } \vec{b} \in \mathbb{H}.$$

In what follows, we will be constructing an upper bound for $F(\vec{a}^{(i)}, \mathbf{K})$. It will be inequality (19).

Denote for fixed i

$$\alpha_i = \frac{(\sqrt{F(\vec{a}^{(i-1)}, \mathbf{K})} - \sqrt{\min F})\sqrt{F(\vec{a}^{(i-1)}, \mathbf{K})}}{(\sqrt{F(\vec{a}^{(i-1)}, \mathbf{K})} - \sqrt{\min F})^2 + d^2}.$$

Then

$$1 - \alpha_i = \frac{d^2 - (\sqrt{F(\vec{a}^{(i-1)}, \mathbf{K})} - \sqrt{\min F})\sqrt{\min F}}{(\sqrt{F(\vec{a}^{(i-1)}, \mathbf{K})} - \sqrt{\min F})^2 + d^2}.$$

Taking into account the relations $0 \leqslant \min F \leqslant F(\vec{a}^{(i-1)}, \mathbf{K}) \leqslant d^2$, we obtain the inequality $0 \leqslant \alpha_i \leqslant 1/2$.

Next auxiliary result also will be applied to get (19). Namely, we construct the upper bound for $F((1-\alpha_i)\vec{a}^{(i-1)} + \alpha_i \vec{a}_*, a^{(i-1)}, \mathbf{K})$. For every $1 \leqslant m \leqslant N$,

$$\begin{aligned}
&\big\|(1-\alpha_i)\vec{a}^{(i-1)} + \alpha_i \vec{a}_* - \vec{k}_{m\bullet}\big\|_m \\
&\quad \leqslant (1-\alpha_i)\big\|\vec{a}^{(i-1)} - \vec{k}_{m\bullet}\big\|_m + \alpha_i \|\vec{a}_* - \vec{k}_{m\bullet}\|_m \\
&\quad \leqslant (1-\alpha_i)\sqrt{F(\vec{a}^{(i-1)}, \mathbf{K})} + \alpha_i \sqrt{\min F} \\
&\quad = \frac{d^2\sqrt{F(\vec{a}^{(i-1)}, \mathbf{K})}}{(\sqrt{F(\vec{a}^{(i-1)}, \mathbf{K})} - \sqrt{\min F})^2 + d^2},
\end{aligned}$$

and

$$\begin{aligned}
&\big\|(1-\alpha_i)\vec{a}^{(i-1)} + \alpha_i \vec{a}_* - \vec{a}^{(i-1)}\big\|_{\perp m} \\
&\quad = \alpha_i \big\|\vec{a}_* - \vec{a}^{(i-1)}\big\|_{\perp m} \leqslant \alpha_i d \\
&\quad = \frac{d(\sqrt{F(\vec{a}^{(i-1)}, \mathbf{K})} - \sqrt{\min F})\sqrt{F(\vec{a}^{(i-1)}, \mathbf{K})}}{(\sqrt{F(\vec{a}^{(i-1)}, \mathbf{K})} - \sqrt{\min F})^2 + d^2}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\big\|(1-\alpha_i)\vec{a}^{(i-1)} + \alpha_i \vec{a}_* - \vec{k}_{m\bullet}\big\|_m^2 + \big\|(1-\alpha_i)\vec{a}^{(i-1)} + \alpha_i \vec{a}_* - \vec{a}^{(i-1)}\big\|_{\perp m}^2 \\
&\quad \leqslant \frac{d^4 F(\vec{a}^{(i-1)}, \mathbf{K}) + d^2(\sqrt{F(\vec{a}^{(i-1)}, \mathbf{K})} - \sqrt{\min F})^2 F(\vec{a}^{(i-1)}, \mathbf{K})}{((\sqrt{F(\vec{a}^{(i-1)}, \mathbf{K})} - \sqrt{\min F})^2 + d^2)^2} \\
&\quad = \frac{d^2 F(\vec{a}^{(i-1)}, \mathbf{K})}{(\sqrt{F(\vec{a}^{(i-1)}, \mathbf{K})} - \sqrt{\min F})^2 + d^2}.
\end{aligned} \tag{16}$$

