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Coupled fractals in complete metric spaces*

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Abstract. The aim of this paper is to present fixed set theorems, collage type and anticollage type results for single-valued operators $T : X \times X \to X$ in the framework of a complete metric space X. Based on the coupled fixed point theory, existence of fixed sets, collage type and anticollage type results for iterated function systems are also presented. The results are closely related to self-similar sets theory and the mathematics of fractals. Several examples of coupled fractals illustrate our results.

Keywords: set-to-set operator, fixed point, self-similar set, coupled fixed point, collage theorem.

1 Introduction

The notion of coupled fixed point appeared for the first time in some papers of Amann [1] and Opoitsev [13], while a large development of the field started after the works of Guo and Lakshmikantham [6] and Bhaskar and Lakshmikantham [4]. If (X, d) is a metric space and $T : X \times X \to X$ is an operator, then, by definition, a coupled fixed point for T is a pair $(x, y) \in X \times X$ satisfying

$$x = T(x, y), \qquad y = T(y, x). \tag{1}$$

There are many applications of the coupled fixed point theorems for solving different problems related to systems of integral and differential equations, see [2, 4, 6, 9, 22].

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Generalizations of this problem to tripled or N-upled fixed point results were also given in the recent literature, see, for example, [3,5,14,17,20,21]. More general contexts (coupled coincidence problems, coupled fixed point problems in cone metric spaces etc.) were also considered, see, for example, [18, 19].

The aim of this paper is to present some fixed set and coupled fixed set theorems for single-valued operators and iterated function systems consisting of a finite number of self operators in complete metric spaces. Some connections with the mathematics of self-similar sets are also discussed.

More precisely, if $T : X \times X \to X$ is a given operator, we are interested under which conditions there exists a (unique) pair $(A, B) \in P_{\mathrm{cp}}(X) \times P_{\mathrm{cp}}(X)$ such that

$$A = T(A, B), \qquad B = T(B, A), \tag{2}$$

where $T(A, B) := \{T(a, b) \mid a \in A,\ b \in B\}$.

A pair $(A, B) \in P_{\mathrm{cp}}(X) \times P_{\mathrm{cp}}(X)$ with the above property will be called a coupled fixed set or a pair of coupled self-similar sets. By this approach new examples of self-similar constructions are given, and the classical theory of self-similar sets and fractals is extended to operators working on a Cartesian product of metric spaces.

2 Preliminaries

In this short section, we will present some auxiliary notions. Let (X, d) be a metric space, and $P(X)$ be the family of all nonempty subsets of X. We also denote by $P_{\mathrm{cp}}(X)$ the family of all nonempty compact subsets of X.

If $f : X \to X$, then we denote $\mathrm{Fix}(f) := \{x \in X\colon\ x = f(x)\}$ the fixed point set for f. Moreover, the fractal operator generated by a continuous mapping $f : X \to X$ is denoted by $\hat{f}$ and is defined $\hat{f} : P_{\mathrm{cp}}(X) \to P_{\mathrm{cp}}(X)$, $\hat{f}(Y) := \bigcup_{y \in Y} f(y) = f(Y)$ for all $Y \in P_{\mathrm{cp}}(X)$.

When $X := \mathbb{R}^n$, a fixed point of $\hat{f}$ (i.e., a fixed set for f) is called a self-similar set for f.

The following functionals are well known in the field of set-valued analysis:

- the gap functional generated by d

$$D_d(A, B) := \inf\bigl\{d(a, b) \bigm| a \in A,\ b \in B\bigr\}, \quad A, B \in P(X);$$

- the Hausdorff–Pompeiu functional generated by d

$$H_d(A, B) = \max\Bigl\{\sup_{a \in A} D_d(a, B), \sup_{b \in B} D_d(b, A)\Bigr\}, \quad A, B \in P(X).$$

3 Coupled self-similar sets for contractions type operators

We recall first some notions and a fixed point result for contractions type operators in vector-valued metric spaces.

We denote by $M_{mm}(\mathbb{R}_+)$ the set of all $m \times m$ matrices with positive elements, by I the identity $m \times m$ matrix, and by O_m the null $m \times m$ matrix. If $x, y \in \mathbb{R}^m$, $x = (x_1, \dots, x_m)$

and $y = (y_1, \dots, y_m)$, then, by definition, we write $x \leqslant y$ if and only if $x_i \leqslant y_i$ for $i \in \{1, 2, \dots, m\}$. Throughout this paper, we will make an identification between row and column vectors in $\mathbb{R}^m$.

Let us recall first some important preliminary concepts and results. Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}^m_+$ is called a vector-valued metric on X if all the axioms of the classical metric are satisfied with respect to the component-wise partial order. Moreover, a nonempty set X endowed with a vector-valued metric d is called a generalized metric space in the sense of Perov (in short, a generalized metric space), and it will be denoted by (X, d). The usual notions of analysis (such as convergent sequence, Cauchy sequence, completeness, open subset, closed set, open and closed ball, etc.) are defined in a similar to the case of metric spaces.

Moreover, the vector-valued Hausdorff–Pompeiu metric on $P_{\mathrm{cp}}(X)$ generated by a vector-valued metric $\bar{d} : X \times X \to \mathbb{R}^m_+$ given by $\bar{d}(x, y) := (d_1(x, y), \dots, d_m(x, y))$ will be denoted by $H_{\bar{d}}$ and represented as $H_{\bar{d}}(A, B) := (H_{d_1}(A, B), \dots, H_{d_m}(A, B))$. We notice that $(P_{\mathrm{cp}}(X), H_{\bar{d}})$ is a generalized metric space in the sense of Perov, i.e., $H_{\bar{d}}$ satisfies all the axioms of the vector-valued metric on $P_{\mathrm{cp}}(X)$. We point out that $(P_{\mathrm{cp}}(X), H_{\bar{d}})$ is complete if the vector-valued metric $\bar{d}$ is complete. We also mention that the generalized metric space in the sense of Perov is a particular case of the so-called cone metric spaces (or K-metric space), see [26].

A square matrix A of real positive numbers is said to be convergent to zero if and only if all the eigenvalues of A are in the open unit disc (see, for example, [16]).

A classical result in matrix analysis is the following theorem (see, for example, [16]).

Theorem 1. *Let $A \in M_{mm}(\mathbb{R}_+)$. The following assertions are equivalent*:

(i) *A is convergent towards zero*;

(ii) *$A^n \to O_m$ as $n \to \infty$*;

(iii) *the matrix $(I - A)$ is nonsingular, and*
$$(I - A)^{-1} = I + A + \cdots + A^n + \cdots ; \tag{3}$$

(iv) *the matrix $(I - A)$ is nonsingular, and $(I - A)^{-1}$ has nonnegative elements.*

We recall now Perov's fixed point theorem [15].

