Iterative unique positive solutions for singular $p$-Laplacian fractional differential equation system with several parameters

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Received: May 3, 2017 / Revised: December 17, 2017 / Published online: February 12, 2018

Abstract. By using the method of reducing the order of a derivative, the higher-order fractional differential equation is transformed into the lower-order fractional differential equation and combined with the mixed monotone operator, a unique positive solution is obtained in this paper for a singular $p$-Laplacian boundary value system with the Riemann–Stieltjes integral boundary conditions. This equation system is very wide because there are many parameters, which can be changeable in the equation system in this paper, and the nonlinearity is allowed to be singular in regard to not only the time variable but also the space variable. Moreover, the unique positive solution that we obtained in this paper is dependent on $\lambda$, and an iterative sequence and convergence rate are given, which are important for practical application. An example is given to demonstrate the application of our main results.

Keywords: fractional differential equation system, singular $p$-Laplacian, integral boundary condition; iterative positive solution, mixed monotone operator.

1 Introduction

During the last decades, boundary value problems for nonlinear fractional differential equations have gained its popularity and significance due to its distinguished applications.

*This research was supported by the National Natural Science Foundation of China (11371221, 11571296), the natural Science Foundation of Shandong Province of China (ZR2015AL002), and Project of Shandong Province Higher Educational Science and Technology Program (J15LI16).

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as valuable tools in different areas of applied different areas such as physics, chemistry, electrical networks, economics, rheology, biology chemical, image processing, and so on. Fractional calculus has been shown to be more accurate and realistic than integer-order models, and it also provides an excellent tool to describe the hereditary properties of material and processes, particularly in viscoelasticity, electrochemistry, porous media, and so on. There has been a significant development in the study of fractional differential equations in recent years. For an extensive collection of such literature, readers can refer to [4,6–8,10–13,15,17,18,21–25,27–32], and there are a lot of methods to study differential equations such as degree theory (see [14]), mixed monotone operator (see [8,13,15,29]), bifurcation method (see [14,20]), spectral analysis (see [2,19,27,31]), and so on. For some differential equation in which fractional derivatives are involved in the nonlinear terms, reader can refer to [6–8,28,29,31]. In order to meet the needs, the p-Laplacean equation is introduced in some boundary value problems. Fractional differential equation system of p-Laplacean with the Riemann–Stieltjes integral boundary conditions is a type of equation system that is very wide, and the general equation systems are special cases of p-Laplacean equation system. We refer the reader to [3,9,12,16,20,26,27] for some relevant work.

In [27], the authors considered the following fractional differential equation:

\[-D_0^\beta (\varphi_p(D_0^\alpha x))(t) = \lambda f(t, x(t)), \quad 0 < t < 1,\]
\[x(0) = 0, \quad D_0^\alpha x(0) = 0, \quad x(1) = \int_0^1 x(s) \, dA(s),\]

where \(\alpha, \beta \in \mathbb{R}_+ = [0, +\infty), 0 < \beta \leq 1, 1 < \alpha \leq 2, \lambda > 0\) is a parameter, \(A\) is a function of bounded variation, \(\int_0^1 x(s) \, dA(s)\) denotes the Riemann–Stieltjes integral of \(x\) with respect to \(A\), \(f(t, x) : (0,1) \times (0, +\infty) \to (0, +\infty)\) is continuous and may be singular at \(t = 0,1\) and \(x = 0\). \(D_0^{\alpha p}, D_0^{\beta p}\) are the Riemann–Liouville differential fractional derivatives of order \(\alpha, \beta\), and the p-Laplacean operator \(\varphi_p\) is defined as \(\varphi_p(s) = |s|^{p-2}s, p > 1\). The authors obtained the existence of positive solution by the upper and lower solutions and Schauder fixed-point theorems. In [12], the authors discussed a Hadamard fractional differential equation boundary value problem with p-Laplacean operator

\[-D_0^\beta (\varphi_p(D_0^\alpha x))(t) = f(t, x(t)), \quad 1 < t < e,\]
\[x(1) = x'(1) = x'(e) = 0, \quad D_0^\alpha x(1) = D_0^\alpha x(e) = 0,\]

where \(\alpha, \beta \in \mathbb{R}_+, 2 < \alpha \leq 3, 1 < \beta \leq 2, \varphi_p(s) = |s|^{p-2}s, p > 1\), and \(f : [1, e] \times \mathbb{R}_+ \to \mathbb{R}_+\) is a positive continuous function, \(D_0^{\alpha e}, D_0^{\beta e}\) are the Riemann–Liouville differential fractional derivative of order \(\alpha, \beta\). The authors obtained the existence and the uniqueness of positive solutions by using the Leray–Schauder-type alternative and the Guo–Krasnoselskii fixed-point theorem. In [23], the authors investigated the singular problem

\[-D_0^\alpha u(t) + \lambda f(t, u(t)), D_0^\beta u(t), v(t) = 0, \quad 0 < t < 1,\]
\[-D_1^\gamma v(t) + \lambda g(t, u(t)) = 0, \quad 0 < t < 1,\]
\[
D_t^\beta u(0) = D_t^{\beta + 1} u(0) = 0, \quad D_t^\alpha v(0) = 0, \quad v(1) = \int_0^1 v(s) \, dB(s),
\]

where \(\alpha, \beta, \gamma \in R_+^1\), \(2 < \alpha, \gamma \leq 3\), \(0 < \beta < 1\), \(u\) denotes the number of uninfected \(CD4^+\) T cells and \(v\) denotes the number of infected cells, \(\lambda > 0\) is a parameter, \(\alpha - \beta > 2\), \(\int_0^1 D_t^\alpha u(s) \, dA(s)\) and \(\int_0^1 v(s) \, dB(s)\) denote the Riemann–Stieltjes integrals of \(u, v\) with respect to \(A, B\), respectively, \(A, B\) are bounded variations, \(f : (0, 1) \times R_+^1 \rightarrow R\), \(g : (0, 1) \times R_+ \rightarrow R\) are two continuous functions and may be singular at \(t = 0, 1\), \(D_t^\lambda, D_t^\beta, D_t^\alpha\) are the standard Riemann–Liouville derivatives. The authors obtained the existence of positive solution by the fixed-point theorem.

Motivated by the excellent results above, in this paper, we will devote to considering the following singular \(p\)-Laplacian fractional differential equation (PFDE) with the Riemann–Stieltjes integral boundary conditions:

\[
\begin{align*}
D_0^{n_{-1}} (\varphi_p (D_0^\gamma u(t))) + \mu P_{a-1} f(t, u(t), D_0^\alpha u(t), D_0^\beta u(t), \ldots, \\
D_0^{n_{-1}} u(t), v(t)) &= 0, \quad 0 < t < 1, \\
D_0^{n_{-1}} (\varphi_p (D_0^\delta v(t))) + \mu P_{a-1} g(t, u(t), D_0^\alpha u(t), D_0^\beta u(t), \ldots, \\
D_0^{n_{-1}} u(t)) &= 0, \quad 0 < t < 1, \\
u(0) &= D_0^\alpha u(0) = 0, \quad D_0^\gamma u(0) = D_0^\gamma u(0) = 0, \quad i = 1, 2, \ldots, n - 2, \\
D_0^{n_{-1}} (1) &= \chi \int_0^\eta h(t) D_0^{n_{-1}} u(t) \, dA(t), \\
v(0) &= D_0^\delta v(0) = 0, \quad D_0^\delta v(0) = D_0^\delta v(0) = 0, \quad i = 1, 2, \ldots, m - 2, \\
D_0^{n_{-1}} (1) &= \iota \int_0^\vartheta a(t) D_0^{n_{-1}} v(t) \, dB(t),
\end{align*}
\]

where \(\alpha, \beta, \gamma, \delta, \mu, k, \eta \in R_+ (\kappa = 1, 2, \ldots, n - 1; \rho = 1, 2, \ldots, m - 1)\), \(n, m \in N\) (natural number set), \(n, m \geq 2\), and \(1/2 < \alpha, \beta \leq 1, n - 1 < \gamma \leq n, m - 1 < \delta \leq m\), \(p\)-Laplacian operator \(\varphi_p\) is defined as \(\varphi_p(s) = |s|^{p-2} s, p, q > 1, 1/p + 1/q = 1\), \(n - k < \gamma < \mu_k \leq n + 1 - \kappa, m - \rho < \delta - \eta \leq m + 1 - \rho (\kappa = 1, 2, \ldots, n - 1; \rho = 1, 2, \ldots, m - 1)\), \(0 < \eta, \vartheta \leq 1, \lambda, \mu, \chi, \iota > 0\) are parameters, \(f \in C((0, 1) \times (0, +\infty)^{n+1}, R_+), f(t, x_1, x_2, \ldots, x_{n+1})\) has singularity at \(x_i = 0 (i = 1, 2, \ldots, n + 1)\), \(t = 0, 1, g \in C([0, 1] \times R_+^m, R_+)\), \(h, a \in C(0, 1), A, B\) are functions of bounded variation, \(\int_0^\eta h(t) D_0^{n_{-1}} u(t) \, dA(t), \int_0^\vartheta a(t) D_0^{n_{-1}} v(t) \, dB(t)\) denote the Riemann–Stieltjes integral with respect to \(A, B\).
Positive solutions for system of singular boundary value system