Take the maximum (over $m = 1, \dots, N$) in the left-hand side of (16) and obtain

$$F\big((1-\alpha_i)\vec{a}^{(i-1)} + \alpha_i \vec{a}_*,\, \vec{a}^{(i-1)},\, \mathbf{K}\big) \leqslant \frac{d^2 F(\vec{a}^{(i-1)}, \mathbf{K})}{(\sqrt{F(\vec{a}^{(i-1)}, \mathbf{K})} - \sqrt{\min F})^2 + d^2}.$$

Now we continue with upper bound for $F(\vec{a}^{(i)}, \mathbf{K})$. With Proposition 1, we get an inequality

$$\begin{aligned} F\bigl(\vec{a}^{(i)}, \mathbf{K}\bigr) &\leqslant F\bigl(\vec{a}^{(i)}, \vec{a}^{(i-1)}, \mathbf{K}\bigr) \leqslant F\bigl((1-\alpha_i)\vec{a}^{(i-1)} + \alpha_i \vec{a}_*,\, a^{(i-1)},\, \mathbf{K}\bigr) \\ &\leqslant \frac{d^2 F(\vec{a}^{(i-1)}, \mathbf{K})}{(\sqrt{F(\vec{a}^{(i-1)}, \mathbf{K})} - \sqrt{\min F})^2 + d^2} \leqslant F\bigl(\vec{a}^{(i-1)}, \mathbf{K}\bigr). \end{aligned} \tag{17}$$

The sequence $\{F(\vec{a}^{(i)}, \mathbf{K}),\ i \geqslant 0\}$ is decreasing and bounded, more exactly, $0 \leqslant \min F \leqslant F(\vec{a}^{(i)}, \mathbf{K}) \leqslant F(\vec{a}^{(0)}, \mathbf{K})$. Hence, it is convergent,

$$0 \leqslant \min F \leqslant \lim_{i\to\infty} F\bigl(\vec{a}^{(i)}, \mathbf{K}\bigr) < \infty. \tag{18}$$

From inequality (17) we obtain the desired upper bound

$$F\bigl(\vec{a}^{(i)}, \mathbf{K}\bigr) \leqslant \frac{d^2 F(\vec{a}^{(i-1)}, \mathbf{K})}{(\sqrt{F(\vec{a}^{(i-1)}, \mathbf{K})} - \sqrt{\min F})^2 + d^2}. \tag{19}$$

In (19), take the limit as $i \to \infty$:

$$\lim_{i\to\infty} F\bigl(\vec{a}^{(i)}, \mathbf{K}\bigr) \leqslant \frac{d^2 \lim_{i\to\infty} F(\vec{a}^{(i)}, \mathbf{K})}{(\sqrt{\lim_{i\to\infty} F(\vec{a}^{(i)}, \mathbf{K})} - \sqrt{\min F})^2 + d^2}.$$

Therefore,

$$\Bigl(\sqrt{\lim_{i\to\infty} F(\vec{a}^{(i)}, \mathbf{K})} - \sqrt{\min F}\Bigr)^2 \lim_{i\to\infty} F\bigl(\vec{a}^{(i)}, \mathbf{K}\bigr) \leqslant 0,$$

whence either $\sqrt{\lim_{i\to\infty} F(\vec{a}^{(i)}, \mathbf{K})} - \sqrt{\min F} = 0$ or $\lim_{i\to\infty} F(\vec{a}^{(i)}, \mathbf{K}) \leqslant 0$. In either case, $\lim_{i\to\infty} F(\vec{a}^{(i)}, \mathbf{K}) = \min F$ (here we use inequality (18) in the latter case $\lim_{i\to\infty} F(\vec{a}^{(i)}, \mathbf{K}) \leqslant 0$). The first statement of the theorem is proved.