Theorem 2 [Perov theorem]. *Let (X, d) be a complete generalized metric space, and let $f : X \to X$ be a contraction with matrix A, i.e., $A \in M_{mm}(\mathbb{R}_+)$ converges towards zero, and*
$$d\bigl(f(x), f(y)\bigr) \leqslant A d(x, y) \quad \forall x, y \in X.$$

Then:

(i) $\operatorname{Fix} f = \{x^*\}$;

(ii) *the sequence of successive approximations $(x_n)_{n \in \mathbb{N}}$, $x_n := f^n(x_0)$ is convergent in X to x^* for all $x_0 \in X$*;

(iii) *one has the following estimation*:
$$d(x_n, x^*) \leqslant A^n (I - A)^{-1} d(x_0, x_1). \tag{4}$$

Remark 1. If, in the above theorem, we take $m = 1$, then we obtain the well-known Banach's contraction principle with $A := a \in (0,1)$.

The following theorem is our first main result. The proof is based on the application of Banach's contraction principle on $X \times X$ endowed with a scalar type metric.

Theorem 3. *Let (X,d) be a complete metric space. Let $T : X \times X \to X$ be an operator. Assume that there exists $k_1, k_2 \in \mathbb{R}_+$ with $k := \max\{k_1, k_2\} \in (0,1)$ such that*

$$\begin{aligned} &d\bigl(T(x,y), T(u,v)\bigr) + d\bigl(T(y,x), T(v,u)\bigr) \\ &\quad \leqslant k_1 d(x,u) + k_2 d(y,v) \quad \forall (x,y),(u,v) \in X \times X. \end{aligned}$$

Then the following conclusions hold:

(i) *there exists a unique self-similar set $U^* \in P_{\mathrm{cp}}(X \times X)$ of the operator*

$$F : X \times X \to X \times X, \quad (x,y) \mapsto \bigl(T(x,y), T(y,x)\bigr),$$

and, for any $U_0 \in P_{\mathrm{cp}}(X \times X)$, the sequence $(U_n)_{n\in\mathbb{N}}$ defined by $U_{n+1} = F(U_n)$, $n \in \mathbb{N}$, converges in $(P_{\mathrm{cp}}(X \times X), H_{\tilde{d}})$ to U^* as $n \to \infty$, where $\tilde{d}((x,y),(u,v)) := d(x,u) + d(y,v)$;

(ii) *the following estimation holds*:

$$H_{\tilde{d}}(U_n, U^*) \leqslant \frac{k^n}{1-k} \cdot H_{\tilde{d}}(U_0, U_1) \quad \forall n \in \mathbb{N};$$

(iii) (*the collage theorem*)

$$H_{\tilde{d}}(U, U^*) \leqslant \frac{1}{1-k} \cdot H_{\tilde{d}}\bigl(U, F(U)\bigr) \quad \forall U \in P_{\mathrm{cp}}(X \times X);$$

(iv) (*the anti-collage theorem*)

$$H_{\tilde{d}}(U, U^*) \geqslant \frac{1}{1+k} \cdot H_{\tilde{d}}\bigl(U, F(U)\bigr) \quad \forall U \in P_{\mathrm{cp}}(X \times X).$$

Proof. We introduce on $Z := X \times X$ the functional $\tilde{d} : Z \times Z \to \mathbb{R}_+$ defined by

$$\tilde{d}\bigl((x,y),(u,v)\bigr) := d(x,u) + d(y,v).$$

Notice that $\tilde{d}$ is a complete metric on Z.

We consider now the operator $F : Z \to Z$ given by $F(x,y) := (T(x,y), T(y,x))$. It is easy to prove (see [24]) that F is a contraction in $(Z, \tilde{d})$ with constant $k \in (0,1)$, i.e.,

$$\tilde{d}\bigl(F(z), F(w)\bigr) \leqslant k\tilde{d}(z,w) \quad \forall z,w \in Z.$$

As a consequence, F is continuous from $(Z,\tilde{d})$ to $(Z,\tilde{d})$.

Let us consider on $P_{\mathrm{cp}}(Z)$ the fractal operator $U \mapsto \hat{F}(U)$ generated by F. Notice that, by the continuity of F, the fractal operator is well defined, i.e., $\hat{F} : P_{\mathrm{cp}}(Z) \to P_{\mathrm{cp}}(Z)$.

Moreover, since F is a k-contraction on $(Z,\tilde{d})$, we immediately get (see [12]) that $\hat{F}$ is a k-contraction on $(P_{\mathrm{cp}}(Z), H_{\tilde{d}})$. Hence, by Banach's contraction principle $\hat{F}$ has a unique fixed point in $P_{\mathrm{cp}}(Z)$, i.e., there exists $U^* \in P_{\mathrm{cp}}(Z)$ such that $U^* = \hat{F}(U^*)$. Moreover, by the contraction principle we also have that

$$H_{\tilde{d}}(U_n, U^*) \leqslant \frac{k^n}{1-k} \cdot H_{\tilde{d}}(U_0, U_1), \tag{5}$$

where $U_0 \in P_{\mathrm{cp}}(X)$ is arbitrary and $U_1 := \hat{F}(U_0)$.

The conclusion (iii) follows by (ii), while for (iv), we notice that $H_{\tilde{d}}(U, F(U)) \leqslant H_{\tilde{d}}(U, U^*) + H_{\tilde{d}}(U^*, F(U)) \leqslant (1+k) H_{\tilde{d}}(U, U^*)$. $\square$

Remark 2. Using the same approach, a similar result can be obtained working on $X \times X$ with the metric $\hat{d}((x,y),(u,v)) := \max\{d(x,u), d(y,v)\}$. For related coupled fixed point theorems in this framework, see [23].

In the next part of this section, we will illustrate the vector-valued metric approach in coupled fixed point theory. For the proof of our next result, we need the following lemma, which is itself a result with a good potential, see Remark 3.

Lemma 1. *Let (X,d) be a generalized metric space in the sense of Perov with $d: X \times X \to \mathbb{R}^2_+$ given by*

$$d(x,y) := \begin{pmatrix} d_1(x,y) \\ d_2(x,y) \end{pmatrix}.$$

Let $f: X \to X$ be a contraction with a matrix A convergent to zero of the following form:

$$A := \begin{pmatrix} k_1 & k_2 \\ k_2 & k_1 \end{pmatrix}.$$

Then the fractal operator $\hat{f}$ generated by f is a (k_1+k_2)-contraction on $(P_{\mathrm{cp}}(X), H_{\tilde{d}})$, where $\tilde{d}(x,y) := d_1(x,y) + d_2(x,y)$.

