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A note on the asymptotics for the randomly stopped weighted sums*

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Abstract. Let $\{X_i,\, i \geqslant 1\}$ be a sequence of identically distributed real-valued random variables with common distribution F_X; let $\{\theta_i,\, i \geqslant 1\}$ be a sequence of identically distributed, nonnegative and nondegenerate at zero random variables; and let τ be a positive integer-valued counting random variable. Assume that $\{X_i,\, i \geqslant 1\}$, $\{\theta_i,\, i \geqslant 1\}$ and τ are mutually independent. In the presence of heavy-tailed X_i's, this paper investigates the asymptotic tail behavior for the maximum of randomly weighted sums $M_\tau = \max_{1\leqslant k\leqslant \tau} \sum_{i=1}^k \theta_i X_i$ under the condition that $\{\theta_i,\, i \geqslant 1\}$ satisfy a general dependence structure.

Keywords: asymptotics, randomly weighted sums, heavy tail, widely orthant dependence.

1 Introduction

Let $\{X_i,\, i \geqslant 1\}$ be a sequence of identically distributed real-valued random variables (r.v.s) with common distribution F_X; let $\{\theta_i,\, i \geqslant 1\}$ be a sequence of nonnegative and nondegenerate at zero r.v.s, which may be arbitrarily dependent; and let τ be a positive integer-valued r.v. Assume that $\{X_i,\, i \geqslant 1\}$, $\{\theta_i,\, i \geqslant 1\}$ and τ are mutually independent. Many researchers have focused themselves on the asymptotic tail behavior for the randomly weighted sums, denoted by

$$S_\tau = \sum_{i=1}^{\tau} \theta_i X_i, \tag{1}$$

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and their maximum $M_\tau = \max_{1\leqslant k\leqslant \tau} \sum_{i=1}^{k} \theta_i X_i$. Under the conditions that $\{X_i,\, i \geqslant 1\}$ are independent and identically distributed (i.i.d.) and $\{\theta_i,\, i \geqslant 1\}$ are arbitrarily dependent, some earlier works have been devoted to the investigation of the asymptotic tail behavior for the finite or infinite number of randomly weighted sums or their maximum, i.e., $\mathbf{P}(S_n > x)$ and $\mathbf{P}(M_n > x)$ (or $\mathbf{P}(S_\infty > x)$ and $\mathbf{P}(M_\infty > x)$) as $x \to \infty$. [7] considered the case that the θ_i's are bounded and obtained the precise asymptotic formula

$$\mathbf{P}(M_n > x) \sim \mathbf{P}(S_n > x) \sim \sum_{i=1}^{n} \mathbf{P}(\theta_i X_i > x) \tag{2}$$

for any fixed $n \geqslant 1$ under the condition that X_i's have a common subexponential tail. Later, [11] derived relation (2) without the restriction of bounded θ_i's, however, they required that X_i's distributions are long-tailed and dominatedly-varying-tailed. We remark that the condition of bounded θ_i's is apparently too restrictive for applications; see [5] for more discussions in the field of actuarial science. Recently, [9] extended the results in [7] by allowing θ_i's to have a zero lower bound and keeping that X_i's have a common subexponential tail. The study of infinite number of random weighted sums was initiated by [8] in the presence of extendedly-regularly-varying-tailed X_i's, in which they assumed, further, that θ_i is the product of a series of i.i.d. r.v.s $\{Y_j,\, j \geqslant 1\}$, and derived relation (2) with $n = \infty$ and $\theta_i = \prod_{j=1}^{i} Y_j$. For the general θ_i's, [10] obtained the asymptotic equivalence formula (2) with $n = \infty$. In the above works, the assumption of independence among X_i's is for mathematical convenience, but far too unrealistic for applied problems. A recent new trend of study is to introduce various dependence structures to look into the randomly weighted sums by allowing a certain dependence structure among X_i's and arbitrary dependence among θ_i's, but keeping the two sequences mutually independent; see [3, 16, 17] and [13] among others, in which they considered all kinds of dependent X_i's and derived relation (2) holding for any fixed $n \geqslant 1$ or $n = \infty$.