(ii) We apply Lemma A.1. We take the restriction of $F(\cdot, \mathbf{K})$ onto the set $\mathbb{H}$ for the function f. From the proof of the first part of Theorem 2, we take the following: $\vec{a}_* \in \mathbb{H}$ (at least one point of minimum of $F(\cdot, \mathbf{K})$ belongs to $\mathbb{H}$; since the point of minimum is unique, it must belong to $\mathbb{H}$), all elements of the sequence $\{\vec{a}^{(i)},\ i \geqslant 1\}$ belong to $\mathbb{H}$, and $\lim_{i\to\infty} F(\vec{a}^{(i)}, \mathbf{K}) = \min F = \min_{\vec{a}\in\mathbb{H}} F(\vec{a}, \mathbf{K})$. Hence, the convergence $\vec{a}^{(i)} \to a_*$ follows. □

Now we perform the implementation of computation of $\min F$ and the minimizing vector. We take the bottom row of the matrix K as the initial approximation, that is $\vec{a}^{(0)} = \vec{k}_{N\bullet}$. Then we iteratively perform minimization in (14) using Chebyshev center algorithm:

$$\begin{aligned} \vec{a}^{(i)} &= \operatorname*{arg\,min}_{\vec{a}\in\mathbb{R}^N} F\bigl(\vec{a}, \vec{a}^{(i-1)}, \mathbf{K}\bigr) \\ &= \operatorname{ChebyCenter}\bigl(\vec{k}_1\bigl(\vec{a}^{(i-1)}\bigr), \vec{k}_2\bigl(\vec{a}^{(i-1)}\bigr), \dots, \vec{k}_N\bigl(\vec{a}^{(i-1)}\bigr)\bigr). \end{aligned}$$

For evaluating Chebyshev center, we adapted an algorithm presented in [6, 10].

If we are evaluating $\min F$, we stop iterations when $U_{i-1} \leqslant p_1$, where

$$U_{i-1} = \begin{cases} 2d\sqrt{\frac{F(\vec{a}^{(i-1)},\mathbf{K})(F(\vec{a}^{(i-1)},\mathbf{K})-F(\vec{a}^{(i)},\mathbf{K}))}{F(\vec{a}^{(i)},\mathbf{K})}} - \frac{d^2(F(\vec{a}^{(i-1)},\mathbf{K})-F(\vec{a}^{(i)},\mathbf{K}))}{F(\vec{a}^{(i)},\mathbf{K})} \\ \quad \text{if } F(\vec{a}^{(i-1)},\mathbf{K}) \geqslant \frac{d^2(F(\vec{a}^{(i-1)},\mathbf{K})-F(\vec{a}^{(i)},\mathbf{K}))}{F(\vec{a}^{(i)},\mathbf{K})}, \\ \frac{d^2(F(\vec{a}^{(i-1)},\mathbf{K})-F(\vec{a}^{(i)},\mathbf{K}))}{F(\vec{a}^{(i)},\mathbf{K})} \\ \quad \text{if } F(\vec{a}^{(i-1)},\mathbf{K}) \leqslant \frac{d^2(F(\vec{a}^{(i-1)},\mathbf{K})-F(\vec{a}^{(i)},\mathbf{K}))}{F(\vec{a}^{(i)},\mathbf{K})} \end{cases}$$

is the upper bound for $F(a^{(i-1)}) - \min F$. If we are evaluating the minimizing vector $\vec{a}_*$, we stop iterations when both inequalities $U_{i-1} \leqslant p_1$ and $\|\vec{a}^{(i)} - \vec{a}^{(i-1)}\| \leqslant p_2$ hold true. Here $p_1 > 0$ and $p_2 > 0$ are thresholds. Finally, we take $\vec{a}^{(i)}$ as an approximation of the point of minimum.

4 Computation of the minimizing function

Figure 1 displays the graph of the approximation of the optimal function $a(t)$ and the square-distance

$$f_a(t) = \int_0^t \bigl(a(s) - z(t,s)\bigr)^2 \,\mathrm{d}s = \mathsf{E}\bigl(X_t - B_t^H\bigr)^2,$$

for $H = 1/3$ and for $H = 2/3$.

We also display the graph of approximate density of the random variable ξ in the representation

$$a(s) = \mathsf{E}\bigl[z(\xi, s),\, \xi \geqslant s\bigr],$$

see [17, Thm. 4]. This density is obtained as follows.