Proof. Since f is a contraction with matrix A, we have, for all $x, y \in X$, that

$$d_1\bigl(f(x), f(y)\bigr) \leqslant k_1 d_1(x,y) + k_2 d_2(x,y),$$
$$d_2\bigl(f(x), f(y)\bigr) \leqslant k_2 d_1(x,y) + k_1 d_2(x,y).$$

Thus,

$$\tilde{d}\bigl(f(x), f(y)\bigr) \leqslant (k_1+k_2)\tilde{d}(x,y) \quad \forall x, y \in X.$$

Since the matrix A converges to zero, we get that $k_1 + k_2 < 1$. Thus, f is a (k_1+k_2)-contraction on $(X,\tilde{d})$. By the classical result of Nadler [12] we obtain that the fractal operator $\hat{f}$ generated by f is a (k_1+k_2)-contraction on $(P_{\mathrm{cp}}(X), H_{\tilde{d}})$. $\square$

Remark 3. Let (X,d) be a generalized metric space in the sense of Perov. It is an open question to establish when a contraction condition with a matrix A (convergent to zero, for example) on a single-valued operator $f: X \to X$ implies that the fractal operator $\hat{f}: P_{\mathrm{cp}}(X) \to P_{\mathrm{cp}}(X)$ generated by f is a contraction with matrix A on $(P_{\mathrm{cp}}(X), H_d)$.

The proof of our second result involves Lemma 1 and the vector-valued metric approach.

Theorem 4. *Let (X,d) be a complete metric space. Let $T: X \times X \to X$ be an operator. Assume that there exists $k_1, k_2 \in \mathbb{R}_+$ with $k_1 + k_2 < 1$ such that*

$$d\bigl(T(x,y), T(u,v)\bigr) \leqslant k_1 d(x,u) + k_2 d(y,v) \quad \forall (x,y),(u,v) \in X \times X.$$

Then the following conclusions hold:

(i) *there exists a unique self-similar set $U^* \in P_{\mathrm{cp}}(X \times X)$ of the operator*

$$F: X \times X \to X \times X, \quad (x,y) \mapsto \bigl(T(x,y), T(y,x)\bigr),$$

and, for any $U_0 \in P_{\mathrm{cp}}(X \times X)$, the sequence $(U_n)_{n \in \mathbb{N}}$ defined by $U_{n+1} = F(U_n)$, $n \in \mathbb{N}$, converges in $(P_{\mathrm{cp}}(X \times X), H_{\tilde{d}})$ to U^* as $n \to \infty$, where $\tilde{d}: (X \times X) \times (X \times X) \to \mathbb{R}_+$ is given by

$$\tilde{d}\bigl((x,y),(u,v)\bigr) := d(x,u) + d(y,v);$$

(ii) *the following estimation holds*:

$$H_{\tilde{d}}(U_n, U^*) \leqslant \frac{k^n}{1-k} \cdot H_{\tilde{d}}(U_0, U_1), \quad n \in \mathbb{N},$$

where $H_{\tilde{d}}$ is the Hausdorff–Pompeiu generalized metric induced by $\tilde{d}$ and $k := k_1 + k_2$;

(iii) (*the collage theorem*)

$$H_{\tilde{d}}(U, U^*) \leqslant \frac{1}{1-k} \cdot H_{\tilde{d}}\bigl(U, F(U)\bigr) \quad \forall U \in P_{\mathrm{cp}}(X \times X);$$

(iv) (*the anti-collage theorem*)

$$H_{\tilde{d}}(U, U^*) \geqslant \frac{1}{1+k} \cdot H_{\tilde{d}}\bigl(U, F(U)\bigr) \quad \forall U \in P_{\mathrm{cp}}(X \times X).$$

Proof. We introduce on $Z := X \times X$ the functional $\bar{d}: Z \times Z \to \mathbb{R}^2_+$ defined by

$$\bar{d}\bigl((x,y),(u,v)\bigr) := \begin{pmatrix} d(x,u) \\ d(y,v) \end{pmatrix}.$$

Notice that $\bar{d}$ is a complete generalized metric (in the sense of Perov) on Z.

We will prove now that the operator $F: Z \to Z$ given by $F(x,y) := (T(x,y), T(y,x))$ is a contraction in $(Z, \bar{d})$ with a convergent to zero matrix

$$A := \begin{pmatrix} k_1 & k_2 \\ k_2 & k_1 \end{pmatrix},$$

i.e., $\bar{d}(F(z), F(w)) \leqslant A\bar{d}(z,w)$ for all $z, w \in Z$.

Indeed, by the contraction condition on T we also get that

$$d\big(T(y,x),T(v,u)\big) \leqslant k_2 d(x,u) + k_1 d(y,v) \quad \forall (y,x),(v,u) \in X \times X.$$

Thus, for all $z = (x,y)$, $w = (u,v) \in Z$, we have

$$\bar{d}\big(F(x,y),F(u,v)\big) = \begin{pmatrix} d(T(x,y),T(u,v)) \\ d(T(y,x),T(v,u)) \end{pmatrix} \leqslant A\bar{d}\big((x,y),(u,v)\big).$$

On the other hand, an easy calculation shows that the eigenvalues of the matrix A are in the unit open disc. Hence, F is a contraction on $(Z,\bar{d})$ with matrix A.

Let us now consider the fractal operator $\hat{F} : P_{\mathrm{cp}}(Z) \to P_{\mathrm{cp}}(Z)$, $U \mapsto \hat{F}(U)$ generated by F. By the continuity of F (with respect to $\bar{d}$) the fractal operator is well defined. Moreover, since F is a contraction with matrix A on $(Z,\bar{d})$, we get (see Lemma 1) that $\hat{F}$ is a (k_1+k_2)-contraction on $(P_{\mathrm{cp}}(Z), H_{\tilde{d}})$. Hence, by Banach's contraction principle $\hat{F}$ has a unique fixed point in $P_{\mathrm{cp}}(Z)$, i.e., there exists $U^* \in P_{\mathrm{cp}}(Z)$ such that $U^* = \hat{F}(U^*)$. Additionally, by the same theorem we also have that

$$H_{\tilde{d}}(U_n,U^*) \leqslant \frac{k^n}{1-k} \cdot H_{\tilde{d}}(U_0,U_1), \tag{6}$$

where $U_0 \in P_{\mathrm{cp}}(X)$ is arbitrary, and $U_1 := \hat{F}(U_0)$.

Notice that conclusions (iii) and (iv) follow in a similar manner as above. □

Remark 4. It is worth to mention that Theorem 4 also follows directly by Theorem 3. Indeed, by the hypothesis we have, for all $(x,y),(u,v) \in X \times X$, that

$$d\big(T(x,y),T(u,v)\big) + d\big(T(y,x),T(v,u)\big) \leqslant (k_1+k_2)\big[d(x,u)+d(y,v)\big].$$

We will discuss now the case of coupled self-similar sets. We need first another auxiliary result.