In this paper, we are interested in the asymptotic behavior for the tail probability of a random number of randomly weighted sums and their maximum defined in (1), which is motivated by two recent papers [15] and [4]. The former considered a random number of randomly weighted sums and their maximum and studied the tail behaviors of S_τ and M_τ under a restrictive dependence structure. [4] further specified some conditions in [15] to make the results more clear and verifiable by assuming $\{\theta_i,\, i \geqslant 1\}$ to be upper extended negatively dependent and identically distributed. Following the work [4], in the present paper, we aim to consider the case that $\{\theta_i,\, i \geqslant 1\}$ follow the more general widely upper orthant dependence structure.

We remark that the topic on randomly weighted sums are closely related to non-life insurance mathematical models. In a discrete-time risk model, X_k can be interpreted as the net loss of an insurance company within the time period k, while the weight θ_k can be regarded as the discount factor from time k to time 0. In such a situation, the tail probability $\mathbf{P}(M_\tau > x)$ can be understood as the random-time ruin probability, where $x \geqslant 0$ is the initial capital reserve of the insurance company. Specially, if $\tau = n$ for any fixed $n \geqslant 1$ or $\tau = \infty$, then $\mathbf{P}(M_n > x)$ and $\mathbf{P}(M_\infty > x)$ are the so-called finite-time and infinite-time ruin probabilities, respectively.

The rest of this paper is organized as follows. In Section 2, we give some preliminaries on some classes of heavy-tailed distributions and dependence structures used in the paper, as well as our main results. In Section 3, we prove the main results after a series of lemmas.

2 Preliminaries and main results

Throughout this paper, without special statement, all the limit relationships hold for x tending to ∞. For two positive functions $f(x)$ and $g(x)$, we write $f(x) \sim g(x)$ if $\lim f(x)/g(x) = 1$; write $f(x) \prec g(x)$ if $\limsup f(x)/g(x) \leqslant 1$; $f(x) = o(g(x))$ if $\lim f(x)/g(x) = 0$; $f(x) = O(g(x))$ if $\limsup f(x)/g(x) < \infty$; and $f(x) \asymp g(x)$ if $0 < \liminf f(x)/g(x) \leqslant \limsup f(x)/g(x) < \infty$. For a real number x, denote its positive part by $x^+ = \max\{x, 0\}$, and denote by $\lfloor x \rfloor$ the greatest integer smaller than or equal to x. The indicator function of a set A is denoted by $\mathbf{1}_A$.

We will restrict ourselves to some classes of heavy-tailed distributions. A r.v. ξ or its distribution V is said to be heavy-tailed if $\mathbf{E}[\mathrm{e}^{t\xi}] = \infty$ for any $t > 0$. An important class of heavy-tailed distributions is the class $\mathcal{D}$ consisting of dominatedly-varying-tailed distributions in the sense that $\overline{V}(x) = 1 - V(x) > 0$ for all $x \in (-\infty, \infty)$ and

$$\limsup_{x\to\infty} \frac{\overline{V}(xy)}{\overline{V}(x)} < \infty$$

holds for any $0 < y < 1$. A smaller class is $\mathcal{C}$ of consistently-varying-tailed distributions. A distribution V belongs to the class $\mathcal{C}$ if $\lim_{y\downarrow 1} \liminf_{x\to\infty} \overline{V}(xy)/\overline{V}(x) = 1$. Both of these two classes include the class $\mathcal{R}_{-\alpha}$, for some $\alpha > 0$, of regularly-varying-tailed distributions satisfying $\lim_{x\to\infty} \overline{V}(xy)/\overline{V}(x) = y^{-\alpha}$ for any $y > 0$.

For a distribution V, denote its upper and lower Matuszewska indexes, respectively, by

$$J_V^+ = -\lim_{y\to\infty} \frac{\log \overline{V}_*(y)}{\log y} \quad \text{with } \overline{V}_*(y) := \liminf_{x\to\infty} \frac{\overline{V}(xy)}{\overline{V}(x)} \quad \text{for } y > 1,$$
$$J_V^- = -\lim_{y\to\infty} \frac{\log \overline{V}^*(y)}{\log y} \quad \text{with } \overline{V}^*(y) := \limsup_{x\to\infty} \frac{\overline{V}(xy)}{\overline{V}(x)} \quad \text{for } y > 1.$$

Define another important parameter $L_V = \lim_{y\downarrow 1} \overline{V}_*(y)$. The parameter L_V and the Matuszewska indices are three important quantities for the characterization of the class $\mathcal{D}$. The following four statements are equivalent: (i) $V \in \mathcal{D}$; (ii) $\overline{V}_*(y) > 0$ for $y > 1$; (iii) $L_V > 0$; (iv) $J_V^+ < \infty$; see, e.g., [1].