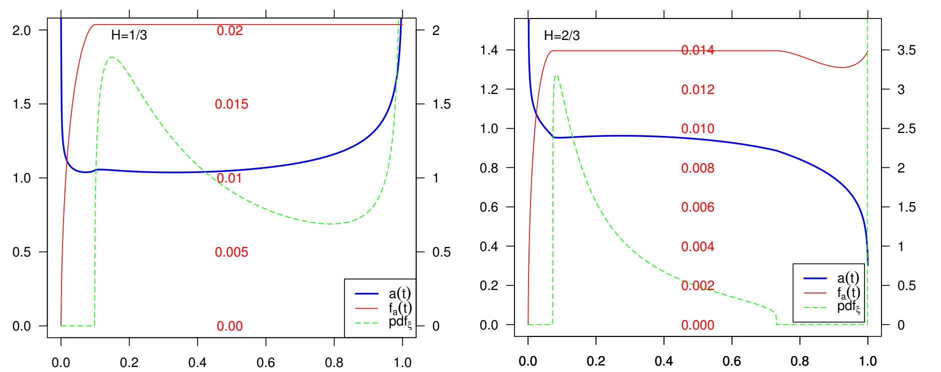

Figure 1. The minimizing function $a(t)$, the square distance $f_a(t)$ and the probability density of random variable ξ for $H = 1/3$ and for $H = 2/3$.

The Chebyshev center algorithm calculates the weights w_m in the representation $\vec{a}^{(i)} = \sum_{m=1}^{N} w_m \vec{b}_m(\vec{a}^{(i-1)})$. These weights determine the discrete distribution, which is concentrated on the finite subset $\{m/N,\ 1 \leqslant m \leqslant N\} \subset [0,1]$. This distribution on the last iteration of the iterative minimization algorithm is taken as the approximation of the distribution of ξ.

According to our discretized minimization, if $H = 1/3$, then the random variable ξ is concentrated in interval $[0.1, 1]$. If $H = 2/3$, then ξ is concentrated in $[0.074, 0.731] \cup \{1\}$ with $\mathsf{P}(\xi \in [0.074, 0.731]) = 0.6365$ and $\mathsf{P}(\xi = 1) = 0.3635$.

Appendix

A.1 Calculus: Convergence lemmas

The next two results can be proved by methods of standard calculus, therefore their proofs are omitted.

Lemma A.1. *Let* $\mathbb{H}$ *be a nonempty compact set* (in $\mathbb{R}^N$)*, and* f *be a continuous function* $\mathbb{H} \to \mathbb{R}$. *Assume that the function* f *attains its minimum at the unique point* x_**, that is the equality* $f(x) = \min_{y\in\mathbb{H}} f(y)$ *holds true if and only if* $x = x_*$. *If* $\{x_n,\ n \geqslant 1\}$ *is a sequence of elements of* $\mathbb{H}$ *and* $\lim_{n\to\infty} f(x_n) = \min_{x\in\mathbb{H}} f(x)$*, then* $\{x_n,\ n \geqslant 1\}$ *converges to* x_*.

Lemma A.2. *Let* $b = \{b_m,\ m \geqslant 1\}$ *be a sequence of real numbers and* $\{c_{Nm},\ N \geqslant m \geqslant 1\}$ *be a triangular array. If* $\lim_{m\to\infty} b_m = 0$*,* $\sup_{N\geqslant m} |c_{Nm}| < \infty$ *and, for all* $m \in \mathbb{N}$*,* $\lim_{N\to\infty} c_{Nm} = 0$*, then*

$$\lim_{N\to\infty} \max_{1\leqslant m\leqslant N} c_{Nm} b_m = 0.$$

A.2 Piecewise-constant approximation of the Molchan–Golosov kernel

Recall the definitions from Section 2. Let $N \in \mathbb{N}$. Molchan–Golosov kernel $z(t,s)$, nonnegative numbers $k_{mn}^{(N)}$ and a function $z^{(N)}(t,s)$ are defined in (2), (5) and (6), respectively. Obviously, Molchan–Golosov kernel has the following homogeneity property:

$$z(kt, ks) = k^{H-1/2} z(t,s), \quad k > 0,\ 0 < s < t. \tag{A.1}$$

Relation (A.1) can be directly verified using (2).