Lemma 2. *Let (X,d) be a metric space, and $T : X \times X \to X$ be an operator. Assume that there exists $k > 0$ such that*

$$d\big(T(x,y),T(u,v)\big) \leqslant \frac{k}{2}\big[d(x,u)+d(y,v)\big] \quad \forall (x,y),(u,v) \in X \times X.$$

Then the following conclusions hold:

(i) $d(T(y,x),T(v,u)) \leqslant (k/2)[d(x,u)+d(y,v)]$ *for all* $(x,y),(u,v) \in X \times X$;
(ii) $H_d(T(A,B),T(U,V)) \leqslant (k/2)[H_d(A,U)+H_d(B,V)]$ *for all* $A,B,U,V \in P_{\mathrm{cp}}(X)$;
(iii) $H_d(T(B,A),T(V,U)) \leqslant (k/2)[H_d(A,U)+H_d(B,V)]$ *for all* $A,B,U,V \in P_{\mathrm{cp}}(X)$;
(iv) $H_d(T(A,B),T(U,V)) + H_d(T(B,A),T(V,U)) \leqslant k \cdot [H_d(A,U)+H_d(B,V)]$ *for all* $A,B,U,V \in P_{\mathrm{cp}}(X)$.

Proof. (i), (iii), and (iv) follow immediately. Let us show (ii). For this purpose, it is enough to prove that, for all $c \in T(A, B)$, there exists $w \in T(U, V)$ such that

$$d(c, w) \leqslant \frac{k}{2}\big[H_d(A, U) + H_d(B, V)\big],$$

and that, for all $s \in T(U, V)$, there exists $f \in T(A, B)$ such that

$$d(s, f) \leqslant \frac{k}{2}\big[H_d(A, U) + H_d(B, V)\big].$$

Let $c \in T(A, B)$. Then there exists $a \in A$ and $b \in B$ such that $c = T(a, b)$. For $a \in A$, there exists $u \in U$ such that $d(a, u) \leqslant H_d(A, U)$. In a similar way, for $b \in B$, there exists $v \in V$ such that $d(b, v) \leqslant H_d(B, V)$. Define $w := T(u, v)$. Thus,

$$\begin{aligned} d(c, w) = d\big(T(a, b), T(u, v)\big) &\leqslant \frac{k}{2}\big[d(a, u) + d(b, v)\big] \\ &\leqslant \frac{k}{2}\big[H_d(A, U) + H_d(B, V)\big], \end{aligned}$$

proving the first relation from above. In a similar way, we can prove the second relation, and the conclusion follows. □

Our third main result is the following coupled self-similar set theorem.

Theorem 5. *Let (X, d) be a complete metric space, and $T : X \times X \to X$ be an operator. Assume that there exists $k \in (0, 1)$ such that*

$$d\big(T(x, y), T(u, v)\big) \leqslant \frac{k}{2}\big[d(x, u) + d(y, v)\big] \quad \forall (x, y), (u, v) \in X \times X.$$

Then the following conclusions hold:

(i) *there exists a unique pair of coupled self-similar sets $(A^*, B^*) \in P_{\mathrm{cp}}(X) \times P_{\mathrm{cp}}(X)$, and, for any starting point $(A_0, B_0) \in P_{\mathrm{cp}}(X) \times P_{\mathrm{cp}}(X)$, the sequences $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$ defined, for $n \in \mathbb{N}$, by*

$$A_{n+1} = T(A_n, B_n), \qquad B_{n+1} = T(B_n, A_n) \tag{7}$$

converge (*with respect to H_d*) *to A^* and respectively to B^* as $n \to \infty$*;

(ii) *for each $n \in \mathbb{N}$, the following estimation holds*:

$$\begin{aligned} &H_d(A_n, A^*) + H_d(B_n, B^*) \\ &\quad \leqslant \frac{k^n}{1-k} \cdot \big[H_d\big(A_0, T(A_0, B_0)\big) + H_d\big(B_0, T(B_0, A_0)\big)\big]; \end{aligned}$$

(iii) (*the collage theorem*) *for all $A, B \in P_{\mathrm{cp}}(X)$, we have*

$$H_d(A, A^*) + H_d(B, B^*) \leqslant \frac{1}{1-k} \cdot \big[H_d\big(A, T(A, B)\big) + H_d\big(B, T(B, A)\big)\big];$$

(iv) (*the anti-collage theorem*) *for all* $A, B \in P_{\mathrm{cp}}(X)$*, we have*

$$H_d(A, A^*) + H_d(B, B^*) \geqslant \frac{1}{1+k} \cdot \bigl[H_d\bigl(A, T(A, B)\bigr) + H_d\bigl(B, T(B, A)\bigr)\bigr].$$

Proof. Let $\hat{T} : P_{\mathrm{cp}}(X) \times P_{\mathrm{cp}}(X) \to P_{\mathrm{cp}}(X)$ be defined by $(A, B) \mapsto T(A, B)$. Since T is continuous, $\hat{T}$ is well defined, i.e., $\hat{T}(A, B) \in P_{\mathrm{cp}}(X)$ for all $A, B \in P_{\mathrm{cp}}(X)$. By (iv) in Lemma 2 we get that $\hat{T}$ satisfies all the assumptions of Theorem 3.7 in [24]. Hence, there exists a unique pair $(A^*, B^*) \in P_{\mathrm{cp}}(X) \times P_{\mathrm{cp}}(X)$ such that $A^* = \hat{T}(A^*, B^*)$ and $B^* = \hat{T}(B^*, A^*)$. These relations show that (A^*, B^*) defines a pair of coupled self-similar sets for T. The second conclusion also follows by [24, Thm. 3.7]. Conclusion (iii) follows by (ii), while (iv) is a consequence of the following estimations:

$$\begin{aligned} &H_d\bigl(A, T(A, B)\bigr) + H_d\bigl(B, T(B, A)\bigr) \\ &\quad \leqslant H_d(A, A^*) + H_d\bigl(A^*, T(A, B)\bigr) + H_d(B, B^*) + H_d\bigl(B^*, T(B, A)\bigr) \\ &\quad \leqslant (1+k)\bigl[H_d(A, A^*) + H_d(B, B^*)\bigr]. \end{aligned}$$

□

A similar approach can be considered by working with the following contraction condition on the operator T:

$$d\bigl(T(x, y), T(u, v)\bigr) \leqslant k \max\bigl\{d(x, u), d(y, v)\bigr\} \quad \forall (x, y), (u, v) \in X \times X,$$

where $k \in (0, 1)$. For example, we have the following results.