$D_{0+}^\alpha u$, $D_{0+}^\alpha v$, $D_{0+}^\beta u$, $D_{0+}^\beta v$, $D_{0+}^\gamma u$, $D_{0+}^\gamma v$ are the standard Riemann–Liouville derivative. In this paper, the existence of positive solutions is obtained by means of mixed monotone operator in cones.

In this paper, we investigate the existence of positive solutions for a singular $p$-Laplacian boundary value system with the Riemann–Stieltjes integral boundary conditions. A vector $(u, v) \in C[0, 1] \times C[0, 1]$ is said to be a positive solution of system (1) if and only if $(u, v) \in C[0, 1] \times C[0, 1]$ and $u(t) > 0$, $v(t) > 0$ for any $t \in (0, 1]$.

Compared with the above results, our work presented in this paper has the following several new features. Firstly, fractional derivatives are involved in the nonlinear terms of fractional differential equation (1). Secondly, the method we used in this paper is reducing the order of derivative, that is, higher-order fractional differential equation is transformed into lower-order fractional differential equation. Thirdly, the uniqueness positive solution of Eq. (1) is dependent on $\lambda$.

For convenience in presentation, here we list some conditions to be used throughout the paper.

(S1) $f(t, x_1, x_2, \ldots, x_{n+1}) = \phi(t, x_1, x_2, \ldots, x_{n+1}) + \psi(t, x_1, x_2, \ldots, x_{n+1})$, where $\phi : (0, 1) \times (0, +\infty)^{n+1} \to \mathbb{R}_+$ is continuous, $\phi(t, x_1, x_2, \ldots, x_{n+1})$ may be singular at $t = 0$ and is nondecreasing on $x_i > 0$ ($i = 1, 2, \ldots, n + 1$). $\psi : (0, 1) \times (0, +\infty)^{n+1} \to \mathbb{R}_+$ is continuous, $\psi(t, x_1, x_2, \ldots, x_{n+1})$ may be singular at $t = 0$, $1$, $x_i = 0$ and is nonincreasing on $x_i > 0$ ($i = 1, 2, \ldots, n + 1$).

(S2) There exists $0 < \sigma < 1$ such that, for all $x_i > 0$ ($i = 1, 2, \ldots, n + 1$) and $t, l \in (0, 1)$,

$$\phi(t, tx_1, tx_2, \ldots, tx_{n+1}) \geq t^{\sigma/(q-1)} \phi(t, x_1, x_2, \ldots, x_{n+1}),$$
$$\psi(t, lx_1, lx_2, \ldots, lx_{n+1}) \geq l^{\sigma/(q-1)} \psi(t, x_1, x_2, \ldots, x_{n+1}).$$

(S3) $g \in C((0, 1) \times (0, +\infty)^m, \mathbb{R}_+)$, $g(t, x_1, x_2, \ldots, x_m)$ is nondecreasing on $x_i > 0$ ($i = 1, 2, \ldots, m$), and $g(t, 1, 1, \ldots, 1) \neq 0$, $t \in (0, 1)$. Moreover, there exists $\varsigma \in (0, 1)$ such that, for all $x_i > 0$ ($i = 1, 2, \ldots, m$) and $t, l \in (0, 1)$,

$$g(t, lx_1, lx_2, \ldots, lx_m) \geq l^{\varsigma/(q-1)} g(t, x_1, x_2, \ldots, x_m).$$

(S4) $0 < \int_0^1 \phi^2(\tau, 1, 1, \ldots, 1) \, d\tau < +\infty$,

$$0 < \int_0^1 \tau^{-2(\gamma-1)\sigma/(q-1)} \psi^2(\tau, 1, 1, \ldots, 1) \, d\tau < +\infty,$$

$$0 < \int_0^1 g^2(\tau, 1, 1, \ldots, 1) \, d\tau < +\infty.$$

Here $q$ is defined by (1).

Remark 1. According to (S2) and (S3), for all $x_i > 0$ ($i = 1, 2, \ldots, n+1$), $\sigma, \varsigma, t \in (0, 1)$, we have
$$\phi(t, lx_1, lx_2, \ldots, lx_{n+1}) \leq l^{\sigma_{1/(q-1)}} \phi(t, x_1, x_2, \ldots, x_{n+1}),$$
$$\psi(t, l^{-1}x_1, l^{-1}x_2, \ldots, l^{-1}x_{n+1}) \leq l^{\sigma_{1/(q-1)}} \psi(t, x_1, x_2, \ldots, x_{n+1}),$$
$$g(t, lx_1, lx_2, \ldots, lx_m) \leq l^{\varsigma_{1/(q-1)}} g(t, x_1, x_2, \ldots, x_m).$$

2 Preliminaries and lemmas

For some basic definitions and lemmas about the theory of fractional calculus, the reader can refer to the recent literature such as [17, 18].

Lemma 1. (See [17, 18].) Assume that $u \in C^n(0, 1) \cap L(0, 1)$, then
$$I_0^\alpha D_0^\alpha u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n},$$
where $n = [\alpha] + 1, C_i \in \mathbb{R}$ ($i = 1, 2, \ldots, n$).

Lemma 2. (See [10].)
(i) If $x \in L(0, 1), v > \sigma > 0$, then
$$I_0^\sigma D_0^\sigma x(t) = \Gamma(\sigma)(\Gamma(\sigma - v)) t^{\sigma-v-1}.$$  

(ii) If $v > 0, \sigma > 0$, then $D_0^\sigma, t^{\sigma-v} = (\Gamma(\sigma)/\Gamma(\sigma - v)) t^{\sigma-v-1}$.

Lemma 3. Let $\rho \in L^1(0, 1) \cap C(0, 1)$, then the equation of the BVPs
$$-D_0^{\gamma_{0-n}} x(t) = \rho(t), \quad 0 < t < 1,$$
$$x(0) = 0, \quad x(1) = \chi \int_0^1 h(t)x(t) \, dA(t),$$
$$-D_0^{\sigma_{0-n}} y(t) = \rho(t), \quad 0 < t < 1,$$
$$y(0) = 0, \quad y(1) = \iota \int_0^1 a(t)y(t) \, dB(t)$$

have integral representation
$$x(t) = \int_0^1 G(t, s) \rho(s) \, ds = \int_0^1 (G_1(t, s) + G_2(t, s)) \rho(s) \, ds,$$
$$y(t) = \int_0^1 H(t, s) \rho(s) \, ds = \int_0^1 (H_1(t, s) + H_2(t, s)) \rho(s) \, ds,$$  

