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Fixed point results for ordered S-G-contractions in ordered metric spaces

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Abstract. In this paper, we prove existence and uniqueness of fixed point in the setting of ordered metric spaces. Precisely, we combine the recent notions of (F,φ)-contraction and $\mathcal{Z}$-contraction in order to introduce the notion of ordered S-G-contraction. Then we use the notion of ordered S-G-contraction to show existence and uniqueness of fixed point. We stress that the notion of ordered S-G-contraction includes different types of ordered contractive conditions in the existing literature. Also, we give some examples and additional results in ordered partial metric spaces to support the new theory.

Keywords: ordered S-G-contraction, fixed point, ordered metric space, nonlinear contraction.

1 Introduction

The metric fixed point theory furnishes useful tools in the study of various practical problems, which appear in the framework of applied sciences. The results of Nieto and Rodriguez-Lopez (see [7]) are fundamental in ordered metric spaces (see also [10]). Such results can be, for example, used to solve problems of integro-differential type, which can be transformed in problems of fixed point type.

The notion of ordered S-G-contraction, that we introduce here, joins the notions of (F,φ)-contraction (see [3]) and $\mathcal{Z}$-contraction (see [1, 4, 6]). In this paper, we prove the existence and uniqueness of fixed point of an ordered S-G-contraction. Further, we prove that the fixed point of an ordered S-G-contraction belongs to the zero-set of a given function (see [13, 14]). We notice as our main results allows to infer, as a particular case, some of the most known results in the literature of ordered metric spaces. Also, we give some examples and additional results in ordered partial metric spaces to support the new theory.

2 Preliminaries

In this section, we recall the notions and fix the notation, which we will use in the sequel.

In [3], Jleli et al. take into account the family $\mathcal{G}$ of functions $G : [0,+\infty[^3 \to [0,+\infty[$ satisfying the following conditions:

(G1) $\max\{\delta,\theta\} \leqslant G(\delta,\theta,\lambda)$ for all $\delta,\theta,\lambda \in [0,+\infty[$;
(G2) $G(0,0,0) = 0$;
(G3) G is continuous.

Example 1. The following are examples of functions, which belong to $\mathcal{G}$:

(i) $G(\delta,\theta,\lambda) = \delta + \theta + \lambda$;
(ii) $G(\delta,\theta,\lambda) = \max\{\delta,\theta\} + \lambda$;
(iii) $G(\delta,\theta,\lambda) = \delta + \theta + \delta\theta + \lambda$.

By using the family $\mathcal{G}$, Jleli et al. introduce a new type of contraction (see [3, Def. 2.4]) useful to get the existence and uniqueness of fixed point belonging to the zero-set of a given function in the setting of metric spaces.

In the sequel, we denote by $\mathcal{S}$ the family of functions $S : [0,+\infty[^2 \to \mathbb{R}$ satisfying the following conditions (see [1,4]):

(S1) $S(\alpha,\beta) < \beta - \alpha$ for all $\alpha,\beta > 0$;
(S2) If $\{\alpha_m\}, \{\beta_m\}$ are sequences in $]0,+\infty[$ such that

$$\lim_{m\to+\infty} \alpha_m = \lim_{m\to+\infty} \beta_m = \gamma \in]0,+\infty[,$$

then $\limsup_{m\to+\infty} S(\alpha_m,\beta_m) < 0$.

Example 2. The following are examples of functions in $\mathcal{S}$:

(i) Let $\xi : [0,+\infty[\to [0,+\infty[$ be a lower semicontinuous function with $\xi^{-1}(0) = \{0\}$. So, $S \in \mathcal{S}$ if $S(\alpha,\beta) = \beta - \xi(\beta) - \alpha$ for all $\alpha,\beta \geqslant 0$.
(ii) Let $\xi : [0,+\infty[\to [0,1[$ be such that $\limsup_{\beta\to s^+} \xi(\beta) < 1$ for all $s > 0$. So, $S \in \mathcal{S}$ if $S(\alpha,\beta) = \beta\,\xi(\beta) - \alpha$ for all $\alpha,\beta \geqslant 0$.
(iii) Let $\xi : [0,+\infty[\to [0,+\infty[$ be an upper semicontinuous function with $\xi(\beta) < \beta$ for all $\beta > 0$ and $\xi(0) = 0$. So, $S \in \mathcal{S}$ if $S(\alpha,\beta) = \xi(\beta) - \alpha$ for all $\alpha,\beta \geqslant 0$.
(iv) Let $f_1, f_2 : [0,+\infty[\times[0,+\infty[\to]0,+\infty[$ be two continuous functions such that $f_1(\alpha,\beta) > f_2(\alpha,\beta)$ for all $\alpha,\beta > 0$. So, $S \in \mathcal{S}$ if $S(\alpha,\beta) = \beta - (f_1(\alpha,\beta)/f_2(\alpha,\beta))\alpha$ for all $\alpha,\beta \in [0,+\infty[$.

Argoubi et al. in [1] use the family $\mathcal{S}$ to establish coincidence and common fixed point results in the setting of ordered metric spaces. We also stress that in [4,6] the family $\mathcal{S}$ is used to obtain existence and uniqueness of fixed point in the setting of metric spaces.

Here, if (Y,d) is a metric space and $(Y,\preccurlyeq)$ is a partially ordered set, then we say that $(Y,d,\preccurlyeq)$ is an ordered metric space. Two elements $z,w \in Y$ are said comparable if $z \preccurlyeq w$ or $w \preccurlyeq z$ holds. The mapping $g : (Y,\preccurlyeq) \to (Y,\preccurlyeq)$ is called to be nondecreasing if $gz \preccurlyeq gw$ whenever $z \preccurlyeq w$. A sequence $\{z_m\}$ is nondecreasing if $z_{m-1} \preccurlyeq z_m$ for all $m \in \mathbb{N}$.