Lemma 3. *Let* (X, d) *be a metric space, and let* $T : X \times X \to X$ *be an operator. Assume that there exists* $k > 0$ *such that*

$$d\bigl(T(x, y), T(u, v)\bigr) \leqslant k \max\bigl\{d(x, u), d(y, v)\bigr\} \quad \forall (x, y), (u, v) \in X \times X.$$

Then the following conclusions hold:

(i) $d(T(y, x), T(v, u)) \leqslant k \max\{d(x, u), d(y, v)\}$ *for all* $(x, y), (u, v) \in X \times X$;

(ii) *for all* $A, B, U, V \in P_{\mathrm{cp}}(X)$*, we have*

$$H_d\bigl(T(A, B), T(U, V)\bigr) \leqslant k \max\bigl\{H_d(A, U), H_d(B, V)\bigr\};$$

(iii) *for all* $A, B, U, V \in P_{\mathrm{cp}}(X)$*, we have*

$$H_d\bigl(T(B, A), T(V, U)\bigr) \leqslant k \max\bigl\{H_d(A, U), H_d(B, V)\bigr\};$$

(iv) *for all* $A, B, U, V \in P_{\mathrm{cp}}(X)$*, we have*

$$\begin{aligned} &\max\bigl\{H_d\bigl(T(A, B), T(U, V)\bigr), H_d\bigl(T(B, A), T(V, U)\bigr)\bigr\} \\ &\quad \leqslant k \cdot \max\bigl\{H_d(A, U), H_d(B, V)\bigr\}. \end{aligned}$$

Proof. We will prove conclusion (ii). For this purpose, it is enough to prove that, for all $c \in T(A,B)$, there exists $w \in T(U,V)$ such that

$$d(c,w) \leqslant k \max\bigl\{H_d(A,U), H_d(B,V)\bigr\}$$

and that, for all $s \in T(U,V)$, there exists $f \in T(A,B)$ such that

$$d(s,f) \leqslant k \max\bigl\{H_d(A,U), H_d(B,V)\bigr\}.$$

Let $c \in T(A,B)$. Then there exists $a \in A$ and $b \in B$ such that $c = T(a,b)$. For $a \in A$, there exists $u \in U$ such that $d(a,u) \leqslant H_d(A,U)$. In a similar way, for $b \in B$, there exists $v \in V$ such that $d(b,v) \leqslant H_d(B,V)$. Define $w := T(u,v)$. Thus,

$$\begin{aligned} d(c,w) &= d\bigl(T(a,b), T(u,v)\bigr) \leqslant k \max\bigl\{d(a,u), d(b,v)\bigr\} \\ &\leqslant k \max\bigl\{H_d(A,U), H_d(B,V)\bigr\}, \end{aligned}$$

proving the first relation from above. In a similar way, we can prove the above second relation and the conclusion follows. □

For our next theorems, we need the following result, which was essentially proved in [23], see Theorem 5.

Theorem 6. *Let (X,d) be a complete metric space. Let $T : X \times X \to X$ be an operator. Assume that there exists $k \in (0,1)$ such that, for all $(x,y), (u,v) \in X \times X$, we have*

$$\max\bigl\{d\bigl(T(x,y), T(u,v)\bigr), d\bigl(T(y,x), T(v,u)\bigr)\bigr\} \leqslant k \max\bigl\{d(x,u), d(y,v)\bigr\}.$$

Then the following conclusions hold:

(i) *there exists a unique solution $(x^*, y^*) \in X \times X$ of the coupled fixed point problem* (1)*, and, for any initial point $(x_0, y_0) \in X \times X$, the sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ defined, for $n \in \mathbb{N}$, by*

$$x_{n+1} = T(x_n, y_n), \qquad y_{n+1} = T(y_n, x_n) \tag{8}$$

converge to x^* and respectively to y^* as $n \to \infty$;

(ii) *for all $n \in \mathbb{N}^*$, the following estimation holds*:

$$\begin{aligned} &\max\bigl\{d\bigl(T^n(x_0,y_0), x^*\bigr), d\bigl(T^n(y_0,x_0), y^*\bigr)\bigr\} \\ &\quad\leqslant \frac{k^n}{1-k} \cdot \max\bigl\{d\bigl(x_0, T(x_0,y_0)\bigr), d\bigl(y_0, T(y_0,x_0)\bigr)\bigr\}. \end{aligned}$$

Based on Lemma 3 and Theorem 6, we can prove the following coupled self-similar set theorem.

Theorem 7. *Let (X,d) be a complete metric space and $T : X \times X \to X$ be an operator. Assume that there exists $k \in (0,1)$ such that*

$$d\bigl(T(x,y), T(u,v)\bigr) \leqslant k \max\bigl\{d(x,u), d(y,v)\bigr\} \quad \forall (x,y), (u,v) \in X \times X.$$

Then the following conclusions hold:

(i) *there exists a unique coupled self-similar pair* $(A^*, B^*) \in P_{\mathrm{cp}}(X) \times P_{\mathrm{cp}}(X)$*, and, for any starting point* $(A_0, B_0) \in P_{\mathrm{cp}}(X) \times P_{\mathrm{cp}}(X)$*, the sequences* $(A_n)_{n\in\mathbb{N}}$*,* $(B_n)_{n\in\mathbb{N}}$ *defined, for* $n \in \mathbb{N}$*, by*

$$A_{n+1} = T(A_n, B_n), \qquad B_{n+1} = T(B_n, A_n) \tag{9}$$

converge (*with respect to* H_d) *to* A^* *and respectively to* B^* *as* $n \to \infty$;

(ii) *the following estimation holds*:

$$\begin{aligned}&\max\bigl\{H_d(A_n, A^*), H_d(B_n, B^*)\bigr\}\\ &\quad\leqslant \frac{k^n}{1-k}\cdot\max\bigl\{H_d\bigl(A_0, T(A_0, B_0)\bigr), H_d\bigl(B_0, T(B_0, A_0)\bigr)\bigr\} \quad \forall n \in \mathbb{N}^*;\end{aligned}$$

(iii) (*the collage theorem*)

$$\begin{aligned}&\max\bigl\{H_d(A, A^*), H_d(B, B^*)\bigr\}\\ &\quad\leqslant \frac{1}{1-k}\cdot\max\bigl\{H_d\bigl(A, T(A, B)\bigr), H_d\bigl(B, T(B, A)\bigr)\bigr\} \quad \forall A, B \in P_{\mathrm{cp}}(X);\end{aligned}$$

(iv) (*the anti-collage theorem*)

$$\begin{aligned}&\max\bigl\{H_d(A, A^*), H_d(B, B^*)\bigr\}\\ &\quad\geqslant \frac{1}{1+k}\cdot\max\bigl\{H_d\bigl(A, T(A, B)\bigr), H_d\bigl(B, T(B, A)\bigr)\bigr\} \quad \forall A, B \in P_{\mathrm{cp}}(X).\end{aligned}$$