In this paper, we allow $\{X_i,\, i \geqslant 1\}$ and $\{\theta_i,\, i \geqslant 1\}$ follow some certain dependence structures. For r.v.s $\{\xi_i,\, i \geqslant 1\}$, if there exists a sequence of finite and positive constants $\{g_U^\xi(i),\, i \geqslant 1\}$ such that for each integer $n \geqslant 1$ and for all $x_i \in (-\infty, \infty)$, $1 \leqslant i \leqslant n$,

$$\mathbf{P}\left(\bigcap_{i=1}^n \{\xi_i > x_i\}\right) \leqslant g_U^\xi(n) \prod_{i=1}^n \mathbf{P}(\xi_i > x_i), \tag{3}$$

then the r.v.s $\{\xi_i,\, i \geqslant 1\}$ are said to be widely upper orthant dependent (WUOD) with dominating coefficients $\{g_U^\xi(i),\, i \geqslant 1\}$.

We remind that such dependence structures contain a lot of commonly used ones. Indeed, if $g_U^\xi(i) = M$ for all $i \geqslant 1$ and some constant $M > 0$ in (3), then $\{\xi_i,\, i \geqslant 1\}$ are said to be upper extended negatively dependent (UEND). The UEND dependence structure was introduced by [6]. Some useful properties of these structures were later analyzed in [2]. Note that the UEND structure has been widely investigated, which contains not only the negative dependence structure but also some positive dependence structures.

Before introducing our main result, we first restate one result in [15], which obtained that $\mathbf{P}(M_\tau > x)$ is weakly tail-equivalent to $\mathbf{E}[\sum_{k=1}^{\tau} \mathbf{P}(\theta_k X_k > x)]$.

Theorem A. *Let $\{X, X_i,\, i \geqslant 1\}$ be a sequence of UEND real-valued r.v.s with common distribution $F_X \in \mathcal{D}$ such that $J_{F_X}^- > 0$ and $F_X(-x) = o(\overline{F_X}(x))$; let $\{\theta_i,\, i \geqslant 1\}$ be a sequence of nonnegative and nondegenerate at zero r.v.s (not necessarily i.i.d.); and let τ be a positive integer-valued r.v. Assume that $\{X, X_i,\, i \geqslant 1\}$, $\{\theta_i,\, i \geqslant 1\}$ and τ are mutually independent. If $Z_\tau := \sum_{i=1}^{\tau} \theta_i$ satisfies*

$$\mathbf{P}(Z_\tau > x) = o\bigl(\overline{F_X}(x)\bigr), \tag{4}$$

and there exists some $\epsilon \in (0, J_{F_X}^-)$ such that $\mathbf{E}[(X^+)^{1+\epsilon}] < \infty$, $\mathbf{E}[\sum_{i=1}^{\tau} \theta_i^{J_{F_X}^+ - \epsilon}] < \infty$ and $\mathbf{E}[\sum_{i=1}^{\tau} \theta_i^{J_{F_X}^+ + \epsilon}] < \infty$, then

$$L_{F_X} \mathbf{E}\left[\sum_{i=1}^{\tau} \mathbf{P}(\theta_i X_i > x)\right] \prec \mathbf{P}(M_\tau > x) \prec L_{F_X}^{-1} \mathbf{E}\left[\sum_{i=1}^{\tau} \mathbf{P}(\theta_i X_i > x)\right]. \tag{5}$$

Based on Theorem A, [4] considered a more specific case, in which θ_i's are assumed to be UEND and identically distributed r.v.s with common distribution $F_\theta \in \mathcal{D}$, and their result makes Theorem A more clear and verifiable. In the present paper, we aim to consider the case that $\{\theta_i,\, i \geqslant 1\}$ follow the more general WUOD structure and obtain the weakly asymptotic formula (5). Now we are ready to state our main results.