Lemma A.3. *For* d_{mN} *defined in* (7)*, we have the convergence*

$$\lim_{N\to\infty} \max_{1\leqslant m\leqslant N} d_{mN} = 0.$$

Proof. Since the function $z(1,\cdot) \in L_2[0,1]$, we have that

$$\lim_{m\to\infty} d_{mm} = \lim_{m\to\infty} \int_0^1 \bigl(z^{(m)}(1,s) - z(1,s)\bigr)^2 \,\mathrm{d}s = 0.$$

For the short proof, see [12, Lemma A.1].

Due to homogeneity (A.1) of $z(t,s)$, we have

$$z\left(\frac{m}{N}, s\right) = \left(\frac{m}{N}\right)^{H-1/2} z\left(1, \frac{Ns}{m}\right) \quad \text{if } 0 < s < \frac{m}{N}, \tag{A.2}$$

$$k_{mn}^{(N)} = \left(\frac{m}{N}\right)^{H} k_{mn}^{(m)} \quad \text{if } 1 \leqslant n \leqslant m \leqslant N, \tag{A.3}$$

$$z^{(N)}\left(\frac{m}{N}, s\right) = \left(\frac{m}{N}\right)^{H-1/2} z^{(m)}\left(1, \frac{sN}{m}\right) \quad \text{if } 0 < s < \frac{m}{N}. \tag{A.4}$$

We also have $z(m/N, s) = z^{(N)}(m/N, s) = 0$ if $s > m/N$, and $k_{mn}^{(N)} = 0$ if $1 \leqslant m < n \leqslant N$. From (A.2) and (A.4) it follows that

$$d_{mN} = \left(\frac{m}{N}\right)^{2H} d_{mm}.$$

Applying Lemma A.2 with $b_m = d_{mm}$ and $c_{Nm} = (m/N)^{2H}$, we get the proof. □

Lemma A.4. *For $k_{mn}^{(N)}$ defined in* (5)*, we have the convergence*

$$\lim_{N\to\infty} \max_{1\leqslant m,n\leqslant N} k_{mn}^{(N)} = 0.$$

Proof. By Cauchy–Schwarz inequality,

$$\begin{aligned}\left(k_{Nn}^{(N)}\right)^2 &= N\left(\int\limits_{(n-1)/N}^{n/N} z(1,s)\mathrm{d}s\right)^2 \leqslant N \int\limits_{(n-1)/N}^{n/N} \mathrm{d}u \int\limits_{(n-1)/N}^{n/N} z(1,s)^2\mathrm{d}s \\ &= \int\limits_{(n-1)/N}^{n/N} z(1,s)^2\,\mathrm{d}s.\end{aligned}$$

As we know, $z(1,\cdot) \in L_2[0,1]$. Therefore, $\max_{1\leqslant n\leqslant N} (k_{Nn}^{(N)})^2 \to 0$ as $N \to \infty$. Due to (A.3), we have

$$\max_{1\leqslant m,n\leqslant N} k_{mn}^{(N)} = \max_{1\leqslant m\leqslant N}\left(\left(\frac{m}{N}\right)^{H} \max_{1\leqslant n\leqslant m} k_{mn}^{(m)}\right).$$

Applying Lemma A.2 with $b_m = \max_{1\leqslant n\leqslant m} k_{mn}^{(m)}$ and $c_{Nm} = (m/N)^H$, we complete the proof. □

Lemma A.5. *Let $a : [0,1] \to \mathbb{R}$ be a piecewise-constant nonrandom function,*

$$a(s) = \text{const} \quad \textit{for all } s \in \left(\frac{n-1}{N}, \frac{n}{N}\right),\ 1 \leqslant n \leqslant N$$