Proof. As in the proof of Theorem 5, the operator $\hat{T} : P_{\mathrm{cp}}(X) \times P_{\mathrm{cp}}(X) \to P_{\mathrm{cp}}(X)$ defined by $(A, B) \mapsto T(A, B)$ is well defined. By (iv) in Lemma 3 we get that $\hat{T}$ satisfies all the assumptions of Theorem 6. Hence, there exists a unique pair $(A^*, B^*) \in P_{\mathrm{cp}}(X) \times P_{\mathrm{cp}}(X)$ such that $A^* = \hat{T}(A^*, B^*)$ and $B^* = \hat{T}(B^*, A^*)$. These relations show that (A^*, B^*) defines a pair of coupled self-similar sets for T. The rest of the conclusions also follows, in a similar way to the proof of Theorem 5, by Theorem 6. Notice that (iv) is a consequence of the following estimations:

$$\begin{aligned}&\max\bigl\{H_d\bigl(A, T(A, B)\bigr), H_d\bigl(B, T(B, A)\bigr)\bigr\}\\ &\quad\leqslant \max\bigl\{H_d(A, A^*) + H_d\bigl(A^*, T(A, B)\bigr),\ H_d(B, B^*) + H_d\bigl(B^*, T(B, A)\bigr)\bigr\}\\ &\quad\leqslant (1+k)\max\bigl\{H_d(A, A^*), H_d(B, B^*)\bigr\}.\end{aligned}$$

□

4 Iterated function systems and coupled self-similar pairs

The purpose of this section is to discuss coupled fixed point properties for iterated function system of operators $f_i : X \times X \to X$, where $i \in \{1, 2, \dots, m\}$.

Let (X, d) be a metric space, and $f_i : X \times X \to X$ (where $i \in \{1, 2, \dots, m\}$) be continuous operators. We denote by $F : P_{\mathrm{cp}}(X) \times P_{\mathrm{cp}}(X) \to P_{\mathrm{cp}}(X)$ given by

$$F(A, B) := \bigcup_{i=1}^{m} f_i(A, B),$$

where $f_i(A,B) := \{f_i(a,b) \mid a \in A,\, b \in B\}$ for $i \in \{1,2,\dots,m\}$. We will call F the fractal operator generated by the iterated function system $(f_1,\dots,f_m)$.

By the continuity of f_i the operator F is well defined. Then we have the following result.

Theorem 8. *Let (X,d) be a complete metric space, and $f_i : X \times X \to X$ be operators such that, for each $i \in \{1,2,\dots,m\}$, there exists $k_i \in (0,1)$ satisfying*

$$d\bigl(f_i(x,y), f_i(u,v)\bigr) \leqslant k_i \max\bigl\{d(x,u), d(y,v)\bigr\} \quad \forall (x,y),(u,v) \in X \times X.$$

Then the following conclusions hold:

(i) *the fractal operator F generated by the iterated function system $(f_1,\dots,f_m)$ satisfies the following condition*:

$$\max\bigl\{H_d\bigl(F(A,B),F(U,V)\bigr),\, H_d\bigl(F(B,A),F(V,U)\bigr)\bigr\} \\ \leqslant \max_{1\leqslant i\leqslant m} k_i \cdot \max\bigl\{H_d(A,U), H_d(B,V)\bigr\} \quad \forall A,B,U,V \in P_{\mathrm{cp}}(X);$$

(ii) *there exists a unique pair $(A^*,B^*) \in P_{\mathrm{cp}}(X) \times P_{\mathrm{cp}}(X)$ such that*

$$A^* = F(A^*,B^*), \qquad B^* = F(B^*,A^*); \tag{10}$$

(iii) *for any pair $(A_0,B_0) \in P_{\mathrm{cp}}(X) \times P_{\mathrm{cp}}(X)$, the sequence $((A_n,B_n))_{n\in\mathbb{N}} \subset P_{\mathrm{cp}}(X) \times P_{\mathrm{cp}}(X)$ given by*

$$A_{n+1} = F(A_n,B_n), \qquad B_{n+1} = F(B_n,A_n) \tag{11}$$

converges to $(A^*,B^*) \in P_{\mathrm{cp}}(X) \times P_{\mathrm{cp}}(X)$ as $n \to \infty$.

Proof. Notice first that, by the contraction condition, each operator f_i is continuous for each $i \in \{1,2,\dots,m\}$.

(i) For all $A,B,U,V \in P_{\mathrm{cp}}(X)$, using Lemma 3, we successively can write

$$H_d\bigl(F(A,B),F(U,V)\bigr) \leqslant \max_{1\leqslant i\leqslant m} H_d\bigl(f_i(A,B), f_i(U,V)\bigr) \\ \leqslant \max_{1\leqslant i\leqslant m} k_i \cdot \max\bigl\{H_d(A,U),\, H_d(B,V)\bigr\}.$$

Thus, we also have

$$H_d(F(B,A),F(V,U)) \leqslant \max_{1\leqslant i\leqslant m} H_d\bigl(f_i(B,A), f_i(V,U)\bigr) \\ \leqslant \max_i k_i \cdot \max\bigl\{H_d(A,U), H_d(B,V)\bigr\}.$$

Hence,

$$\max\bigl\{H_d\bigl(F(A,B),F(U,V)\bigr),\, H_d\bigl(F(B,A),F(V,U)\bigr)\bigr\} \\ \leqslant \max_{1\leqslant i\leqslant m} k_i \cdot \max\bigl\{H_d(A,U),\, H_d(B,V)\bigr\}.$$

(ii)–(iii) The conclusions follow by Theorem 6. □

A similar result takes place for the case of the additive metric.

Theorem 9. *Let (X,d) be a complete metric space, and $f_i : X \times X \to X$ be operators for which there exists $k_i \in (0,1)$ such that, for each $i \in \{1,2,\dots,m\}$, we have*

$$d\big(f_i(x,y), f_i(u,v)\big) \leqslant \frac{k_i}{2}\big(d(x,u)+d(y,v)\big) \quad \forall (x,y),(u,v) \in X \times X.$$

Then the following conclusions hold:

(i) *the fractal operator F generated by the iterated function system $(f_1,\dots,f_m)$ satisfies, for all $A,B,U,V \in P_{\mathrm{cp}}(X)$, the following condition*:

$$H_d\big(F(A,B),F(U,V)\big) + H_d\big(F(B,A),F(V,U)\big) \leqslant \max_i k_i \cdot \big(H_d(A,U)+H_d(B,V)\big);$$

(ii) *there exists a unique pair $(A^*,B^*) \in P_{\mathrm{cp}}(X) \times P_{\mathrm{cp}}(X)$ such that relations* (10) *hold*;

(iii) *for any pair $(A_0,B_0) \in P_{\mathrm{cp}}(X) \times P_{\mathrm{cp}}(X)$, the sequence $((A_n,B_n))_{n\in\mathbb{N}} \subset P_{\mathrm{cp}}(X) \times P_{\mathrm{cp}}(X)$ given by relations* (11) *converges to $(A^*,B^*) \in P_{\mathrm{cp}}(X) \times P_{\mathrm{cp}}(X)$ as $n \to \infty$.*

Remark 5.