(2)
respectively, where

\[ G_1(t, s) = \frac{1}{\Gamma(\gamma - \mu_{n-1})} \]
\[ \quad \times \begin{cases} t^{\gamma - \mu_{n-1} - 1}(1 - s)^{\gamma - \mu_{n-1} - 1} - (t - s)^{\gamma - \mu_{n-1} - 1}, & 0 \leq s \leq t \leq 1, \\ t^{\gamma - \mu_{n-1} - 1}(1 - s)^{\gamma - \mu_{n-1} - 1}, & 0 \leq t \leq s \leq 1, \end{cases} \]
\[ G_2(t, s) = \frac{1}{1 - F} j_A(s), \quad F = \chi \int_0^\eta t^{\gamma - \mu_{n-1} - 1} h(t) \, dA(t), \]
\[ H_1(t, s) = \frac{1}{\Gamma(\delta - \eta_{m-1})} \]
\[ \quad \times \begin{cases} t^{\delta - \eta_{m-1} - 1}(1 - s)^{\delta - \eta_{m-1} - 1} - (t - s)^{\delta - \eta_{m-1} - 1}, & 0 \leq s \leq t \leq 1, \\ t^{\delta - \eta_{m-1} - 1}(1 - s)^{\delta - \eta_{m-1} - 1}, & 0 \leq t \leq s \leq 1, \end{cases} \]
\[ H_2(t, s) = \frac{t^{\delta - \eta_{m-1} - 1}}{1 - C} j_B(s), \quad C = t \int_0^\varphi t^{\delta - \eta_{m-1} - 1} a(t) \, dB(t), \]
\[ j_A(s) = \int_0^\eta h(t) G_1(t, s) \, dA(t), \quad j_B(s) = \int_0^\varphi a(t) H_1(t, s) \, dB(t). \]

**Proof.** The proof is similar to that for Lemma 2.3 in [25], we omit it here. \( \square \)

**Lemma 4.** Let \( 0 \leq C, F < 1, \) and \( j_A(s), j_B(s) \geq 0 \) for \( s \in [0, 1], \) then the Green functions defined by (2) satisfy:

(i) \( G, H : [0, 1] \times [0, 1] \to \mathbb{R}_+ \) are continuous, and \( G(t, s), H(t, s) > 0 \) for all \( t, s \in (0, 1); \)

(ii) There exist four positive constants \( a_*, a^*, b_*, b^* \) such that, for all \( t, s \in [0, 1], \)

\[ a_* t^{\gamma - \mu_{n-1} - 1} j_A(s) \leq G(t, s) \leq a^* t^{\gamma - \mu_{n-1} - 1}, \]
\[ b_* t^{\delta - \eta_{m-1} - 1} j_B(s) \leq H(t, s) \leq b^* t^{\delta - \eta_{m-1} - 1}, \]

where

\[ a_* = \frac{\chi}{1 - F}, \quad a^* = \frac{\chi \| j_A(s) \|}{1 - F} + \frac{1}{\Gamma(\gamma - \mu_{n-1} - 1)}, \]
\[ b_* = \frac{\varphi}{1 - C}, \quad b^* = \frac{\varphi \| j_B(s) \|}{1 - C} + \frac{1}{\Gamma(\delta - \eta_{m-1} - 1)}. \]

**Proof.** The proof is similar to that for Lemma 2.2 in [27], we omit it here. \( \square \)

To study the PFDE (1), in what follows, we consider the associated linear PFDE:

\[
D^\alpha_0 \left( \varphi_p \left( D^\gamma_{0^+} x \right) \right)(t) + \rho(t) = 0, \quad 0 < t < 1,
\]

\[
x(0) = 0, \quad D^\gamma_{0^+} x(0) = 0, \quad x(1) = \chi \int_0^1 h(t)x(t) \, dA(t),
\]

(3)

\[
D^\beta_{0^+} \left( \varphi_p \left( D^\delta_{0^+} y \right) \right)(t) + \rho(t) = 0, \quad 0 < t < 1,
\]

\[
y(0) = 0, \quad D^\delta_{0^+} y(0) = 0, \quad y(1) = \mu \int_0^1 a(t)y(t) \, dB(t).
\]

(4)

**Lemma 5.** The PFDE (3), (4) have the unique positive solution

\[
x(t) = \int_0^1 G(t,s) \left( \int_0^s \alpha(s-\tau)^{\alpha-1} \rho(\tau) \, d\tau \right)^{q-1} ds, \quad t \in [0,1],
\]

(5)

\[
y(t) = \int_0^1 H(t,s) \left( \int_0^s b(s-\tau)^{\beta-1} \rho(\tau) \, d\tau \right)^{q-1} ds, \quad t \in [0,1],
\]

(6)

respectively, where \( \overline{\alpha} = 1/\Gamma(\alpha) , \overline{\beta} = 1/\Gamma(\beta) . \)

**Proof.** Let \( h = D^\gamma_{0^+} x \), \( k = \varphi_p(h) \), then the solution of the initial value problem

\[
D^\alpha_0 k(t) + \rho(t) = 0, \quad 0 < t < 1, \quad k(0) = 0
\]

is given by \( k(t) = C_1 t^{\alpha-1} - I^\alpha_{0^+} \rho(t) , t \in [0,1] \). By the relations \( k(0) = 0 \), we have \( C_1 = 0 \), and hence

\[
k(t) = -I^\alpha_{0^+} \rho(t) , \quad t \in [0,1].
\]

(7)

By \( D^\gamma_{0^+} x = h = \varphi_p^{-1}(k) \), we have from (7) that the solution of (3) satisfies

\[
D^\gamma_{0^+} x(t) = \varphi_p^{-1} \left( -I^\alpha_{0^+} \rho(t) \right) , \quad 0 < t < 1,
\]

\[
x(0) = 0, \quad x(1) = \chi \int_0^1 h(t)x(t) \, dA(t).
\]

(8)

By (2), the solution of Eq. (8) can be written as

\[
x(t) = -\int_0^1 G(t,s) \varphi_p^{-1} \left( -I^\alpha_{0^+} \rho(s) \right) ds, \quad t \in [0,1].
\]

(9)

Since \( \rho(s) \geq 0, s \in [0,1] \), we have \( \varphi_p^{-1} \left( -I^\alpha_{0^+} \rho(s) \right) = -\left( I^\alpha_{0^+} \rho(s) \right)^{q-1} , s \in [0,1] \), which implies that the solution of Eq. (3) is (5). Similarly, the solution of Eq. (4) is (6). \( \square \)

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Let \( u(t) = I_{0+}^{\nu_{n-1}} x(t), \) \( v(t) = I_{0+}^{\nu_{m-1}} y(t), \) problem (1) can turn into the following modified problem of the PFDE:

\[
\begin{align*}
D_{0+}^\alpha (\varphi_\nu (D_{0+}^{\delta_{n-1}} x)) (t) + \lambda^{1/(q-1)} f(t, I_{0+}^{\mu_{n-1}} x(t), I_{0+}^{\mu_{n-1} - \mu_1} x(t), \ldots, \\
I_{0+}^{\mu_{n-1} - \mu_2} x(t), x(t), I_{0+}^{\nu_{m-1}} y(t)) = 0, \quad 0 < t < 1, \\
D_{0+}^\beta (\varphi_\eta (D_{0+}^{\delta_{m-1}} y)) (t) + \mu^{1/(q-1)} g(t, I_{0+}^{\mu_{m-1}} x(t), I_{0+}^{\mu_{m-1} - \eta_1} x(t), \ldots, \\
I_{0+}^{\mu_{m-1} - \eta_2} x(t)) = 0, \quad 0 < t < 1,
\end{align*}
\]

\[ x(0) = 0, \quad D_{0+}^\gamma x(0) = 0, \quad x(1) = \chi (0^\eta \int_0^\eta h(t)x(t)\,dA(t), \quad y(0) = 0, \quad D_{0+}^\delta y(0) = 0, \quad y(1) = \mu (0^\varphi \int_0^\varphi a(t)y(t)\,dB(t). \]

**Lemma 6.** Let \( u(t) = I_{0+}^{\nu_{n-1}} x(t), \) \( v(t) = I_{0+}^{\nu_{m-1}} y(t), \) \( x(t), y(t) \in C[0, 1]. \) Then (1) can be transformed into (10). Moreover, if \((x, y) \in C[0, 1] \times C[0, 1] \) is a positive solution of problem (10), then \((I_{0+}^{\nu_{n-1}} x, I_{0+}^{\nu_{m-1}} y) \) is a positive solution of problem (1).