(*we do not make any assumption about behaviour of* $a(s)$ *at points* n/N *except that* $a(n/N)$ *are real, finite nonrandom numbers). Consider the fBm* $B_t^H = \int_0^t z(t,s)\,\mathrm{d}W_s$ *and a Gaussian martingale* $B_t^H = \int_0^t a(s)\,\mathrm{d}W_s$*, which are adapted w.r.t. the same filtration* $\mathcal{F}^W$*. Then*

$$\Bigl(\max_{t\in[0,1]} \mathsf{E}\bigl(X_t - B_t^H\bigr)^2\Bigr)^{1/2} \leqslant \Bigl(\max_{n=1,\dots,N} \mathsf{E}(X_{n/N} - B_{n/N}^H)^2\Bigr)^{1/2} + \frac{1}{(2N)^H} + \frac{1}{\sqrt{2N}} \max_{s\in[0,1]} \bigl|a(s)\bigr|. \tag{A.5}$$

Proof. Let $t_1 \in [0,1]$. Then there exists an integer n_1, $0 \leqslant n_1 \leqslant N$, such that $|t_1 - n_1/N| \leqslant 1/(2N)$.

We have the following bounds:

$$\mathsf{E}\bigl(B_{t_1}^H - B_{n_1/N}^H\bigr)^2 = \left|t_1 - \frac{n_1}{N}\right|^{2H} \leqslant \frac{1}{(2N)^{2H}},$$

$$\begin{aligned}\mathsf{E}(X_{t_1} - X_{n_1/N})^2 &= \mathsf{E}\Biggl(\int_{t_1}^{n_1/N} a(s)\,\mathrm{d}W_s\Biggr)^2 = \Biggl|\int_{t_1}^{n_1/N} a(s)^2\,\mathrm{d}s\Biggr| \\ &= \left|t_1 - \frac{n_1}{N}\right| \max_{s\in[0,1]} a(s)^2 \leqslant \frac{1}{2N} \max_{s\in[0,1]} a(s)^2.\end{aligned}$$

Obviously,

$$\begin{aligned}&\bigl(\mathsf{E}\bigl(X_{t_1} - B_{t_1}^H\bigr)^2\bigr)^{1/2} \\ &\quad\leqslant \bigl(\mathsf{E}\bigl(X_{t_1} - X_{n_1/N}\bigr)^2\bigr)^{1/2} + \bigl(\mathsf{E}\bigl(X_{n_1/N} - B_{n_1/N}^H\bigr)^2\bigr)^{1/2} \\ &\qquad+ \bigl(\mathsf{E}\bigl(B_{n_1/N}^H - B_{t_1}^H\bigr)^2\bigr)^{1/2} \\ &\quad\leqslant \frac{1}{\sqrt{2N}} \max_{s\in[0,1]} \bigl|a(s)\bigr| + \bigl(\mathsf{E}\bigl(X_{n_1/N} - B_{n_1/N}^H\bigr)^2\bigr)^{1/2} + \frac{1}{(2N)^H}.\end{aligned} \tag{A.6}$$

Since the stochastic processes X_t and B_t^H are mean-square continuous, the maximum $\sup_{t_1\in[0,1]}(\mathsf{E}(X_{t_1} - B_{t_1}^H)^2)^{1/2}$ is attained. Maximizing (A.6) over $t_1 \in [0,1]$, we obtain the inequality

$$\begin{aligned}&\max_{t_1\in[0,1]} \bigl(\mathsf{E}\bigl(X_{t_1} - B_{t_1}^H\bigr)^2\bigr)^{1/2} \\ &\quad\leqslant \frac{1}{\sqrt{2N}} \max_{s\in[0,1]} \bigl|a(s)\bigr| + \max_{n=0,\dots,N} \bigl(\mathsf{E}\bigl(X_{n/N} - B_{n/N}^H\bigr)^2\bigr)^{1/2} + \frac{1}{(2N)^H},\end{aligned}$$