1. We also refer to [11] for an extension of this study to a discussion on some qualitative properties (such as well-posedness or Ulam–Hyers stability) of a fixed point problem.
2. The above approach can be generalized to the case of n-tuples self-similar sets using the methods presented in [5, 14, 17, 20] and [21]. For the tripled fixed point problem, see [3]. It could be also of interest to consider the above study for some extensions of the coupled fixed point problem, see, for example, [18, 19].
3. A unified treatment of the coupled fixed point problem (and its consequences such as coupled fixed sets) can be given by considering, instead of the ℓ^1-type metric $\tilde{d}$ or the ℓ^∞-type metric $\hat{d}$, the ℓ^p-type metric on the Cartesian product $X \times X$. It is known that this general ℓ^p-product metric was often employed in the fixed point theory, see [8, 10, 25]. Another research direction on this topic could involve the symmetric product of the sets, see [7].

5 Examples

Example 1. Let $f_1, f_2 : [0,1] \times [0,1] \to [0,1]$ given by

$$f_1(x,y) := \frac{x+y}{6}, \qquad f_2(x,y) = 1 - \frac{x+y}{6}.$$

Then f_i, $i \in \{1,2\}$, satisfy, for all $(x,y),(u,v) \in [0,1]^2$, the following relations:

$$\big|f_i(x,y) - f_i(u,v)\big| \leqslant \frac{1}{6}\big(|x-u|+|y-v|\big),$$
$$\big|f_i(x,y) - f_i(u,v)\big| \leqslant \frac{1}{3}\max\big\{|x-u|,|y-v|\big\}.$$

Consider the iterated function system $f = (f_1, f_2)$ and the fractal operator F generated by f, i.e., $F : P_{\mathrm{cp}}([0,1]) \times P_{\mathrm{cp}}([0,1]) \to P_{\mathrm{cp}}([0,1])$ given by

$$F(A,B) := f_1(A,B) \cup f_2(A,B).$$

If we take, for example, $A_0 = B_0 = [0,1]$, then the sequences (A_n) and (B_n) given by

$$A_{n+1} = F(A_n, B_n), \qquad B_{n+1} = F(B_n, A_n) \tag{12}$$

both take, for $n \geqslant 1$, the values $A_n = B_n = [0,1/3] \cup [2/3,1] = A^* = B^*$. In this case, even the iterated function system f is of Cantor type, the sets A^*, B^* have not a self-similar structure.

Example 2 (*Cantor-type coupled fixed points*). Let $f_1, f_2 : [-1,1] \times [-1,1] \to [-1,1]$ given by

$$f_1(x,y) = \begin{cases} \frac{1}{3}\max(|x|,|y|) & \text{if } x = 0, \\ \operatorname{sign} x \frac{1}{3}\max(|x|,|y|) & \text{if } x \neq 0, \end{cases}$$

$$f_2(x,y) = \begin{cases} \frac{2}{3} + \frac{1}{3}\max(|x|,|y|) & \text{if } x = 0, \\ \operatorname{sign} x (\frac{2}{3} + \frac{1}{3}\max(|x|,|y|)) & \text{if } x \neq 0. \end{cases}$$

Consider the iterated function system $f = (f_1, f_2)$ and the fractal operator F generated by f, i.e., $F : P_{\mathrm{cp}}([-1,1]) \times P_{\mathrm{cp}}([-1,1]) \to P_{\mathrm{cp}}([-1,1])$ given by $F(A,B) := f_1(A,B) \cup f_2(A,B)$. Consider $A_0 = B_0 = [-1,1]$ and, for $n \geqslant 1$, the sequences (A_n) and (B_n) given by]

$$A_{n+1} = F(A_n, B_n), \qquad B_{n+1} = F(B_n, A_n). \tag{13}$$

Then $A^*, B^* \in P_{\mathrm{cp}}([-1,1])$ have a Cantor-type self-similar structure, see the approximations of it in Fig. 1.

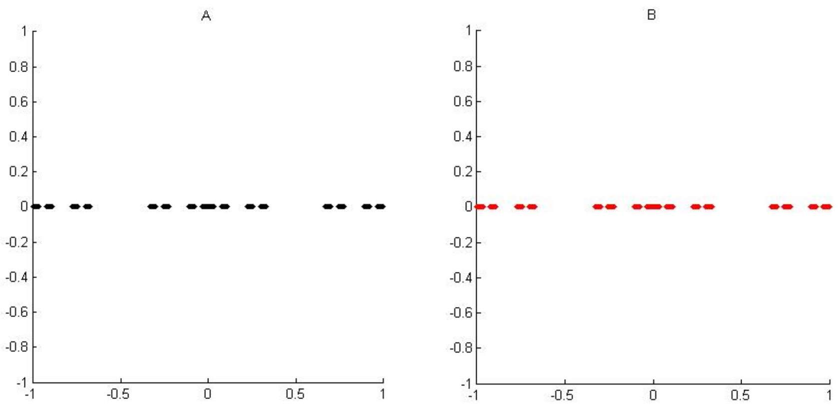

Figure 1. A_3 and B_3 from Example 2.

Example 3 (*Sierpinski gasket-type coupled fixed points*). Let us consider $f_1, \dots, f_8 : [-1,1]^2 \times [-1,1]^2 \to [-1,1]^2$ as follows:

$$\begin{aligned}
f_1\big((x,y),(u,v)\big) &= \left(\operatorname{sign} x \frac{1}{3}\max\big(|x|,|u|\big),\ \operatorname{sign} y \frac{1}{3}\max\big(|y|,|v|\big)\right),\\
f_2\big((x,y),(u,v)\big) &= \left(\operatorname{sign} x \left(\frac{1}{3}+\frac{1}{3}\max\big(|x|,|u|\big)\right),\ \operatorname{sign} y \frac{1}{3}\max\big(|y|,|v|\big)\right),\\
f_3\big((x,y),(u,v)\big) &= \left(\operatorname{sign} x \left(\frac{2}{3}+\frac{1}{3}\max\big(|x|,|u|\big)\right),\ \operatorname{sign} y \frac{1}{3}\max\big(|y|,|v|\big)\right),\\
f_4\big((x,y),(u,v)\big) &= \left(\operatorname{sign} x \left(\frac{2}{3}+\frac{1}{3}\max\big(|x|,|u|\big)\right),\ \operatorname{sign} y \left(\frac{1}{3}+\frac{1}{3}\max\big(|y|,|v|\big)\right)\right),\\
f_5\big((x,y),(u,v)\big) &= \left(\operatorname{sign} x \left(\frac{2}{3}+\frac{1}{3}\max\big(|x|,|u|\big)\right),\ \operatorname{sign} y \left(\frac{2}{3}+\frac{1}{3}\max\big(|y|,|v|\big)\right)\right),\\
f_6\big((x,y),(u,v)\big) &= \left(\operatorname{sign} x \left(\frac{1}{3}+\frac{1}{3}\max\big(|x|,|u|\big)\right),\ \operatorname{sign} y \left(\frac{2}{3}+\frac{1}{3}\max\big(|y|,|v|\big)\right)\right),\\
f_7\big((x,y),(u,v)\big) &= \left(\operatorname{sign} x \left(\frac{1}{3}\max\big(|x|,|u|\big)\right),\ \operatorname{sign} y \left(\frac{2}{3}+\frac{1}{3}\max\big(|y|,|v|\big)\right)\right),\\
f_8\big((x,y),(u,v)\big) &= \left(\operatorname{sign} x \left(\frac{1}{3}\max\big(|x|,|u|\big)\right),\ \operatorname{sign} y \left(\frac{1}{3}+\frac{1}{3}\max\big(|y|,|v|\big)\right)\right).
\end{aligned}$$

Consider the iterated function system $f = (f_1, \dots, f_8)$ and the fractal operator F generated by f, i.e., $F : P_{\mathrm{cp}}([-1,1]^2) \times P_{\mathrm{cp}}([-1,1]^2) \to P_{\mathrm{cp}}([-1,1]^2)$ given by $F(A,B) := f_1(A,B) \cup \dots \cup f_8(A,B)$. Consider $A_0 = B_0 = [-1,1]^2$ and, for $n \geqslant 1$, the sequences (A_n) and (B_n) given by

$$A_{n+1} = F(A_n, B_n), \qquad B_{n+1} = F(B_n, A_n). \tag{14}$$

Then the coupled self-similar sets $A^*, B^* \in P_{\mathrm{cp}}([-1,1]^2)$ of the iterated function system $f = (f_1, \dots, f_8)$ are approximated in Fig. 2.

Example 4. If, in the same context as before, working with the same initial sets A_0, B_0, we choose the following iterated function system:

$$\begin{aligned}
f_1\big((x,y),(u,v)\big) &= \left(\operatorname{sign} x \frac{1}{3}|x|,\ \operatorname{sign} y \frac{1}{3}|y|\right),\\
f_2\big((x,y),(u,v)\big) &= \left(\operatorname{sign} x \left(\frac{1}{3}+\frac{1}{3}|x|\right),\ \operatorname{sign} y \frac{1}{3}|y|\right)\\
f_3\big((x,y),(u,v)\big) &= \left(\operatorname{sign} x \left(\frac{2}{3}+\frac{1}{3}|x|\right),\ \operatorname{sign} y \frac{1}{3}|y|\right),\\
f_4\big((x,y),(u,v)\big) &= \left(\operatorname{sign} x \left(\frac{2}{3}+\frac{1}{3}|x|\right),\ \operatorname{sign} y \left(\frac{1}{3}+\frac{1}{3}|y|\right)\right),
\end{aligned}$$

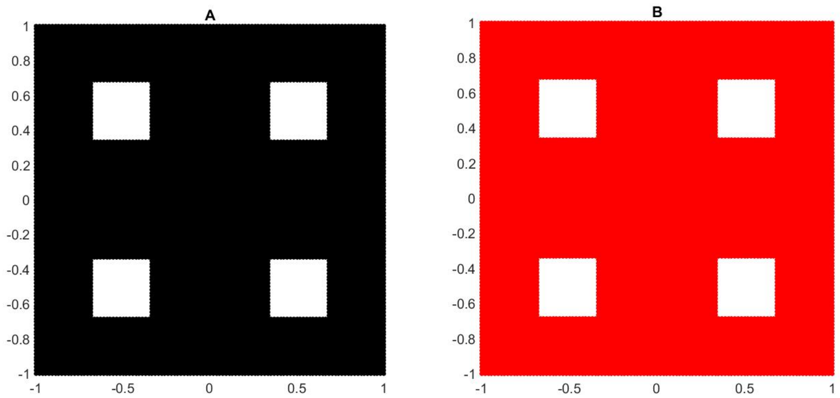

Figure 2. A_3 and B_3 from Example 3.

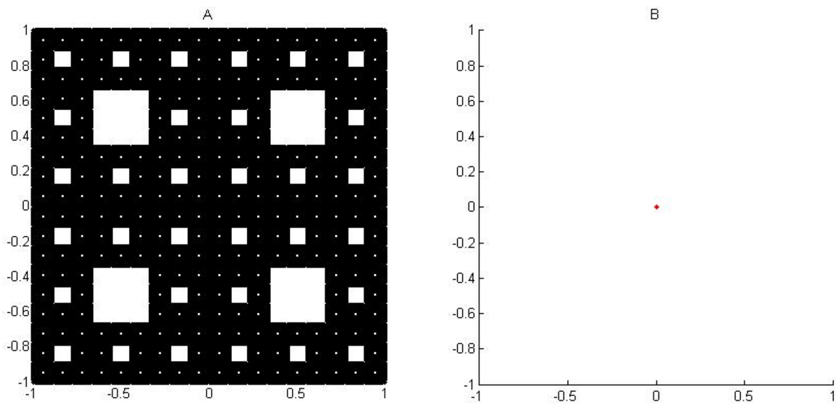

Figure 3. A_3 and B_3 from Example 4.

$$f_5\big((x,y),(u,v)\big) = \left(\operatorname{sign} x\left(\frac{2}{3}+\frac{1}{3}|x|\right), \operatorname{sign} y\left(\frac{2}{3}+\frac{1}{3}|y|\right)\right),$$
$$f_6\big((x,y),(u,v)\big) = \left(\operatorname{sign} x\left(\frac{1}{3}+\frac{1}{3}|x|\right), \operatorname{sign} y\left(\frac{2}{3}+\frac{1}{3}|y|\right)\right),$$
$$f_7\big((x,y),(u,v)\big) = \left(\operatorname{sign} x\left(\frac{1}{3}|x|\right), \operatorname{sign} y\left(\frac{2}{3}+\frac{1}{3}|y|\right)\right),$$
$$f_8\big((x,y),(u,v)\big) = \left(\operatorname{sign} x\left(\frac{1}{3}|x|\right), \operatorname{sign} y\left(\frac{1}{3}+\frac{1}{3}|y|\right)\right),$$

then the fixed point will be the pair (Sierpinski carpet $\times\{(0,0)\}$), see Fig. 3.

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