An ordered metric space $(Y, d, \preccurlyeq)$ is said regular if the following holds:

- For every nondecreasing sequence $\{z_m\} \subset Y$ such that $z_m \to \zeta \in Y$, we have $z_{m-1} \preccurlyeq \zeta$ for all $m \in \mathbb{N}$.

Let $(Y, d, \preccurlyeq)$ be an ordered metric space, we say Y has the following property:

(H) If for each pair of not comparable elements $\zeta, \eta \in Y$, then there exists $\theta \in Y$ such that $\zeta \preccurlyeq \theta$ and $\eta \preccurlyeq \theta$.

Let Y be a nonempty set and $g : Y \to Y$ a given mapping. Fixed $z_0 \in Y$, let $z_m = gz_{m-1}$ for all $m \in \mathbb{N}$. The sequence $\{z_m\}$ is known as the sequence of Picard starting at z_0.

3 Fixed points for S-G-contractions

In this section, we use the following notion of ordered S-G-contraction to establish results of existence and uniqueness of fixed point in the setting of ordered metric spaces.

Definition 1. Let $(Y, d, \preccurlyeq)$ be an ordered metric space, and let g be a self-mapping on Y. The mapping g is an ordered S-G-contraction if there exist three functions $G \in \mathcal{G}$, $S \in \mathcal{S}$, and $\Lambda : Y \to [0, +\infty[$ such that, for all $z, w \in Y$, $z \preccurlyeq w$,

$$S\bigl(G(d(gz, gw), \Lambda(gz), \Lambda(gw)), G\bigl(d(z, w), \Lambda(z), \Lambda(w)\bigr)\bigr) \geqslant 0. \tag{1}$$

Firstly, we give some useful remarks to get our main results.

Remark 1. Let $(Y, d, \preccurlyeq)$ be an ordered metric space, and let $g : Y \to Y$ be an ordered S-G-contraction with respect to the functions $G \in \mathcal{G}$, $S \in \mathcal{S}$, and $\Lambda : Y \to [0, +\infty[$. If $\zeta \in Y$ is a fixed point of g, then $\Lambda(\zeta) = 0$.

In fact, if we suppose that $\Lambda(\zeta) > 0$, then

$$0 < \Lambda(\zeta) \leqslant G\bigl(d(\zeta, \zeta), \Lambda(\zeta), \Lambda(\zeta)\bigr) \quad \text{by (G1)}.$$

Using (1) with $z = w = \zeta$ and (S1), we get

$$\begin{aligned} 0 &\leqslant S\bigl(G(d(g\zeta, g\zeta), \Lambda(g\zeta), \Lambda(g\zeta)\bigr), G\bigl(d(\zeta, \zeta), \Lambda(\zeta), \Lambda(\zeta)\bigr)\bigr) \\ &= S\bigl(G\bigl(d(\zeta, \zeta), \Lambda(\zeta), \Lambda(\zeta)\bigr), G\bigl(d(\zeta, \zeta), \Lambda(\zeta), \Lambda(\zeta)\bigr)\bigr) \\ &< G\bigl(d(\zeta, \zeta), \Lambda(\zeta), \Lambda(\zeta)\bigr) - G\bigl(d(\zeta, \zeta), \Lambda(\zeta), \Lambda(\zeta)\bigr) = 0. \end{aligned}$$

This is not possible, and so, $\Lambda(\zeta) = 0$.

Remark 2. Let $(Y, d, \preccurlyeq)$ be an ordered metric space, and let $g : Y \to Y$ be an ordered S-G-contraction with respect to the functions $G \in \mathcal{G}$, $S \in \mathcal{S}$, and $\Lambda : Y \to [0, +\infty[$. If $\zeta, \eta \in Y$ are two fixed points of g, then $\zeta \neq \eta$ if and only if ζ and η are not comparable.

In fact, if ζ and η are comparable, we say $\zeta \preccurlyeq \eta$ and $\zeta \neq \eta$, then

$$0 < d(\zeta, \eta) \leqslant G\bigl(d(\zeta, \eta), \Lambda(\zeta), \Lambda(\eta)\bigr) \quad \text{(by (G1))}.$$

Using (1) with $z = \zeta$ and $w = \eta$ and (S1), we obtain

$$\begin{aligned} 0 &\leqslant S\bigl(G\bigl(d(g\zeta, g\eta), \Lambda(g\zeta), \Lambda(g\eta)\bigr), G\bigl(d(\zeta, \eta), \Lambda(\zeta), \Lambda(\eta)\bigr)\bigr) \\ &< G\bigl(d(\zeta, \eta), \Lambda(\zeta), \Lambda(\eta)\bigr) - G\bigl(d(\zeta, \eta), \Lambda(\zeta), \Lambda(\eta)\bigr) = 0, \end{aligned}$$

which is a contradiction.

Lemma 1. *Let $(Y, d, \preccurlyeq)$ be an ordered metric space, and let $g : Y \to Y$ be a nondecreasing ordered S-G-contraction with respect to the functions $G \in \mathcal{G}$, $S \in \mathcal{S}$, and $\Lambda : Y \to [0, +\infty[$. If there exists a point $z_0 \in Y$ such that $z_0 \preccurlyeq gz_0$, then the Picard sequence $\{z_m\}$ starting at z_0 is Cauchy.*

Proof. Let $z_0 \in Y$ be such that $z_0 \preccurlyeq gz_0$, and let $\{z_m\}$ be the sequence of Picard starting at z_0. If $z_j = z_{j-1}$ for some $j \in \mathbb{N}$, then $z_m = z_j$ for all $m \geqslant j$, and hence $\{z_m\}$ is a Cauchy sequence.