Theorem 1. *Let $\{X, X_i,\, i \geqslant 1\}$ be a sequence of UEND real-valued r.v.s with common distribution $F_X \in \mathcal{D}$ such that $J_{F_X}^- > 0$ and $F_X(-x) = o(\overline{F_X}(x))$; let $\{\theta, \theta_i,\, i \geqslant 1\}$ be a sequence of WUOD nonnegative and non-degenerate at zero r.v.s with common distribution $F_\theta \in \mathcal{D}$ and dominating coefficients $\{g_U^\theta(i),\, i \geqslant 1\}$ such that $\mathbf{E}[\theta^{J_{F_X}^+ + \epsilon}] < \infty$ for some $0 < \epsilon < J_{F_X}^-$; and let τ be a positive integer-valued r.v. with distribution F_τ. Assume that $\{X, X_i,\, i \geqslant 1\}$, $\{\theta, \theta_i,\, i \geqslant 1\}$ and τ are mutually independent. If $\mathbf{E}[(X^+)^{1+\epsilon}] < \infty$ for some $\epsilon \in (0, J_{F_X}^-)$, $\overline{F_\theta}(x) \asymp \overline{F_\tau}(x)$ and*

$$\lim_{i\to\infty} g_U^\theta(i)\bigl(i\overline{F_\theta}(i)\bigr)^p = 0 \tag{6}$$

for any $0 < p < 1$, then

$$L_{F_X} \mathbf{E}\tau \mathbf{P}(\theta X > x) \prec \mathbf{P}(M_\tau > x) \prec L_{F_X}^{-1} \mathbf{E}\tau \mathbf{P}(\theta X > x). \tag{7}$$

Under some specific conditions, we can make Theorem 1 more verifiable and computable.

Corollary 1.

(i) *Under the conditions of Theorem* 1*, if* $F_X \in \mathcal{C}$*, then*

$$\mathbf{P}(M_\tau > x) \sim \mathbf{E}\tau\mathbf{P}(\theta X > x). \tag{8}$$

(ii) *In particular, if* $F_X \in \mathcal{R}_{-\alpha}$ *for some* $\alpha > 0$*, then*

$$\mathbf{P}(M_\tau > x) \sim \mathbf{E}\theta\mathbf{E}\tau\overline{F_X}(x).$$

3 Proof of the main results

In order to prove our main result Theorem 1, it is essential to investigate the asymptotic behavior of $\mathbf{P}(Z_\tau > x) = \mathbf{P}(\sum_{i=1}^{\tau}\theta_i > x)$. We start this section by introducing a series of lemmas.

Lemma 1. *Let* $\{\theta_i,\, i \geqslant 1\}$ *be a sequence of WUOD nonnegative r.v.s with common distribution* $F_\theta \in \mathcal{D}$ *and dominating coefficients* $\{g_U^\theta(i),\, i \geqslant 1\}$*. For any fixed* $n \geqslant 1$*, it holds that*

$$\mathbf{P}(Z_n > x) \prec L_{F_\theta}^{-1} n\overline{F_\theta}(x). \tag{9}$$

Proof. Relation (9) is trivial for $n = 1$. Hereafter, we assume $n \geqslant 2$. For any fixed $0 < \varepsilon < 1$,

$$\begin{aligned}
\mathbf{P}(Z_n > x) &= \mathbf{P}\biggl(Z_n > x,\, \bigcup_{1\leqslant i<j\leqslant n}\biggl\{\theta_i > \frac{\varepsilon x}{n},\, \theta_j > \frac{\varepsilon x}{n}\biggr\}\biggr)\\
&\quad + \mathbf{P}\biggl(Z_n > x,\, \bigcap_{1\leqslant i<j\leqslant n}\biggl(\biggl\{\theta_i \leqslant \frac{\varepsilon x}{n}\biggr\}\bigcup\biggl\{\theta_j \leqslant \frac{\varepsilon x}{n}\biggr\}\biggr)\biggr)\\
&=: I_1 + I_2.
\end{aligned} \tag{10}$$