**Proof.** The proof is similar to that for Lemma 2.5 in [8], we omit it here. \( \square \)

The vector \((u, v)\) is a solution of system (1) if and only if \((x, y) \in C[0, 1] \times C[0, 1] \) is a solution of the following nonlinear integral equation system:

\[
\begin{align*}
x(t) &= \lambda \int_0^1 G(t, s) \left( \int_0^s \overline{\varphi}(s-\tau)^{\alpha-1} f(\tau, I_{0+}^{\mu_{n-1}} x(\tau), I_{0+}^{\mu_{n-1} - \mu_1} x(\tau), \ldots, \\
I_{0+}^{\mu_{n-1} - \mu_2} x(\tau), x(\tau), I_{0+}^{\nu_{m-1}} y(\tau)) \,d\tau \right)^{\gamma-1} ds, \quad t \in [0, 1], \\
y(\tau) &= \mu \int_0^1 H(\tau, s) \left( \int_0^s \overline{\varphi}(s-w)^{\beta-1} g(w, I_{0+}^{\mu_{m-1}} x(w), I_{0+}^{\mu_{m-1} - \eta_1} x(w), \ldots, \\
I_{0+}^{\mu_{m-1} - \eta_2} x(w)) \,dw \right)^{\varrho-1} dw, \quad t \in [0, 1].
\end{align*}
\]

Obviously, system (11) is equivalent to the following integral equation:

\[
\begin{align*}
x(t) &= \lambda \int_0^1 G(t, s) \left( \int_0^s \overline{\varphi}(s-\tau)^{\alpha-1} f(\tau, I_{0+}^{\mu_{n-1}} x(\tau), I_{0+}^{\mu_{n-1} - \mu_1} x(\tau), \ldots, \\
I_{0+}^{\mu_{n-1} - \mu_2} x(\tau), x(\tau)), \quad t \in [0, 1].
\end{align*}
\]
\[
I^{\eta_{m-1}}_{0^+} \left[ \mu \int_0^1 H(\tau, s) \left( \int_0^s b(s - w) \beta_{1-s} \gamma_{(\mu - 1)/\gamma - 1} g(w, I^{\mu_{n-1}}_{0^+} x(w), \ldots, \right) d\tau \right] \beta_{1-s} \gamma_{(\mu - 1)/\gamma - 1} \right] ds.
\]

Let \( P \) be a normal cone of a Banach space \( E \), and \( c \in P \), \( e > \theta \), where \( \theta \) is a zero element of \( E \). Define a component of \( P \) by \( Q_{c} = \{ y \in P \mid \text{there exists a constant } C \geq 1 \text{ such that } e/c \leq y \leq C e \} \). \( A : Q_c \times Q_c \to P \) is said to be mixed monotone if \( A(u, y) \) is nondecreasing in \( u \) and nonincreasing in \( y \), i.e., \( u_1 \leq u_2 \left( u_1, u_2 \in Q_c \right) \) implies \( A(u_1, y) \leq A(u_2, y) \) for any \( y \in Q_e \), and \( y_1 \leq y_2 \left( y_1, y_2 \in Q_c \right) \) implies \( A(u, y_1) \geq A(u, y_2) \) for any \( u \in Q_e \). The element \( x^* \in Q_c \) is called a fixed point of \( A \) if \( A(x^*, x^*) = x^* \).

**Lemma 7.** (See \([1, 5]\).) Suppose that \( A : Q_c \times Q_c \to Q_c \) is a mixed monotone operator and there exists a constant \( 0 < \sigma < 1 \) such that
\[
A \left( \lambda x, \frac{1}{\gamma} y \right) \geq t^\sigma A(x, y), \quad x, y \in Q_c, \quad 0 < l < 1,
\]
then \( A \) has a unique fixed point \( x^* \in Q_c \), and for any \( x_0 \in Q_c \), we have \( \lim_{k \to \infty} x_k = x^* \), where \( x_k = A(x_{k-1}, x_{k-1}) \left( k = 1, 2, \ldots \right) \), and the convergence rate is \( \|x_k - x^*\| = o(1 - r^k) \), where \( r \) is a constant dependent on \( x_0 \) and \( 0 < r < 1 \).

**Lemma 8.** (See \([1, 5]\).) Suppose that \( A : Q_c \times Q_c \to Q_c \) is a mixed monotone operator and there exists a constant \( \sigma \in (0, 1) \) such that (12) holds. If \( x_1^\lambda \) is a unique solution of equation \( \lambda A(x, x) = x \), \( \lambda > 0 \) in \( Q_c \), then:

(i) For any \( \lambda_0 \in (0, +\infty) \), \( \|x^\lambda - x^\lambda_0\| \to 0 \), \( \lambda \to \lambda_0 \);

(ii) If \( 0 < \sigma < 1/2 \), then \( 0 < \lambda_1 < \lambda_2 \) implies \( x^\lambda_1 \leq x^\lambda_2 \), \( x^\lambda_1 \neq x^\lambda_2 \);

(iii) If \( 0 < \sigma < 1/2 \), then \( \lim_{\lambda \to +\infty} \|x^\lambda\| = +\infty \), \( \lim_{\lambda \to 0^+} \|x^\lambda\| = 0 \).

Let \( c(t) = t^{\gamma-\mu_{n-1}-1} \) for \( t \in [0, 1] \), we define a normal cone of \( C[0, 1] \) by \( P = \{ x \in C[0, 1] : x(t) \geq 0, \ t \in [0, 1] \} \), and define a component of \( P \) by
\[
Q_c = \left\{ x \in P : \text{there exists } D \geq 1, \ \frac{1}{D^t} c(t) \leq x(t) \leq D c(t), \ t \in [0, 1] \right\}.
\]

**Remark 2.** Let \( s = \tau t \), by simple calculation, we have
\[
I^{\mu_{n-1}}_{0^+} e(t) = I^{\mu_{n-1}}_{0^+} t^{\gamma-\mu_{n-1}-1} = \frac{1}{\Gamma(\mu_{n-1})} \int_0^t (t-s)^{\mu_{n-1}-1} s^{\gamma-\mu_{n-1}-1} ds
\]
\[
= \frac{t^{\gamma-1}}{\Gamma(n-2)} \int_0^1 (1-\tau)^{\mu_{n-1}-1} \tau^{\gamma-\mu_{n-1}-1} d\tau = \frac{\Gamma(\gamma - \mu_{n-1})}{\Gamma(\gamma)} t^{\gamma-1}. \quad (13)
\]

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Similarly, we have
\[
\begin{align*}
I_{0^+}^{\eta_{m-1}}t^{\delta-\eta_{m-1}} &= \frac{\Gamma(\delta - \eta_{m-1})}{\Gamma(\delta)} t^{\delta-1}, \\
I_{0^+}^{\mu_{n-1}-\mu_{n}}e(t) &= \frac{\Gamma(\gamma - \mu_{n-1})}{\Gamma(\gamma - \mu_{n})} t^{\gamma - \mu_{n} - 1}, \quad \kappa = 1, 2, \ldots, n - 2, \quad (14) \\
I_{0^+}^{\mu_{p-1}-\mu_{p}}e(t) &= \frac{\Gamma(\gamma - \mu_{p-1})}{\Gamma(\gamma - \eta_{p})} t^{\gamma - \eta_{p} - 1}, \quad \rho = 1, 2, \ldots, m - 2. \quad (15)
\end{align*}
\]

3 Main results

**Theorem 1.** Suppose that (S1)--(S4) hold. Then for all \( t \in [0, 1] \), the PFDE (1) has a unique positive solution \((u^*_\lambda, v^*_\lambda)\), which satisfies
\[
\begin{align*}
\frac{\Gamma(\gamma - \mu_{n-1})}{\Gamma(\gamma)} t^{\gamma-1} &\leq u^*_\lambda(t) \leq \frac{D\Gamma(\gamma - \mu_{n-1})}{\Gamma(\gamma)} t^{\gamma-1}, \\
\mu b_1 \Gamma(\delta - \eta_{m-1}) t^{\delta-1-\frac{1}{2}} &\leq v^*_\lambda(t) \leq \frac{\Gamma(\delta - \eta_{m-1}) b^* \mu t^{\delta-1-\frac{1}{2}}}{\Gamma(\delta)(2\beta - 1)^{(\eta - 1)/2}} K_1,
\end{align*}
\]
where \( K_1, K_2 \) are two positive constants, and at the same time, \( u^*_\lambda \) satisfies:

(i) For \( \lambda_0 \in (0, \infty), \|u^*_\lambda - u^*_\lambda^0\| \to 0, \lambda \to \lambda_0; \)