Now, we assume that $z_{m-1} \neq z_m$ for all $m \in \mathbb{N}$. We claim that

$$\lim_{m \to +\infty} d(z_{m-1}, z_m) = 0 \quad \text{and} \quad \lim_{m \to +\infty} \Lambda(z_m) = 0. \tag{2}$$

The hypothesis $z_{m-1} \neq z_m$ for all $m \in \mathbb{N}$ and (G1) ensure

$$G\bigl(d(z_{m-1}, z_m), \Lambda(z_{m-1}), \Lambda(z_m)\bigr) \geqslant d(z_{m-1}, z_m) > 0$$

for all $m \in \mathbb{N}$.

Furthermore, the hypothesis g nondecreasing ensures that $z_{m-1} \preccurlyeq z_m$ for all $m \in \mathbb{N}$. Now, if in (1), we choose $z = z_{m-1}$ and $w = z_m$ by using (S1), we infer

$$\begin{aligned} 0 &\leqslant S\bigl(G\bigl(d(z_m, z_{m+1}), \Lambda(z_m), \Lambda(z_{m+1})\bigr), G\bigl(d(z_{m-1}, z_m), \Lambda(z_{m-1}), \Lambda(z_m)\bigr)\bigr) \\ &< G\bigl(d(z_{m-1}, z_m), \Lambda(z_{m-1}), \Lambda(z_m)\bigr) - G\bigl(d(z_m, z_{m+1}), \Lambda(z_m), \Lambda(z_{m+1})\bigr) \end{aligned}$$

for all $m \in \mathbb{N}$. This inequality shows that

$$G\bigl(d(z_m, z_{m+1}), \Lambda(z_m), \Lambda(z_{m+1})\bigr) < G\bigl(d(z_{m-1}, z_m), \Lambda(z_{m-1}), \Lambda(z_m)\bigr)$$

for all $m \in \mathbb{N}$. Consequently, the sequence $\{G(d(z_{m-1}, z_m), \Lambda(z_{m-1}), \Lambda(z_m))\} \subset [0, +\infty[$ is decreasing. So, there exists some $s \geqslant 0$ such that

$$\lim_{m \to +\infty} G\bigl(d(z_{m-1}, z_m), \Lambda(z_{m-1}), \Lambda(z_m)\bigr) = s. \tag{3}$$

We point out that if we assume $s > 0$, using condition (S2) with

$$\alpha_m = G\bigl(d(z_m, z_{m+1}), \Lambda(z_m), \Lambda(z_{m+1})\bigr)$$

and

$$\beta_m = G\bigl(d(z_{m-1}, z_m), \Lambda(z_{m-1}), \Lambda(z_m)\bigr),$$

we get

$$0 \leqslant \limsup_{m\to+\infty} S\bigl(G\bigl(d(z_m, z_{m+1}), \Lambda(z_m), \Lambda(z_{m+1})\bigr), \\ G\bigl(d(z_{m-1}, z_m), \Lambda(z_{m-1}), \Lambda(z_m)\bigr)\bigr) < 0.$$

Obviously, this is a contradiction, and thus, we can assert that $s = 0$. Now, by (G1) we say that

$$\max\bigl\{d(z_{m-1}, z_m), \Lambda(z_{m-1})\bigr\} \leqslant G\bigl(d(z_{m-1}, z_m), \Lambda(z_{m-1}), \Lambda(z_m)\bigr)$$

for all $m \in \mathbb{N}$. So, the previous inequality implies

$$\lim_{m\to+\infty} d(z_{m-1}, z_m) = 0 \quad \text{and} \quad \lim_{m\to+\infty} \Lambda(z_{m-1}) = 0.$$

Next, we show that the sequence $\{z_m\}$ is Cauchy. We assume for way of contradiction that $\{z_m\}$ is not a Cauchy sequence. Then there exist a positive real number δ and two sequences $\{j_n\}$ and $\{i_n\}$ such that $i_n > j_n \geqslant n$ and $d(z_{j_n}, z_{i_n}) \geqslant \delta > d(z_{j_n}, z_{i_n-1})$ for all $n \in \mathbb{N}$. Using the first condition of (2), we deduce

$$\lim_{n\to+\infty} d(z_{j_n}, z_{i_n}) = \lim_{n\to+\infty} d(z_{j_n-1}, z_{i_n-1}) = \delta.$$

Consequently, it is not restrictive to suppose that $d(z_{j_n-1}, z_{i_n-1}) > 0$ for all $n \in \mathbb{N}$. Hence, by appling (G1), we obtain

$$G\bigl(d(z_{j_n-1}, z_{i_n-1}), \Lambda(z_{j_n-1}), \Lambda(z_{i_n-1})\bigr) > 0$$

and

$$G\bigl(d(z_{j_n}, z_{i_n}), \Lambda(z_{j_n}), \Lambda(z_{i_n})) > 0.$$

Now, by using the continuity of the function G, the second condition of (2) and (G1), we get

$$\begin{aligned} &\lim_{n\to+\infty} G\bigl(d(z_{j_n-1}, z_{i_n-1}), \Lambda(z_{j_n-1}), \Lambda(z_{i_n-1})\bigr) \\ &\quad= \lim_{n\to+\infty} G\bigl(d(z_{j_n}, z_{i_n}), \Lambda(z_{j_n}), \Lambda(z_{i_n})\bigr) \\ &\quad= G(\delta, 0, 0) \geqslant \delta > 0. \end{aligned}$$

This allows to use condition (S2) with

$$\alpha_n = G\bigl(d(z_{j_n}, z_{i_n}), \Lambda(z_{j_n}), \Lambda(z_{i_n})\bigr)$$

and

$$\beta_n = G\bigl(d(z_{j_n-1}, z_{i_n-1}), \Lambda(z_{j_n-1}), \Lambda(z_{i_n-1})\bigr)$$

and to deduce

$$\limsup_{n\to+\infty} S\big(G\big(d(z_{j_n}, z_{i_n}), \Lambda(z_{j_n}), \Lambda(z_{i_n})\big), G\big(d(z_{j_n-1}, z_{i_n-1}), \Lambda(z_{j_n-1}), \Lambda(z_{i_n-1})\big)\big) < 0. \tag{4}$$