Since $\{\theta_i,\, i \geqslant 1\}$ are WUOD r.v.s, by $F_\theta \in \mathcal{D}$ we have that

$$\begin{aligned}
I_1 &\leqslant \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\mathbf{P}\biggl(\theta_i > \frac{\varepsilon x}{n},\, \theta_j > \frac{\varepsilon x}{n}\biggr) \leqslant g_U^\theta(n) n^2\biggl(\overline{F_\theta}\biggl(\frac{\varepsilon x}{n}\biggr)\biggr)^2\\
&= o\bigl(\overline{F_\theta}(x)\bigr).
\end{aligned} \tag{11}$$

As for I_2, clearly,

$$\begin{aligned}
I_2 &\leqslant \mathbf{P}\Biggl(Z_n > x,\, \bigcup_{j=1}^{n}\bigcap_{\substack{i=1,\\ i\neq j}}^{n}\biggl\{\theta_i \leqslant \frac{\varepsilon x}{n}\biggr\}\Biggr) \leqslant \mathbf{P}\Biggl(Z_n > x,\, \bigcup_{j=1}^{n}\{Z_n - \theta_j \leqslant \varepsilon x\}\Biggr)\\
&\leqslant \sum_{j=1}^{n}\mathbf{P}\bigl(Z_n > x,\, Z_n - \theta_j \leqslant \varepsilon x\bigr) \leqslant n\overline{F_\theta}\bigl((1-\varepsilon)x\bigr),
\end{aligned} \tag{12}$$

whose idea comes from the proof of Lemma 2 in [14]. Plugging (11) and (12) into (10), and from $F_\theta \in \mathcal{D}$, the desired relation (9) can be obtained by letting firstly $x \to \infty$, then $\varepsilon \downarrow 0$. It ends the proof of the lemma. □

The second lemma is related to the upper bound of the precisely large deviations for WUOD r.v.s.

Lemma 2. *Let* $\{\theta_i,\, i \geqslant 1\}$ *be a sequence of WUOD nonnegative r.v.s with common distribution* $F_\theta \in \mathcal{D}$, *mean* $\mathbf{E}\theta = 0$ *and dominating coefficients* $\{g_U^\theta(i),\, i \geqslant 1\}$. *If* (6) *is satisfied, then for any* $\gamma > 0$ *and* $n \geqslant 1$, *there exists a constant* $c_1 = c_1(\gamma) > 0$, *irrespective to* x *and* n, *such that*

$$\mathbf{P}(Z_n > x) \leqslant c_1 n\overline{F_\theta}(x) \tag{13}$$

holds for all $x \geqslant \gamma n$.

Proof. By the precisely large deviations for WUOD r.v.s (see [12, Cor. 1]) we have that for any $\gamma > 0$,

$$\limsup_{n\to\infty} \sup_{x\geqslant\gamma n} \frac{\mathbf{P}(Z_n > x)}{n\overline{F_\theta}(x)} \leqslant L_{F_\theta}^{-1},$$

which implies that for any $\varepsilon > 0$, there exists a positive integer $n_0 \geqslant 1$ such that

$$\mathbf{P}(Z_n > x) \leqslant (1+\varepsilon) L_{F_\theta}^{-1} n\overline{F_\theta}(x) \tag{14}$$

holds for all $x \geqslant \gamma n$ and $n \geqslant n_0$. It is easy to see that $\{\theta_i,\, 1 \leqslant i \leqslant n_0 - 1\}$ are also UEND. In fact, according to the definition of WUOD, for each integer $1 \leqslant n \leqslant n_0 - 1$,

$$\mathbf{P}\left(\bigcap_{i=1}^{n}\{\theta_i > x_i\}\right) \leqslant g_U^\theta(n)\prod_{i=1}^{n} P(\theta_i > x_i) \leqslant \kappa \prod_{i=1}^{n} \mathbf{P}(\theta_i > x_i)$$

with $\kappa = \max_{1\leqslant i\leqslant n_0-1} g_U^\theta(i)$. Hence, by [4, Lemma 5.2], for the above $\gamma > 0$, there exists a positive constant c_1', irrespective to x and n, such that

$$\mathbf{P}(Z_n > x) \leqslant c_1' n\overline{F_\theta}(x) \tag{15}$$

holds for all $x \geqslant \gamma n$ and $1 \leqslant n \leqslant n_0 - 1$. Therefore, by inequalities (14) and (15) we obtain that for any $\gamma > 0$, there exists a constant $c_1 = \max\{(1+\varepsilon)L_{F_\theta}^{-1}, c_1'\}$, irrespective to x and n, such that (13) holds for all $x \geqslant \gamma n$ and $n \geqslant 1$. This completes the proof of the lemma. □

The third lemma investigates the asymptotic tail behavior of the random sum $\mathbf{P}(Z_\tau > x)$, which extends Proposition 1(i) in [4] and plays an important role in the proof of our main result.