(ii) If \( 0 < \sigma < 1/2 \), then \( 0 < \lambda_1 < \lambda_2 \) implies \( u^*_\lambda \leq u^*_{\lambda_2}, u^*_\lambda \neq u^*_{\lambda_2} \);

(iii) If \( 0 < \sigma < 1/2 \), then \( \lim_{\lambda \to 0} \|u^*_\lambda\| = 0, \lim_{\lambda \to +\infty} \|u^*_\lambda\| = +\infty. \)

Moreover, for any \( u_0 \in Q_\varepsilon \), constructing a successively sequence:
\[
\begin{align*}
&u_{k+1}(t) = I_{0^+}^{\mu_{n-1}} \left\{ \lambda \int_0^1 G(t, s) \left[ \int_0^s \varphi(s - \tau)^{\alpha-1} \left( \phi(\tau, u_k(\tau), D^\mu_{0^+} u_k(\tau), \ldots, \\
&\quad D^\mu_{0^+} u_k(\tau), A u_k^{(\mu_{n-1})}(\tau) \right) + \psi(\tau, u_k(\tau), D^\mu_{0^+} u_k(\tau), \ldots, \\
&\quad D^\mu_{0^+} u_k(\tau), A u_k^{(\mu_{n-1})}(\tau)) \right] d\tau \right)^{q-1} ds, \quad k = 0, 1, 2, \ldots, t \in [0, 1],
\end{align*}
\]
and we have \( \|u_k - u^*_\lambda\| \to 0 \) as \( k \to \infty \), the convergence rate is \( \|u_k - u^*_\lambda\| = o(1 - \tau^r) \), where \( r \) is a constant, \( 0 < r < 1 \), and dependent on \( u_0 \).

**Proof.** We now consider the existence of a positive solution to problem (1). From the discussion in Section 2 we only need to consider the existence of a positive solution to PFDE (11). In order to realize this purpose, define the operator \( A : Q_\varepsilon \to P \) by
\[
\begin{align*}
&Ah(\tau) = I_{0^+}^{\eta_{m-1}} \left\{ \mu \int_0^1 H(\tau, s) \left[ \int_0^s b(s - w)^{\beta-1} g(w, I_{0^+}^{\mu_{n-1}} x(w), I_{0^+}^{\mu_{n-1}-\eta_{m-1}} x(w)), \ldots, \\
&\quad I_{0^+}^{\mu_{n-1}-\eta_{m-1}} x(w)) dw \right]^{q-1} ds, \quad \tau \in [0, 1],
\end{align*}
\]
and define the operator $T_{\lambda} : Q_e \times Q_e \rightarrow P$ by

$$T_{\lambda}(x, z)(t) = \lambda \int_0^1 G(t, s) \left[ \int_0^s \pi(s - \tau)^{\alpha - 1} (\phi(\tau, I_{0+}^{\mu_{n-1}} x(\tau)), \ldots, I_{0+}^{\mu_{n-1}} x(\tau), z(\tau), \ldots) \right] d\tau ds, \quad t \in [0, 1].$$

Now we prove that $T_{\lambda} : Q_e \times Q_e \rightarrow P$ is well defined. For any $x, z \in Q_e$, by (16), (S3), (13), (15), and Remark 1, for all $\tau \in [0, 1]$, we have

$$\mu \int_0^1 H(\tau, s) \left[ \int_0^s b(s - w)^{\beta - 1} g(w, I_{0+}^{\mu_{n-1}} x(w), I_{0+}^{\mu_{n-1}} x(w), \ldots, I_{0+}^{\mu_{n-1}} x(w)) dw \right]^{q-1} ds \leq \mu b^{s} \tau^{\delta} \mu_{n-1}^{q-1} \int_0^1 \left[ \int_0^s b(s - w)^{\beta - 1} g(w, D\Gamma(\gamma - \mu_{n-1}) \Gamma(\gamma) w^{\gamma - 1}, D\Gamma(\gamma - \mu_{n-1}) \Gamma(\gamma - \eta_{n-1}) w^{\gamma - \eta_{n-1}} dw \right]^{q-1} ds \leq \mu b^{s} \tau^{\delta} \mu_{n-1}^{q-1} \int_0^1 \left[ \int_0^s b(s - w)^{\beta - 1} g(w, D\Gamma(\gamma - \mu_{n-1}) \Gamma(\gamma - \eta_{n-1}) + 1, D\Gamma(\gamma - \mu_{n-1}) \Gamma(\gamma - \eta_{n-1}) + 1 dw \right]^{q-1} ds \leq \mu b^{s} \tau^{\delta} \mu_{n-1}^{q-1} \int_0^1 \left[ \int_0^s b(s - w)^{\beta - 1} g(w, 1, 1, \ldots, 1 dw \right]^{q-1} ds,$$  

$$\mu \int_0^1 H(\tau, s) \left[ \int_0^s b(s - w)^{\beta - 1} g(w, I_{0+}^{\mu_{n-1}} x(w), I_{0+}^{\mu_{n-1}} x(w), \ldots, I_{0+}^{\mu_{n-1}} x(w)) dw \right]^{q-1} ds \leq \mu b^{s} \tau^{\delta} \mu_{n-1}^{q-1} \int_0^1 \left[ \int_0^s b(s - w)^{\beta - 1} g(w, 1, 1, \ldots, 1 dw \right]^{q-1} ds,$$  

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Hence, by (16), (17), (18), and the Hölder inequality, for \( \tau \in [0, 1] \), we have

\[
Ax(\tau) = I_{0}^{\sigma_{m-1}} \left( \mu \int_{0}^{1} H(\tau, s) \left[ \int_{0}^{s} b(s - w) \gamma_{1}^{-1} g(w, I_{0}^{\sigma_{m-1}} x(w), \right. \right. \right.

\left. \left. I_{0}^{T_{m-1} - \eta_{1}} x(w), \ldots, I_{0}^{T_{m-1} - \eta_{m-1}} x(w) \right] dw \right]^q ds \right)

\leq \frac{\Gamma(\delta - \eta_{m-1}) b^{*} \mu^{\gamma_1-1} b^{1-\gamma_1}}{\Gamma(\delta)(2\beta - 1)(q-1/2)} \left( \frac{D \Gamma(\gamma - \mu_{n-1})}{\Gamma(\gamma - \eta_{m-1})} + 1 \right)^{\gamma_1 - \gamma_2}

\times \int_{0}^{1} \left[ s^{(2\beta - 1)/2} \left( \int_{0}^{s} g^2(w, 1, 1, \ldots, 1) dw \right)^{1/2} q^{-1} ds \right],

Ax(\tau) = I_{0}^{\sigma_{m-2}} \left( \mu \int_{0}^{1} H(\tau, s) \left[ \int_{0}^{s} b(s - w) \gamma_{2}^{-1} g(w, I_{0}^{\sigma_{m-2}} x(w), \right. \right. \right.

\left. \left. I_{0}^{T_{m-2} - \eta_{2}} x(w), \ldots, I_{0}^{T_{m-2} - \eta_{m-2}} x(w) \right] dw \right]^q ds \right)

\leq \frac{\mu b_{*} \Gamma(\delta - \eta_{m-2}) \tau^{\gamma_2-1} b^{1-\gamma_2}}{\Gamma(\delta)} \left( \frac{D \Gamma(\gamma - \mu_{n-2})}{\Gamma(\gamma - \eta_{m-2})} \right)^{\gamma_2}