Since $z_{j_n-1} \preccurlyeq z_{i_n-1}$ for all $n \in \mathbb{N}$, using (1) with $z = z_{j_n-1}$ and $w = z_{i_n-1}$, we have

$$\limsup_{n\to+\infty} S\big(G\big(d(z_{j_n}, z_{i_n}), \Lambda(z_{j_n}), \Lambda(z_{i_n})\big), G\big(d(z_{j_n-1}, z_{i_n-1}), \Lambda(z_{j_n-1}), \Lambda(z_{i_n-1})\big)\big) \geqslant 0. \tag{5}$$

From (4) and (5) we get a contradiction. Consequently, the sequence $\{z_m\}$ is Cauchy. □

Now, we formulate and prove the first of our main results.

Theorem 1. *Let $(Y, d, \preccurlyeq)$ be a complete ordered metric space, and let $g : Y \to Y$ be a nondecreasing ordered S-G-contraction with respect to the functions $G \in \mathcal{G}$, $S \in \mathcal{S}$, and $\Lambda : Y \to [0, +\infty[$. If there exists a point $z_0 \in Y$ such that $z_0 \preccurlyeq gz_0$ and g is continuous, then g has a fixed point ζ such that $\Lambda(\zeta) = 0$.*

Proof. Let $z_0 \in Y$ such that $z_0 \preccurlyeq gz_0$, and we consider the sequence $\{z_m\}$ of Picard starting at z_0. We stress that if $z_j = z_{j-1}$ for some $j \in \mathbb{N}$, then $z_{j-1} = z_j = gz_{j-1}$, that is, z_{j-1} is a fixed point of g. Hence, by Remark 1, we have $\Lambda(z_{j-1}) = 0$, and the proof is complete. So, it is not restrictive to suppose that $z_j \neq z_{j-1}$ for each $j \in \mathbb{N}$.

We can affirm that the sequence $\{z_m\}$ is Cauchy (see Lemma 1). Furthermore, the completeness of $(Y, d, \preccurlyeq)$ ensures there exists some $\zeta \in Y$ such that

$$\lim_{m\to+\infty} z_m = \zeta. \tag{6}$$

Now, in order to get the claim, it is sufficient to note that the continuity of the mapping g ensures that ζ is a fixed point of g, and then, by Remark 1, $\Lambda(\zeta) = 0$. □

Theorem 2. *Let $(Y, d, \preccurlyeq)$ be a complete ordered metric space, and let $g : Y \to Y$ be a nondecreasing ordered S-G-contraction with respect to the functions $G \in \mathcal{G}$, $S \in \mathcal{S}$ and the lower semicontinuous function $\Lambda : Y \to [0, +\infty[$. If there exists a point $z_0 \in Y$ such that $z_0 \preccurlyeq gz_0$ and Y is regular, then g has a fixed point ζ such that $\Lambda(\zeta) = 0$.*

Proof. Let $z_0 \in Y$ such that $z_0 \preccurlyeq gz_0$, and let $\{z_m\}$ be the sequence of Picard starting at z_0. From the proof of Theorem 1 we say that it is not restrictive to suppose $z_j \neq z_{j-1}$ for each $j \in \mathbb{N}$. Moreover, by Lemma 1, we have that $\{z_m\}$ is a Cauchy sequence. Also, the completeness of $(Y, d, \preccurlyeq)$ ensures that there exists some $\zeta \in Y$ such that

$$\lim_{m\to+\infty} z_m = \zeta. \tag{7}$$

Now, the lower semicontinuity of the function Λ and (2) give

$$0 \leqslant \Lambda(\zeta) \leqslant \liminf_{m\to+\infty} \Lambda(z_m) = 0,$$

that is, $\Lambda(\zeta) = 0$.

We claim that ζ is a fixed point of g. Obviously, ζ is a fixed point of the mapping g if there is a subsequence $\{z_{j_n}\}$ of $\{z_m\}$ such that $z_{j_n} = \zeta$ or $gz_{j_n} = g\zeta$ for all $n \in \mathbb{N}$. If a such subsequence there is not, we can assume that $z_m \neq \zeta$ and $gz_m \neq g\zeta$ for all $m \in \mathbb{N}$. So,

$$G\bigl(d(gz_m, g\zeta), \Lambda(gz_m), \Lambda(g\zeta)\bigr),\ G\bigl(d(z_m, \zeta), \Lambda(z_m), \Lambda(\zeta)\bigr) \in]0, +\infty[$$

for all $m \in \mathbb{N}$.

Since Y is regular, we have that $z_m \preccurlyeq \zeta$ for all $m \in \mathbb{N}$. Now, using (1) with $z = z_m$ and $w = \zeta$, we get

$$\begin{aligned} 0 &\leqslant S\bigl(G\bigl(d(gz_m, g\zeta), \Lambda(gz_m), \Lambda(g\zeta)\bigr), G\bigl(d(z_m, \zeta), \Lambda(z_m), \Lambda(\zeta)\bigr)\bigr) \\ &< G\bigl(d(z_m, \zeta), \Lambda(z_m), \Lambda(\zeta)\bigr) - G\bigl(d(gz_m, g\zeta), \Lambda(gz_m), \Lambda(g\zeta)\bigr) \quad \text{(by (S1))}. \end{aligned}$$