Lemma 3. *Let* $\{\theta_i,\, i \geqslant 1\}$ *be a sequence of WUOD nonnegative r.v.s with common distribution* $F_\theta \in \mathcal{D}$, *mean* $\mathbf{E}\theta < \infty$ *and dominating coefficients* $\{g_U^\theta(i),\, i \geqslant 1\}$; *and let* τ *be a positive integer-valued r.v., independent of* $\{\theta_i,\, i \geqslant 1\}$, *with distribution* F_τ. *If* (6) *is satisfied and* $\overline{F_\theta}(x) \asymp \overline{F_\tau}(x)$, *then* $F_\tau \in \mathcal{D}$, $\mathbf{E}\tau < \infty$, *and*

$$\mathbf{P}(Z_\tau > x) \prec L_{F_\theta}^{-1}\mathbf{E}\tau\overline{F_\theta}(x) + L_{F_\tau}^{-1}\overline{F_\tau}\left(\frac{x}{\mathbf{E}\theta}\right). \tag{16}$$

Proof. We follow the line of the proof of Proposition 1 in [4] to prove this lemma. For any positive integer M and any $\varepsilon \in (0,1)$, we divide the tail probability $\mathbf{P}(Z_\tau > x)$ into three parts as

$$\mathbf{P}(Z_\tau > x) =: K_1 + K_2 + K_3, \tag{17}$$

where $K_1 = \sum_{n=1}^{M} \mathbf{P}(Z_n > x)\mathbf{P}(\tau = n)$, $K_2 = \sum_{n=M+1}^{\lfloor (1-\varepsilon)x/\mathbf{E}\theta \rfloor} \mathbf{P}(Z_n > x)\mathbf{P}(\tau = n)$ and $K_3 = \sum_{n=\lfloor (1-\varepsilon)x/\mathbf{E}\theta \rfloor+1}^{\infty} \mathbf{P}(Z_n > x)\mathbf{P}(\tau = n)$. Clearly, by $F_\theta \in \mathcal{D}$, $\mathbf{E}\theta < \infty$ and $\overline{F_\theta}(x) \asymp \overline{F_\tau}(x)$ we know that $F_\tau \in \mathcal{D}$ and $\mathbf{E}\tau < \infty$. For K_1, by Lemma 1 we have

$$K_1 \prec \overline{F_\theta}(x) L_{F_\theta}^{-1} \sum_{n=1}^{M} nP(\tau = n) = \overline{F_\theta}(x) L_{F_\theta}^{-1} \mathbf{E}\bigl(\tau \mathbf{1}_{\{\tau \leqslant M\}}\bigr). \tag{18}$$

For the second term K_2 in equality (17), by $n \leqslant \lfloor (1-\varepsilon)x/\mathbf{E}\theta \rfloor$ we know $x - n\mathbf{E}\theta > \varepsilon x$. Hence, by Lemma 2 and $F_\theta \in \mathcal{D}$ there exists a constant $c_2 > 0$, irrespective to x and n, such that for sufficiently large x,

$$K_2 \leqslant \sum_{n=M+1}^{\lfloor (1-\varepsilon)x/\mathbf{E}\theta \rfloor} \mathbf{P}(Z_n - nE\theta > \varepsilon x)\mathbf{P}(\tau = n) \leqslant c_2 \overline{F_\theta}(x) \mathbf{E}\bigl[\tau \mathbf{1}_{\{\tau > M\}}\bigr]. \tag{19}$$

As for K_3, we have

$$K_3 \leqslant \overline{F_\tau}\biggl(\frac{(1-\varepsilon)x}{\mathbf{E}\theta}\biggr). \tag{20}$$