\times \int_{0}^{1} j_{B}(s) \left[ \int_{0}^{s} w^{(\gamma_2-1)}(s - w) \gamma_2^{-1} g(w, 1, 1, \ldots, 1) dw \right] q^{-1} ds. \quad (19)
By (S4), we get that $Ax(\tau)$ is well defined. From (13), (14), (19), (S1), and Remark 1 we have

$$
\phi(\tau, I_{0+}^{\mu_0-1} x(\tau), I_{0+}^{\mu_0-1-\mu_1} x(\tau), \ldots, I_{0+}^{\mu_0-1-\mu_{n-2}} x(\tau), x(\tau), Ax(\tau))
\leq \phi \left( \tau, \frac{D \Gamma(\gamma - \mu_{n-1})}{\Gamma(\gamma)} \tau^{\gamma-1} + 1, \ldots, \frac{D \Gamma(\gamma - \mu_{n-1})}{\Gamma(\gamma - \mu_{n-2})} \tau^{\gamma-\mu_{n-2}-1} + 1, \right.
$$

$$
D \tau^{\gamma-\mu_{n-1}-1}, \frac{\Gamma(\delta - \eta_{n-1})}{\Gamma(\delta)} D^{(2\beta - 1)(q-1)/2} \left( \frac{D \Gamma(\gamma - \mu_{n-1})}{\Gamma(\gamma - \mu_{n-1})} + 1 \right)^\xi
$$

$$
\times \left[ \int_0^1 \left( \int_0^s g^\beta(w, 1, 1, \ldots, 1) dw \right)^{1/2} ds + 1 \right]
$$

$$
\leq \phi(\tau, D b + 1, D b + 1, \ldots, D^\xi b + 1)
$$

$$
\leq 2^{\sigma_{n/(q-1)}} b^{\sigma_{n/(q-1)}} D^{\sigma_{n/(q-1)}} \phi(\tau, 1, 1, \ldots, 1), \quad \tau \in (0, 1),
$$

(20)

where $D \geq 1$, $b$ are two positive constants. By (13), (14), (19), (S1), and (S2), we also have

$$
\psi(\tau, I_{0+}^{\mu_0-1} x(\tau), I_{0+}^{\mu_0-1-\mu_1} x(\tau), \ldots, I_{0+}^{\mu_0-1-\mu_{n-2}} x(\tau), x(\tau), Ax(\tau))
\leq \psi \left( \tau, \frac{\Gamma(\gamma - \mu_{n-1})}{\Gamma(\gamma)} \tau^{\gamma-1} + 1, \frac{\Gamma(\gamma - \mu_{n-1}) \tau^{\gamma-\mu_{n-1}}}{\Gamma(\gamma - \mu_{n-2})} \frac{D \Gamma(\gamma - \mu_{n-1})}{\Gamma(\gamma - \mu_{n-1})}, \ldots, \frac{D \Gamma(\gamma - \mu_{n-1})}{\Gamma(\gamma - \mu_{n-1})} \tau^{\gamma-\mu_{n-2}-1}, \right.
$$

$$
\leq \frac{\mu b \Gamma(\delta - \eta_{n-1})}{\Gamma(\delta)} D^{(2\beta - 1)(q-1)/2} \left( \frac{D \Gamma(\gamma - \mu_{n-1})}{\Gamma(\gamma - \mu_{n-1})} + 1 \right)^\xi
$$

$$
\times \left[ \int_0^1 j_B(s) \left[ \int_0^s w^{(\gamma-1)\gamma_{n-1}} (s - w)^{\beta-1} g(w, 1, 1, \ldots, 1) dw \right]^{q-1} ds \right]
$$

$$
\leq \psi \left( \tau, \frac{c}{D} \tau^{\gamma-1}, \frac{c}{D} \tau^{\gamma-1}, \ldots, \frac{c}{D} \tau^{\gamma-1} \right)
$$

$$
= c^{-\sigma_{n/(q-1)}} D^{\sigma_{n/(q-1)}} \tau^{(\gamma-1)\gamma_{n-1}} \psi(\tau, 1, 1, \ldots, 1), \quad \tau \in (0, 1),
$$

(21)

where $c$ is a positive constant. Noting $(c/D)\tau^{\gamma-1} < 1$ and by (13), (14), (19), (S1), and (S2), we have

$$
\phi(\tau, I_{0+}^{\mu_0-1} x(\tau), I_{0+}^{\mu_0-1-\mu_1} x(\tau), \ldots, I_{0+}^{\mu_0-1-\mu_{n-2}} x(\tau), x(\tau), Ax(\tau))
\geq \phi \left( \tau, \frac{\Gamma(\gamma - \mu_{n-1})}{\Gamma(\gamma)} \tau^{\gamma-1} + 1, \frac{\Gamma(\gamma - \mu_{n-1}) \tau^{\gamma-\mu_{n-1}}}{\Gamma(\gamma - \mu_{n-2})} \frac{D \Gamma(\gamma - \mu_{n-1})}{\Gamma(\gamma - \mu_{n-1})}, \ldots, \frac{\Gamma(\gamma - \mu_{n-1}) \tau^{\gamma-\mu_{n-2}-1}}{\Gamma(\gamma - \mu_{n-2})} \frac{D \Gamma(\gamma - \mu_{n-1})}{\Gamma(\gamma - \mu_{n-1})}, \right.
$$

(22)

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For any $x, z \in Q_\epsilon$, it follows from (20), (21) that

\[
T_x(y, z)(t) = \lambda \int_0^1 G(t, s) \left[ \int_0^s \pi(s - r)^{\alpha - 1} \phi(r, I_{0+}^{\mu_{n-1}} x(r), I_{0+}^{\mu_{n-1} - \mu_1} x(r), \ldots, I_{0+}^{\mu_{n-1} - \mu_2} x(r), x(r), Ax(r)) + \psi(r, I_{0+}^{\mu_{n-1}} z(r), I_{0+}^{\mu_{n-1} - \mu_1} z(r), \ldots, I_{0+}^{\mu_{n-1} - \mu_2} z(r), z(r), Az(r)) \right] dr \right] ds \leq \lambda \alpha^s t^{\gamma - \mu_{n-1} - 1} D^{\alpha} \pi^{\alpha - 1} \int_0^1 \left( 2^{\alpha^{1/(q-1)}} b^{\sigma^{1/(q-1)}} \right)^{1/2} \frac{1}{(2\alpha - 1)^{1/2}} s^{(2\alpha - 1)/2} \left( \int_0^s \phi^2(t, 1, 1, \ldots, 1) d\tau \right)^{1/2} + c^{\sigma^{1/(q-1)}} \frac{1}{(2\alpha - 1)^{1/2}} s^{(2\alpha - 1)/2}
\]

Formula (25) imply that
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By (S4), (24), we have that
\[ T(t) \]
At the same time, by (22) and (23), for
\[ \lambda \geq \lambda a \]
Next, we will prove
\[ x, z \]
\[ I(s + t) \]
\[ \int_0^s \int_0^t \phi^2(\tau, 1, 1, \ldots, 1) d\tau \] \[ q^{-1} \]
\[ ds \]
\[ \leq \lambda a \times \left( \int_0^s \int_0^t b \phi^2(\tau, 1, 1, \ldots, 1) d\tau \right) \]
\[ \times \left( \int_0^s \int_0^t c^{-\sigma^1/(q - 1)} \phi^2(\tau, 1, 1, \ldots, 1) d\tau \right) \]
\[ q^{-1} \]
\[ ds \]
\[ < +\infty, \quad t \in [0, 1]. \] \[ (24) \]
By (S4), (24), we have that \( T_\lambda : Q_e \times Q_e \to P \) is well defined.
Next, we will prove \( T_\lambda : Q_e \times Q_e \to Q_e \). Formula (24) imply that
\[ T_\lambda(x, z)(t) \leq DT^{\gamma - n + 1} = De(t), \quad t \in [0, 1]. \]
At the same time, by (22) and (23), for \( t \in [0, 1] \), we have
\[ T_\lambda(x, z)(t) \]
\[ = \lambda \int_0^1 \int G(t, s) \left[ \int_0^s \left( \int_0^t \alpha(s - \tau)^{\alpha - 1} \left( \phi(\tau, I_{0+}^{\mu - 1} \tau, I_{0+}^{\mu - 1} \tau, \ldots, 
\right) I_{0+}^{\mu - 1} \tau, x(\tau), Az(\tau), z(\tau) \right) \right] d\tau \left[ \int_0^s \left( \int_0^t \alpha(s - \tau)^{\alpha - 1} \left( \phi(\tau, I_{0+}^{\mu - 1} \tau, I_{0+}^{\mu - 1} \tau, \ldots, \right) I_{0+}^{\mu - 1} \tau, x(\tau), Az(\tau), z(\tau) \right) \right] d\tau \right] q^{-1} \]
\[ ds \]
\[ \geq \lambda \int_0^1 \int G(t, s) \left[ \int_0^s \left( \int_0^t \alpha(s - \tau)^{\alpha - 1} \left( e^{s^1/(q - 1)} D^{-\sigma^1/(q - 1)} D^{-\gamma \sigma^1/(q - 1)} \psi(\tau, 1, 1, \ldots, 1) \right) d\tau \right] q^{-1} \]
\[ ds \]
\[ \geq \lambda a \times \left( \int_0^1 \int_0^s \int_0^t \phi(\tau, 1, 1, \ldots, 1) + 2^{-\sigma^1/(q - 1)} b^{-\sigma^1/(q - 1)} D^{-\gamma \sigma^1/(q - 1)} \psi(\tau, 1, 1, \ldots, 1) d\tau \right] q^{-1} \]
\[ ds. \] \[ (25) \]
Formula (25) imply that
\[ T_\lambda(x, z)(t) \geq \frac{1}{D} T^{\gamma - n - 1} = \frac{1}{D} e(t), \quad t \in [0, 1]. \]
Hence, \( T_\lambda : Q_e \times Q_e \rightarrow Q_e \). It is easy to prove that \( T_\lambda : Q_e \times Q_e \rightarrow Q_e \) is a mixed monotone operator.