From the previous inequality we infer

$$G\bigl(d(gz_m, g\zeta), \Lambda(gz_m), \Lambda(g\zeta)\bigr) < G\bigl(d(z_m, \zeta), \Lambda(z_m), \Lambda(\zeta)\bigr)$$

for all $m \in \mathbb{N}$, and so,

$$\begin{aligned} d(\zeta, g\zeta) &\leqslant d(\zeta, z_{m+1}) + d(gz_m, g\zeta) \\ &\leqslant d(\zeta, z_{m+1}) + G\bigl(d(gz_m, g\zeta), \Lambda(gz_m), \Lambda(g\zeta)\bigr) \quad \text{(by (G1))} \\ &< d(\zeta, z_{m+1}) + G\bigl(d(z_m, \zeta), \Lambda(z_m), \Lambda(\zeta)\bigr) \end{aligned}$$

for all $m \in \mathbb{N}$. Letting $m \to +\infty$ in the above inequality, taking into account that G is continuous in $(0, 0, 0)$, we deduce that $d(\zeta, g\zeta) \leqslant G(0, 0, 0) = 0$, that is, $\zeta = g\zeta$. □

Lemma 2. *Let $(Y, d, \preccurlyeq)$ be an ordered metric space, and let $g : Y \to Y$ be a nondecreasing ordered S-G-contraction with respect to the functions $G \in \mathcal{G}$, $S \in \mathcal{S}$, and $\Lambda : Y \to [0, +\infty[$. If Y has property* (H)*, then g admits at most one fixed point.*

Proof. Assume that $\zeta, \eta \in Y$ are two distinct fixed points of g. By Remark 2, ζ and η are not comparable. Then property (H) ensures that there exists $\theta \in Y$ such that $\zeta \preccurlyeq \theta$ and $\eta \preccurlyeq \theta$. Let $\{\theta_m\}$ the sequence of Picard starting at $\theta_0 = \theta$. From $\zeta \preccurlyeq \theta_0$ and $\eta \preccurlyeq \theta_0$ it follows $\zeta \preccurlyeq \theta_m$ and $\eta \preccurlyeq \theta_m$ for all $m \in \mathbb{N}$ (it is important to recall that g is nondecreasing). Further, we remark that the hypothesis $\zeta \neq \eta$ ensures $\zeta \prec \theta_m$ and $\eta \prec \theta_m$ for all $m \in \mathbb{N}$. In fact, if one has $\eta = \theta_m$ or $\xi = \theta_m$ for some $m \in \mathbb{N}$, then ζ and η are comparable.

Consequently, by (G1)

$$G\bigl(d(\zeta, \theta_{m-1}), \Lambda(\zeta), \Lambda(\theta_{m-1})\bigr) \geqslant d(\zeta, \theta_{m-1}) > 0$$

for all $m \in \mathbb{N}$. Now, using (1) with $z = \zeta$ and $w = \theta_{m-1}$, we get

$$\begin{aligned} 0 &\leqslant S\bigl(G\bigl(d(g\zeta, g\theta_{m-1}), \Lambda(g\zeta), \Lambda(g\theta_{m-1})\bigr), G\bigl(d(\zeta, \theta_{m-1}), \Lambda(\zeta), \Lambda(\theta_{m-1})\bigr)\bigr) \\ &< G\bigl(d(\zeta, \theta_{m-1}), \Lambda(\zeta), \Lambda(\theta_{m-1})\bigr) - G\bigl(d(\zeta, \theta_m), \Lambda(\zeta), \Lambda(\theta_m)\bigr) \quad \text{(by (S1))}. \end{aligned}$$

The previous inequality ensures that the sequence $\{G(d(\zeta, \theta_{m-1}), \Lambda(\zeta), \Lambda(\theta_{m-1}))\} \subset [0, +\infty[$ is decreasing, and so, there exists $s \in [0, +\infty[$ such that

$$\lim_{m\to+\infty} G\bigl(d(\zeta, \theta_{m-1}), \Lambda(\zeta), \Lambda(\theta_{m-1})\bigr) = s.$$

Next, using (1) and (S2), we infer

$$\begin{aligned} 0 \leqslant \limsup_{m\to+\infty} S\bigl(&G\bigl(d(g\zeta, g\theta_{m-1}), \Lambda(g\zeta), \Lambda(g\theta_{m-1})\bigr), \\ &G\bigl(d(\zeta, \theta_{m-1}), \Lambda(\zeta), \Lambda(\theta_{m-1})\bigr)\bigr) < 0, \end{aligned}$$

which is a contradiction. So, $s = 0$. The property (G1) of the function G implies

$$\lim_{m\to+\infty} d(\zeta, \theta_{m-1}) = 0.$$

Similarly, we prove that

$$\lim_{m\to+\infty} d(\eta, \theta_{m-1}) = 0,$$

and from

$$d(\zeta, \eta) \leqslant d(\zeta, \theta_{m-1}) + d(\theta_{m-1}, \eta),$$

letting $m \to +\infty$, we get $d(\zeta, \eta) = 0$, that is, $\zeta = \eta$. □

From Theorem 1 and Lemma 2 we deduce the following result.

Theorem 3. *Let $(Y, d, \preccurlyeq)$ be a complete ordered metric space, and let $g : Y \to Y$ be a nondecreasing ordered S-G-contraction with respect to the functions $G \in \mathcal{G}$, $S \in \mathcal{S}$, and $\Lambda : Y \to [0, +\infty[$. If there exists a point $z_0 \in Y$ such that $z_0 \preccurlyeq gz_0$ and g is continuous, then g has a fixed point ζ such that $\Lambda(\zeta) = 0$. Moreover, if Y has property* (H)*, then g has a unique fixed point ζ such that $\Lambda(\zeta) = 0$.*

From Theorem 2 and Lemma 2 we deduce the following result.

Theorem 4. *Let $(Y, d, \preccurlyeq)$ be a complete ordered metric space, and let $g : Y \to Y$ be a nondecreasing ordered S-G-contraction with respect to the functions $G \in \mathcal{G}$, $S \in \mathcal{S}$ and the lower semicontinuous function $\Lambda : Y \to [0, +\infty[$. If there exists a point $z_0 \in Y$ such that $z_0 \preccurlyeq gz_0$ and Y is regular, then g has a fixed point ζ such that $\Lambda(\zeta) = 0$. Moreover, if Y has property* (H)*, then g has a unique fixed point ζ such that $\Lambda(\zeta) = 0$.*

4 Consequences

In this section, we point out that the notion of S-G-contraction includes different types of contractive conditions in the existing literature.