which is equivalent to (A.5). □

References

1. O. Banna, Yu. Mishura, Approximation of fractional Brownian motion with associated Hurst index separated from 1 by stochastic integrals of linear power functions, *Theory Stoch. Process.*, **14**(3–4):1–16, 2008.
2. O.L. Banna, Yu.S. Mishura, The simpliest martingales of the best approximation of fractional Brownian motion, *Visn., Mat. Mekh., Kyïv. Univ. Im. Tarasa Shevchenka*, **19**:38–43, 2008.
3. O.L. Banna, Yu.S. Mishura, A bound for the distance between fractional Brownian motion and the space of Gaussian martingales on an interval, *Theory Probab. Math. Stat.*, **83**:13–25, 2011, `https://doi.org/10.1090/S0094-9000-2012-00838-7`.
4. O.L. Banna, Yu.S. Mishura, S.V. Shklyar, Approximation of a Wiener process by integrals with respect to the fractional Brownian motion of power functions of a given exponent, *Theory Probab. Math. Stat.*, **90**:13–22, 2015, `https://doi.org/10.1090/tpms/946`.
5. A. Beck, On the convergence of alternating minimization for convex programming with applications to iteratively reweighted least squares and decomposition schemes, *SIAM J. Optim.*, **25**(1):185–209, 2015.
6. N.D. Botkin, V.L. Turova-Botkina, An algorithm for finding the Chebyshev center of a convex polyhedron, in J. Henry, J.-P. Yvon (Eds.), *System Modelling and Optimization. Proceedings of the 16th IFIP-TC7 Conference, Compiègne, France – July 5–9, 1993*, Lect. Notes Control Inf. Sci. – Proc., Vol. 197, Springer, Berlin, Heidelberg, 1994, pp. 833–844.
7. C.-I Chang, *Hyperspectral Imaging: Techniques for Spectral Detection and Classification*, Kluwer, New York, 2003, `https://doi.org/10.1007/978-1-4419-9170-6`.
8. V. Doroshenko, Yu. Mishura, O. Banna, The distance between fractional Brownian motion and the subspace of martingales with "similar" kernels, *Theory Probab. Math. Stat.*, **87**:41–49, 2013, `https://doi.org/10.1090/S0094-9000-2014-00903-5`.
9. T. Dung, N. Semimartingale approximation of fractional Brownian motion and its applications, *Comput. Math. Appl.*, **61**(7):1844–1854, 2011.
10. K. Fischer, B. Gärtner, M. Kutz, Fast smallest-enclosing-ball computation in high dimensions, in G. Di Battista, U. Zwick (Eds.), *Algorithms—ESA 2003*, Lect. Notes Comput. Sci., Vol. 2832, pp. 630–641, Springer, Berlin, 2003, `https://doi.org/10.1007/978-3-540-39658-1_57`.
11. R.S. Liptser, A.N. Shiryaev, *Statistics of Random Processes. I: General theory*, Stoch. Model. Appl. Probab., Vol. 5, Springer, Berlin, 2001.
12. Yu. Mishura, K. Ralchenko, S. Shklyar, Maximum likelihood estimation for Gaussian process with nonlinear drift, *Nonlinear Anal. Model. Control*, **23**(1):120–140, 2018, `https://doi.org/10.15388/NA.2018.1.9`.
13. Yu.S. Mishura, O.L. Banna, Approximation of fractional Brownian motion by Wiener integrals, *Theory Probab. Math. Stat.*, **79**:107–116, 2009, `https://doi.org/10.1090/S0094-9000-09-00773-X`.
14. G.M. Molchan, Linear problems for a fractional Brownian motion: Group approach, *Theory Probab. Appl.*, **47**(1):69–78, 2003, `https://doi.org/10.1137/S0040585X97979445`.
15. G.M. Molchan, Ju.I. Golosov, Gaussian stationary processes with asymptotic power spectrum, *Sov. Math., Dokl.*, **10**(1):134–137, 1969.

16. I. Norros, E. Valkeila, J. Virtamo, An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions, *Bernoulli*, **5**(4):571–587, 1999.

17. S. Shklyar, G. Shevchenko, Yu. Mishura, V. Doroshenko, O. Banna, Approximation of fractional Brownian motion by martingales, *Methodol. Comput. Appl. Probab.*, **16**(3):539–560, 2014.

18. P. Wolfe, Finding the nearest point in a polytope, *Math. Program.*, **11**:128–149, 1976.