Finally, we show that the operator \( T_\lambda \) satisfies (12). For any \( x, z \in Q_e \) and \( t \in (0, 1) \), by (S2) and Remark 1, for all \( t \in [0, 1] \), we have

\[
\lambda \int_0^1 G(t, s) \left[ \int_0^s \pi(s - \tau)^{\sigma - 1} \left( \phi(\tau, I_0^\mu x(\tau), I_0^\mu x(\tau), \ldots, 
I_0^\mu x(\tau), I_0^\mu x(\tau)) + \psi(\tau, I_0^\mu x(\tau), I_0^\mu x(\tau), \ldots, 
I_0^\mu x(\tau), I_0^\mu x(\tau)) \right) d\tau \right] \right] q-1 ds 
\geq \lambda \int_0^1 G(t, s) \left[ \int_0^s \pi(s - \tau)^{\sigma - 1} \left( \phi(\tau, I_0^\mu x(\tau), I_0^\mu x(\tau), \ldots, 
I_0^\mu x(\tau), I_0^\mu x(\tau)) + \psi(\tau, I_0^\mu x(\tau), I_0^\mu x(\tau), \ldots, 
I_0^\mu x(\tau), I_0^\mu x(\tau)) \right) d\tau \right] q-1 ds.
\]

Formula (26) imply that

\[
T_\lambda \left( Ix, \frac{1}{\tau} z \right) \geq l^\sigma T_\lambda(x, z), \quad x, z \in Q_e.
\]

Hence, Lemma 7 assume that there exists a unique positive solution \( x_\lambda^* \in Q_e \) such that 
\( T_\lambda(x_\lambda^*, x_\lambda^*) = x_\lambda^* \). It is easy to check that \( x_\lambda^* \) is a unique positive solution of (10) for any given \( \lambda > 0 \). Moreover, by Lemma 8, we have:

(i) For any \( \lambda_0 \in (0, +\infty) \), \( \|x_\lambda^* - x_{\lambda_0}^*\| \rightarrow 0 \), \( \lambda \rightarrow \lambda_0 \);
(ii) If \( 0 < \sigma < 1/2 \), then \( 0 < \lambda_1 < \lambda_2 \) implies \( x_{\lambda_2} < x_{\lambda_1} \); \( x_{\lambda_1} \neq x_{\lambda_2} \);
(iii) If \( 0 < \sigma < 1/2 \), then \( \lim_{\lambda \to 0} \|x_\lambda^*\| = 0 \), \( \lim_{\lambda \to +\infty} \|x_\lambda^*\| = +\infty \).

By Lemma 6, for any \( t \in [0, 1] \), we have

\[
u_\lambda^*(t) = I_0^\mu x_\lambda^*(t), \quad \nu_\lambda^*(t) = I_0^\mu y_\lambda^*(t).
\]

Hence, by (27) and the monotonicity and continuity of \( I_0^\mu \), we get:
(i) For any \( \lambda_0 \in (0, +\infty) \), \( \|u^*_\lambda - u^*_\lambda_0\| \to 0 \), \( \lambda \to \lambda_0 \);
(ii) If \( 0 < \sigma < 1/2 \), then \( 0 < \lambda_1 < \lambda_2 \) implies \( u^*_{\lambda_1} \leq u^*_{\lambda_2}, u^*_{\lambda_1} \neq u^*_{\lambda_2} \);
(iii) If \( 0 < \sigma < 1/2 \), then \( \lim_{\lambda \to 0} \|u^*_{\lambda}\| = 0 \), \( \lim_{\lambda \to +\infty} \|u^*_{\lambda}\| = +\infty \).

Moreover, for any \( u_0(t) = I_{0+}^{n-2}x_0 \in Q_\varepsilon \) and \( t \in [0,1] \), by Lemma 7, constructing a successively sequence

\[
x_{k+1}(t) = \lambda \int_0^1 G(t,s) \left[ \int_0^s \pi(s-\tau)^{\alpha-1} \left( \phi(\tau, u_k(\tau), D_{0+}^{\mu} u_k(\tau)), \ldots, \right.ight.
\]

\[
I_{0+}^{\mu-n-2}x_k(\tau), x_k(\tau), Ax_k(\tau) + \psi(\tau, I_{0+}^{\mu-n-1}x_k(\tau), I_{0+}^{\mu-n-1}x_k(\tau)), \ldots,
\]

\[
I_{0+}^{\mu-n-2}x_k(\tau), x_k(\tau), Ax_k(\tau)) \right] q^{-1} ds, \quad k = 0, 1, 2, \ldots,
\]

by \( u_{k+1}(t) = I_{0+}^{\mu-n-1}x_{k+1}(t), \)

\[
u_{k+1}(t) = I_{0+}^{\mu-n-1} \left\{ \lambda \int_0^1 G(t,s) \left[ \int_0^s \pi(s-\tau)^{\alpha-1} \left( \phi(\tau, u_k(\tau), D_{0+}^{\mu} u_k(\tau)), \ldots, \right.ight.
\]

\[
D_{0+}^{\mu-n-1}u_k(\tau), u_k(\tau), Au_k^{(\mu-n-1)}(\tau)) + \psi(\tau, u_k(\tau), D_{0+}^{\mu} u_k(\tau)), \ldots,
\]

\[
D_{0+}^{\mu-n-1}u_k(\tau), u_k(\tau), Au_k^{(\mu-n-1)}(\tau)) \right] q^{-1} ds, \quad k = 0, 1, 2, \ldots, t \in [0,1],
\]

and we have \( \|u_k - u^*_\lambda\| = \|I_{0+}^{\mu-n-1}x_k - I_{0+}^{\mu-n-1}x^*_\lambda\| \to 0 \) as \( k \to \infty \), the convergence rate is

\[
\|u_k - u^*_\lambda\| = \|I_{0+}^{\mu-n-1}x_k - I_{0+}^{\mu-n-1}x^*_\lambda\| = o(1 - r^{\sigma^m}),
\]

where \( r \) is a constant, \( 0 < r < 1 \), and dependent on \( u_0 \). By (11), we easily get

\[
g^*_\lambda(t) = \mu \int_0^1 H(t,s) \left[ \int_0^s 5(s-w)^{\beta-1} g(w, I_{0+}^{\mu-n-1}x^*_\lambda(w), I_{0+}^{\mu-n-1}x^*_\lambda(w)), \ldots,
\]

\[
I_{0+}^{\mu-n-1}x^*_\lambda(w)) dw \right] q^{-1} ds, \quad t \in [0,1].
\]

By (19), (27), (28), and \( x^*_\lambda \in Q_\varepsilon \), we get \( (u^*_\lambda, v^*_\lambda) \), which satisfies (+). Therefore, the proof of Theorem 1 is completed.

**Remark 3.** \( p \)-Laplacian boundary value system is an great extension from general fractional-order differential equation, general fractional-order differential equation is a special case of \( p \)-Laplacian fractional-order differential equation, \( p \)-Laplacian fractional-order differential equation is general fractional-order differential equation when \( q = 2 \).