We start by two results of Jleli et al. type (see [3, Thm. 2.1]) in the setting of ordered metric spaces.

Corollary 1. *Let $(Y, d, \preccurlyeq)$ be a complete ordered metric space, and let $g : Y \to Y$ be a nondecreasing mapping. Suppose that there exist $\sigma \in]0, 1[$, two functions $G \in \mathcal{G}$ and $\Lambda : Y \to [0, +\infty[$ such that*

$$G\big(d(gz, gw), \Lambda(gz), \Lambda(gw)\big) \leqslant \sigma G\big(d(z, w), \Lambda(z), \Lambda(w)\big)$$

for all $z, w \in Y$ with $z \preccurlyeq w$. If there exists a point $z_0 \in Y$ such that $z_0 \preccurlyeq gz_0$ and g is continuous, then g has a fixed point ζ such that $\Lambda(\zeta) = 0$. Moreover, if Y has property (H)*, then g has a unique fixed point ζ such that $\Lambda(\zeta) = 0$.*

Proof. We obtain the claim by using Theorem 3 with respect to the function $S \in \mathcal{S}$ defined by $S(\alpha, \beta) = \sigma\beta - \alpha$ for all $\alpha, \beta \geqslant 0$. $\square$

We notice that if $G(\delta, \theta, \lambda) = \delta + \theta + \lambda$ for all $\delta, \theta, \lambda \geqslant 0$ and $\Lambda(z) = 0$ for all $z \in Y$, then we obtain a result of [7] (see Theorem 2.1). Further, the same choice of S also allows to obtain the following corollary.

Corollary 2. *Let $(Y, d, \preccurlyeq)$ be a complete ordered metric space, and let $g : Y \to Y$ be a nondecreasing mapping. Suppose that there exist $\sigma \in]0, 1[$, a function $G \in \mathcal{G}$, and a lower semicontinuous function $\Lambda : Y \to [0, +\infty[$ such that*

$$G\big(d(gz, gw), \Lambda(gz), \Lambda(gw)\big) \leqslant \sigma\, G\big(d(z, w), \Lambda(z), \Lambda(w)\big)$$

for all $z, w \in Y$ with $z \preccurlyeq w$. If there exists a point $z_0 \in Y$ such that $z_0 \preccurlyeq gz_0$ and Y is regular, then g has a fixed point ζ such that $\Lambda(\zeta) = 0$. Moreover, if Y has property (H)*, then g has a unique fixed point ζ such that $\Lambda(\zeta) = 0$.*

We stress that if $G(\delta, \theta, \lambda) = \delta + \theta + \lambda$ for all $\delta, \theta, \lambda \geqslant 0$ and $\Lambda(z) = 0$ for all $z \in Y$, then we have a result of [7] (see Theorem 2.2).

Now, we give a result of Rhoades type (see [12]) in the setting of ordered metric spaces.

Corollary 3. *Let $(Y, d, \preccurlyeq)$ be a complete ordered metric space, and let $g : Y \to Y$ be a nondecreasing mapping. Suppose that there exist two functions $G \in \mathcal{G}$, $\Lambda : Y \to [0, +\infty[$ and a lower semicontinuous functions $\xi : [0, +\infty[\to [0, +\infty[$ with $\xi^{-1}(0) = \{0\}$ such that*

$$\begin{aligned} &G\big(d(gz, gw), \Lambda(gz), \Lambda(gw)\big) \\ &\quad \leqslant G\big(d(z, w), \Lambda(z), \Lambda(w)\big) - \xi\big(G(d(z, w), \Lambda(z), \Lambda(w))\big) \end{aligned}$$

for all $z, w \in Y$ with $z \preccurlyeq w$. If there exists a point $z_0 \in Y$ such that $z_0 \preccurlyeq gz_0$ and one of the following conditions holds:

(i) *the mapping g is continuous*;
(ii) *Y is regular and Λ is a lower semicontinuous function*;

then g has a fixed point ζ such that $\Lambda(\zeta) = 0$. Moreover, if Y has property (H)*, then g has a unique fixed point ζ such that $\Lambda(\zeta) = 0$.*

Proof. Again, we obtain the claim by Theorems 3 and 4 if we choose $S \in \mathcal{S}$ given by $S(\alpha,\beta) = \beta - \xi(\beta) - \alpha$ for all $\alpha,\beta \geqslant 0$ (see Example 2(i)). □

As consequence of Theorems 3 and 4, we get also the following result (see [11]).

Corollary 4. *Let $(Y,d,\preccurlyeq)$ be a complete ordered metric space, and let $g : Y \to Y$ be a nondecreasing mapping. Suppose that there exist three functions $G \in \mathcal{G}$, $\Lambda : Y \to [0,+\infty[$, and $\xi : [0,+\infty[\to [0,1[$ with $\limsup_{\alpha\to s^+} \xi(\alpha) < 1$ for all $s > 0$ such that*

$$G\bigl(d(gz,gw),\Lambda(gz),\Lambda(gw)\bigr) \leqslant \xi\bigl(G\bigl(d(z,w),\Lambda(z),\Lambda(w)\bigr)\bigr)\,G\bigl(d(z,w),\Lambda(z),\Lambda(w)\bigr)$$

for all $z,w \in Y$ with $z \preccurlyeq w$. If there exists a point $z_0 \in Y$ such that $z_0 \preccurlyeq gz_0$ and one of the following conditions holds:

(i) *the mapping g is continuous*;
(ii) *Y is regular and Λ is a lower semicontinuous function*;

then g has a fixed point ζ such that $\Lambda(\zeta) = 0$. Moreover, if Y has property (H)*, then g has a unique fixed point ζ such that $\Lambda(\zeta) = 0$.*

Proof. By using Theorems 3 and 4 and taking $S \in \mathcal{S}$ given by $S(\alpha,\beta) = \beta\,\xi(\beta) - \alpha$ for all $\alpha,\beta \geqslant 0$ (see Example 2(ii)), we deduce our result. □

The following is a result of Boyd–Wong type (see [2]) in the setting of ordered metric spaces.