Plugging (18)–(20) into (17) yields that

$$\begin{aligned}
&\lim_{\varepsilon \downarrow 0} \lim_{M \to \infty} \limsup_{x \to \infty} \frac{\mathbf{P}(Z_\tau > x)}{L_{F_\theta}^{-1} \mathbf{E}\tau \overline{F_\theta}(x) + L_{F_\tau}^{-1} \overline{F_\tau}(x/\mathbf{E}\theta)} \\
&\quad \leqslant \lim_{\varepsilon \downarrow 0} \lim_{M \to \infty} \limsup_{x \to \infty} \max\left\{ \frac{K_1}{L_{F_\theta}^{-1} \mathbf{E}\tau \overline{F_\theta}(x)}, \frac{K_2}{L_{F_\theta}^{-1} \mathbf{E}\tau \overline{F_\theta}(x)}, \frac{K_3}{L_{F_\tau}^{-1} \overline{F_\tau}(x/\mathbf{E}\theta)} \right\} \\
&\quad \leqslant \lim_{\varepsilon \downarrow 0} \lim_{M \to \infty} \limsup_{x \to \infty} \max\left\{ \frac{\mathbf{E}[\tau \mathbf{1}_{\{\tau \leqslant M\}}]}{\mathbf{E}\tau}, \frac{c_2 \mathbf{E}[\tau \mathbf{1}_{\{\tau > M\}}]}{L_{F_\theta}^{-1} \mathbf{E}\tau}, \right. \\
&\qquad \left. L_{F_\tau}\bigl(\overline{F_{\tau *}}\bigl((1-\varepsilon)^{-1}\bigr)\bigr)^{-1} \right\} \\
&\quad = 1,
\end{aligned}$$

which coincides with the desired relation (16). This completes the proof of the lemma. □

Now we give the proofs of our main results.

Proof of Theorem 1. Obviously, notice that the condition $\mathbf{E}[(X^+)^{1+\epsilon}] < \infty$ implies $J_{F_X}^+ \geqslant 1$, thus, $\mathbf{E}\theta < \infty$ follows from $\mathbf{E}[\theta^{J_{F_X}^+ + \epsilon}] < \infty$. By Lemma 3 and Markov's inequality we have

$$\mathbf{P}(Z_\tau > x) = O\bigl(\overline{F_\theta}(x)\bigr) = O\bigl(x^{-(J_{F_X}^+ + \epsilon)}\bigr) = o\bigl(\overline{F_X}(x)\bigr),$$

which means that (4) is satisfied. In addition, Lemma 3 shows $\mathbf{E}\tau < \infty$, which, together with $\mathbf{E}[\theta^{J^+_{F_X}+\epsilon}] < \infty$, yields that

$$\mathbf{E}\left[\sum_{i=1}^{\tau}\theta_i^{J^+_{F_X}+\epsilon}\right] = \mathbf{E}\big[\theta^{J^+_{F_X}+\epsilon}\big]\,\mathbf{E}\tau < \infty.$$

We further have

$$\begin{aligned}\mathbf{E}\left[\sum_{i=1}^{\tau}\theta_i^{J^-_{F_X}-\epsilon}\right] &= \mathbf{E}\left[\sum_{i=1}^{\tau}\theta_i^{J^-_{F_X}-\epsilon}\mathbf{1}_{\{\theta_i\leqslant 1\}}\right] + \mathbf{E}\left[\sum_{i=1}^{\tau}\theta_i^{J^-_{F_X}-\epsilon}\mathbf{1}_{\{\theta_i> 1\}}\right]\\ &\leqslant \mathbf{E}\tau + \mathbf{E}\left[\sum_{i=1}^{\tau}\theta_i^{J^+_{F_X}+\epsilon}\right] < \infty.\end{aligned}$$

Hence, all conditions in Theorem A are satisfied. Therefore, the desired relation (7) follows from Theorem A. □

Proof of Corollary 1. (i) Relation (8) can be derived by $L_{F_X} = 1$ if $F_X \in \mathcal{C}$.

(ii) By $F_X \in \mathcal{R}_{-\alpha}$ and $\mathbf{E}[\theta^{\alpha+\epsilon}] < \infty$ for some $0 < \alpha < \infty$ and any $0 < \epsilon < \alpha$ (here $J^+_{F_X} = J^-_{F_X} = \alpha$), the desired relation follows from the well-known Breiman's theorem immediately. □

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