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4 An example

Example 1. Consider the following boundary value problem:

\[ D_0^{3/4} \left( \varphi_2 (D_0^{5/2} u) \right)(t) + \lambda^2 f(t, u(t), D_0^{1/2} u(t), v(t)) = 0, \quad 0 < t < 1, \]

\[ D_0^{3/4} \left( \varphi_2 (D_0^{5/2} v) \right)(t) + \mu^2 g(t, u(t)) = 0, \quad 0 < t < 1, \]

\[ u(0) = u'(0) = 0, \quad D_0^2 u(0) = D_0^{1/2} u(0) = 0, \quad u'(1) = \frac{1}{2} \int_0^{3/4} u'(t) \, dA(t), \quad (29) \]

\[ v(0) = v'(0) = 0, \quad D_0^2 v(0) = D_0^{1/2} v(0) = 0, \quad v(1) = \frac{5}{6} \int_0^{3/4} v(t) \, dB(t), \]

where \( \gamma = 5/2, \delta = 3/2, \alpha = \beta = 3/4, h(s) = a(s) = 1, \eta = \vartheta = 3/4, \chi = 1/2, \]
\( \iota = 5/6, p = 3, q = 3/2, \) and

\[ \phi(t, x_1, x_2, x_3) = (t^{-1/4} + \cos t)x_1^{1/9} + 2tx_2^{1/8} + 2x_3^{1/16}, \]

\[ \psi(t, x_1, x_2, x_3) = t^{-1/8}x_1^{-1/8} + x_2^{-1/16} + (2 - t)x_3^{-1/15}, \]

\[ g(t, u) = (3t + t^2)u^{3/5} + (t \sin t + t)u^{2/3}, \]

\[ A(t) = \begin{cases} 0, & t \in [0, 1/2), \\ 6, & t \in [1/2, 3/4), \\ 2, & t \in [3/4, 1]. \end{cases} \]

\[ B(t) = \begin{cases} 0, & t \in [0, 1/2), \\ 4, & t \in [1/2, 3/4), \\ 3, & t \in [3/4, 1]. \end{cases} \]

Hence,

\[ \int_0^{\eta} t^{\gamma-n+1} h(t) \, dA(t) = \frac{1}{2} \int_0^{3/4} t^{1/2} \, dA(t) = 3 \left( \frac{1}{2} \right)^{1/2} - 2 \left( \frac{3}{4} \right)^{1/2} < 1, \]

\[ \int_0^{\delta} t^{\delta-m+1} a(t) \, dB(t) = \frac{5}{6} \int_0^{3/4} t^{1/2} \, dB(t) \approx 0.1117 < 1. \]

Moreover, for any \( (t, x_1, x_2, x_3) \in (0, 1) \times (0, \infty)^3 \) and \( 0 < t < 1, \) we have

\[ \phi(t, lx_1, lx_2, lx_3) = (t^{-1/4} + \cos t)(lx_1)^{1/9} + 2tlx_2^{1/8} + 2l(x_3)^{1/16} \]

\[ \geq l^{1/8} \left( (t^{-1/4} + \cos t)x_1^{1/9} + 2tx_2^{1/8} + 2x_3^{1/16} \right) \]

\[ = l^{1/8} \phi(t, x_1, x_2, x_3) = l^{1/(n-1)} \phi(t, x_1, x_2, x_3), \]

\[ \psi(t, l^{-1}x_1, l^{-1}x_2, l^{-1}x_3) = l^{-1/16}(l^{-1}x_1)^{-1/8} + (l^{-1}x_2)^{-1/16} + (2 - t)(l^{-1}x_3)^{-1/15} \]
Noting $\sigma = 1/(2\sqrt{2}) < 1$, $\varsigma = \sqrt{6}/3$, $\psi(\tau, 1, 1, 1) = \tau^{-1/6} + 3 - \tau$, $\phi(\tau, 1, 1, 1) = \tau^{-1/4} + \cos \tau + 2\tau + 2$, $g(\tau, 1) = 3\tau + \tau^2 + \tau \sin \tau + \tau$, we have

$$
0 < \int_0^1 \phi^2(\tau, 1, 1, \ldots, 1) \, d\tau = \int_0^1 (\tau^{-1/4} + \cos \tau + 2\tau + \tau)^2 \, d\tau \leq 27 + \frac{16}{7}
$$

$$
< +\infty,
$$

$$
0 < \int_0^1 \tau^{-2(\gamma-1)} \psi^2(\tau, 1, 1, \ldots, 1) \, d\tau \leq \int_0^1 \tau^{-3/8} (\tau^{-1/6} + 3)^2 \, d\tau
$$

$$
= 11 + \frac{96}{9} < +\infty,
$$

$$
0 < \int_0^1 g^2(\tau, 1, 1, \ldots, 1) \, d\tau \leq \int_0^1 (3\tau + \tau^2 + \tau \sin \tau + \tau)^2 \, d\tau \leq 36 < +\infty.
$$

Thus, assumptions (S1)–(S4) of Theorem 1 hold. Then Theorem 1 implies that problem (29) has a unique solution. Furthermore, when $\lambda \to \lambda_0$, $\lambda_0 \in (0, +\infty)$, we have $\|x^*_\lambda - x^*_{\lambda_0}\| \to 0$. Since $\sigma = 1/(2\sqrt{2}) \in (0, 1/2)$, $0 < \lambda_1 < \lambda_2$ implies $x^*_{\lambda_1}(t) \leq x^*_{\lambda_2}(t)$, $x^*_{\lambda_1}(t) \neq x^*_{\lambda_2}(t)$, $\lim_{\lambda \to 0} \|x^*_{\lambda}\| = 0$, $\lim_{\lambda \to +\infty} \|x^*_{\lambda}\| = +\infty$.

By $u^*_{\lambda_1} = I^1_{0^+} x^*_{\lambda_1}(t)$, we can easily get that:

(i) $\lambda_0 \in (0, +\infty)$, $\|u^*_{\lambda} - u^*_{\lambda_0}\| \to 0$, $\lambda \to \lambda_0$;

(ii) $0 < \lambda_1 < \lambda_2$ implies $u^*_{\lambda_1} \leq u^*_{\lambda_2}$, $u^*_{\lambda_1} \neq u^*_{\lambda_2}$;

(iii) $\lim_{\lambda \to 0} \|u^*_{\lambda}\| = 0$, $\lim_{\lambda \to +\infty} \|u^*_{\lambda}\| = +\infty$.

In addition, for any initial $u_0 = I^1_{0^+} x_0 \in Q_e$, we construct a successively sequence $x_{k+1}(t)$

$$
x_{k+1}(t) = \lambda \int_0^1 G(t, s) \left[ \int_0^s \pi(s - \tau)^{\alpha-1} \left( \phi(t, I^1_{0^+} x_k(t), I^{1/2}_{0^+} x_k(t), A x_k(t)) + \psi(t, I^1_{0^+} x_k(t), I^{1/2}_{0^+} x_k(t), A x_k(t)) \right) \, d\tau \right]^{q-1} \, ds, \quad t \in [0, 1], \quad k = 0, 1, 2, \ldots,
$$

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by \( u_{k+1}(t) = I_{0+}^1 x_{k+1}(t) \), we have

\[
\begin{aligned}
  u_{k+1}(t) & = I_{0+}^1 \left\{ \int_0^1 \lambda G(t,s) \left[ \int_0^s \mathfrak{G}(s - \tau)^{\alpha - 1} \left( \phi(t, u_k(t), D_{0+}^{1/2} u_k(t), Au_k'(t)) \right) \right]^q \, \, ds \right\}^{q-1} \, \, d\tau, \\
\end{aligned}
\]

\( t \in [0,1], k = 0, 1, 2, \ldots \),

and we have \( \| u_k - u_\lambda^* \| = \| I_{0+}^1 x_k - I_{0+}^1 x_\lambda^* \| \to 0 \) as \( k \to \infty \), the convergence rate is

\[
\| u_k - u_\lambda^* \| = \| I_{0+}^1 x_k - I_{0+}^1 x_\lambda^* \| = o(1 - r^{\sigma_k}),
\]

where \( r \) is a constant dependent on \( u_0 \) and \( 0 < r < 1 \).

**Acknowledgment.** The authors would like to thank the referee for his/her valuable comments and suggestions.

**References**