Corollary 5. *Let $(Y,d,\preccurlyeq)$ be a complete ordered metric space, and let $g : Y \to Y$ be a nondecreasing mapping. Suppose that there exist two functions $G \in \mathcal{G}$, $\Lambda : Y \to [0,+\infty[$ and an upper semicontinuous function $\xi : [0,+\infty[\to [0,+\infty[$ with $\xi(\alpha) < \alpha$ for all $\alpha > 0$ and $\xi(0) = 0$ such that*

$$G\bigl(d(gz,gw),\Lambda(gz),\Lambda(gw)\bigr) \leqslant \xi\bigl(G\bigl(d(z,w),\Lambda(z),\Lambda(w)\bigr)\bigr)$$

for all $z,w \in Y$, $z \preccurlyeq w$. If there exists a point $z_0 \in Y$ such that $z_0 \preccurlyeq gz_0$ and one of the following conditions holds:

(i) *the mapping g is continuous*;
(ii) *Y is regular and Λ is a lower semicontinuous function*;

then g has a fixed point ζ such that $\Lambda(\zeta) = 0$. Moreover, if Y has property (H)*, then g has a unique fixed point ζ such that $\Lambda(\zeta) = 0$.*

Proof. In order to have the claim, we can again use Theorems 3 and 4. It is sufficient to choose $S \in \mathcal{S}$ given by $S(\alpha,\beta) = \xi(\beta) - \alpha$ for all $\alpha,\beta \geqslant 0$ (see Exemple 2(iii)). □

We observe that if we suppose $G(\delta,\theta,\lambda) = \delta + \theta + \lambda$ for all $\delta,\theta,\lambda \geqslant 0$ and $\Lambda(z) = 0$ for all $z \in Y$, then we obtain the ordered version of Boyd–Wong fixed point result.

The following examples show as Theorem 1 is a proper generalization in the setting of ordered metric spaces of Theorem 2.2 of [7].

Example 3. Let $Y = [0,2]$ endowed with the usual metric $d(z,w) = |z - w|$ for all $z, w \in Y$. Also, Y can be equipped with a partial order $\preccurlyeq$ given by

$$z, w \in Y,\ z \preccurlyeq w \quad \text{if } z = w,\ \bigl(z \leqslant w,\ z, w \in [0, 15/8]\bigr),$$
$$\text{or } \bigl(z \in [0,2] \text{ and } w = 2\bigr).$$

Obviously, $(Y, d, \preccurlyeq)$ is an ordered complete metric space, is regular, and has property (H). Consider the nondecreasing function $g : Y \to Y$ given by

$$gz = \begin{cases} \frac{z}{2} & \text{if } z \in [0, \frac{15}{8}], \\ \frac{3}{2} & \text{if } z \in]\frac{15}{8}, 2]. \end{cases}$$

The function g satisfies condition (1) with respect to the function $S \in \mathcal{S}$ defined by

$$S(\alpha, \beta) = \beta - \frac{\alpha + 2}{\alpha + 1}\alpha$$

for all $\alpha, \beta \in [0, +\infty[$, the function $G \in \mathcal{G}$ given by $G(\delta, \theta, \lambda) = \delta + \theta + \lambda$ for all $\delta, \theta, \lambda \in [0, +\infty[$, and the lower semicontinuous function $\Lambda : Y \to [0, +\infty[$ defined by $\Lambda(z) = z$ for all $z \in Y$. Indeed, if $z \preccurlyeq w$ and $z, w \in [0, 15/8]$, then

$$\begin{aligned} &S\bigl(d(gz, gw) + \Lambda(gz) + \Lambda(gw),\ d(z,w) + \Lambda(z) + \Lambda(w)\bigr) \\ &\quad = S(w, 2w) = 2w - \frac{w+2}{w+1}w = \frac{w^2}{w+1} \geqslant 0. \end{aligned}$$

If $z \preccurlyeq w$ with $z \in [0,2]$ and $w = 2$, then

$$\begin{aligned} &S\bigl(d(gz, gw) + \Lambda(gz) + \Lambda(gw),\ d(z,w) + \Lambda(z) + \Lambda(w)\bigr) \\ &\quad = S(3, 4) = 4 - \frac{5}{4}3 = \frac{16 - 15}{4} \geqslant 0. \end{aligned}$$

If $w = z \in]15/8, 2]$, then

$$\begin{aligned} &S\bigl(d(gz, gw) + \Lambda(gz) + \Lambda(gw),\ d(z,w) + \Lambda(z) + \Lambda(w)\bigr) \\ &\quad = S(3, 2w) = 2w - \frac{5}{4}3 = \frac{8w - 15}{4} \geqslant 0. \end{aligned}$$

Since all conditions of Theorem 4 are satisfied, g has a unique fixed point $\zeta = 0$ in Y.

Note that, since g is not continuous, Theorem 1 cannot be used to affirm that g has a fixed point. Furthermore, from

$$d\left(g\frac{15}{8}, g2\right) = \frac{3}{2} - \frac{15}{16} = \frac{9}{16} > \frac{1}{8} = d\left(\frac{15}{8}, 2\right),$$

it is clear that we cannot use Theorem 2.2 of [7] in order to deduce that g has a fixed point.

The next example is aimed to show the usefulness of our contractive condition by illustrating its application to a mapping, which is not nonexpansive.

Example 4. Let $Y = [0,1]$ endowed with the usual metric $d(z,w) = |z-w|$ for all $z,w \in Y$. Clearly, $(Y,d,\preccurlyeq)$ is an ordered complete metric space if $\preccurlyeq$ coincide with the partial order $\leqslant$ on $\mathbb{R}$. Fix $\sigma \in [0,1[$ and consider the function $g: Y \to Y$ defined by

$$gz = \begin{cases} 0 & \text{if } z = 0, \\ \frac{\sigma}{2m} - \sigma\frac{2m-1}{2m}(2mz-1) & \text{if } \frac{1}{2m} \leqslant z \leqslant \frac{1}{2m-1},\ m \in \mathbb{N}, \\ \frac{\sigma}{2m} + \sigma\frac{2m+1}{2m}(2mz-1) & \text{if } \frac{1}{2m+1} \leqslant z \leqslant \frac{1}{2m},\ m \in \mathbb{N}. \end{cases}$$

Now, if we consider the function $\Lambda: Y \to [0,+\infty[$ defined by $\Lambda(z) = z$ for all $z \in Y$ and the function $G \in \mathcal{G}$ given by $G(\delta,\theta,\lambda) = \delta + \theta + \lambda$ for all $\delta,\theta,\lambda \in [0,+\infty[$, then we have

$$\begin{aligned} &G\bigl(d(gz,gw),\Lambda(gz),\Lambda(gw)\bigr) \\ &\quad = d(gz,gw) + \Lambda(gz) + \Lambda(gw) = 2\max\{gz,gw\} \\ &\quad \leqslant 2\max\{\sigma z,\sigma w\} = \sigma 2\max\{z,w\} \\ &\quad = \sigma\bigl[d(z,w) + \Lambda(z) + \Lambda(w)\bigr] = \sigma G\bigl(d(z,w),\Lambda(z),\Lambda(w)\bigr) \end{aligned}$$

for all $z,w \in Y$ with $z \preccurlyeq w$. Since all the conditions of Corollary 1 (or Corollary 2) are satisfied, we obtain that g has a unique fixed point in Y.

We remark that g is not nonexpansive if we choose σ suitably close to 1. In fact, if we choose $z = 1/(2m-1)$ and $w = 1/(m-1)$ with $m \in \mathbb{N}$ odd and $m \geqslant 3$, we get that

$$d(z,w) = \frac{m}{(m-1)(2m-1)} \leqslant \frac{3}{5(m-1)}.$$

Consequently,

$$d(gz,gw) = \frac{\sigma}{m-1} > d(z,w)$$

whenever $\sigma \in]3/5,1[$, and hence, g is not nonexpansive. This ensures that both the Banach contraction principle and Boyd–Wong result cannot be applied to obtain that g has a unique fixed point in the setting of metric space. Clearly, in the setting of ordered metric spaces cannot be applied the results of [7] to infer that g has a unique fixed point.

5 Application to ordered partial metric spaces

In this section, we apply the previous results to get some new fixed point theorems in the ordered partial metric setting. To this aim, we need just a function $S \in \mathcal{S}$. In particular, we establish a Matthews type fixed point theorem. We refer the reader to [5,8,9] and the references therein for more details on partial metric spaces.

Definition 2. Let Y be a non-empty set. A partial metric on Y is a function $p: Y \times Y \to [0,+\infty[$ such that

(p1) $z = w \Leftrightarrow p(z,z) = p(z,w) = p(w,w)$ for all $z, w \in Y$;
(p2) $p(z,z) \leqslant p(z,w)$ for all $z, w \in Y$;
(p3) $p(z,w) = p(w,z)$ for all $z, w \in Y$;
(p4) $p(z,w) \leqslant p(z,u) + p(u,w) - p(u,u)$ for all $z, w, u \in Y$.

The pair (Y,p) is called a partial metric space.

We stress that $z = w$ does not imply $p(z,w) = 0$. Moreover, a classic example of partial metric space is the pair $([0,+\infty[,p)$, where $p(z,w) = \max\{z,w\}$ for all $z, w \in [0,+\infty[$ (see [5] for more details).

Let (Y,p) be a partial metric space. A sequence $\{z_m\}$ in (Y,p) converges to a point $z \in Y$ if and only if $p(z,z) = \lim_{m\to+\infty} p(z,z_m)$. A sequence $\{z_m\}$ in (Y,p) is called a Cauchy sequence if there exists (and it is finite) $\lim_{n,m\to+\infty} p(z_n,z_m)$.

A partial metric space (Y,p) is complete if every Cauchy sequence $\{z_m\}$ in Y converges (with respect to the topology τ_p induced by p) to a point $z \in Y$ such that $p(z,z) = \lim_{n,m\to+\infty} p(z_n,z_m)$.

We note that if p is a partial metric on Y, then the function $d_p : Y \times Y \to [0,+\infty[$ given by

$$d_p(z,w) = 2p(z,w) - p(z,z) - p(w,w) \tag{8}$$

is a metric on Y. Furthermore, $\lim_{m\to+\infty} d_p(z_m,z) = 0$ if and only if

$$p(z,z) = \lim_{m\to+\infty} p(z_m,z) = \lim_{n,m\to+\infty} p(z_n,z_m).$$

Lemma 3. *Let (Y,p) be a partial metric space, and let $\Lambda : Y \to [0,+\infty[$ be defined by $\Lambda(z) = p(z,z)$ for all $z \in Y$. Then the function Λ is lower semicontinuous in the metric space (Y,d_p).*

Proof. Let $\{z_m\} \subset Y$ be a sequence, which converges to $z \in Y$ in the metric space (Y,d_p). Then we have

$$\begin{aligned}\Lambda(z) = p(z,z) &= \lim_{m\to+\infty} p(z_m,z_m)\\ &= \liminf_{m\to+\infty} p(z_m,z_m) = \liminf_{m\to+\infty} \Lambda(z_m).\end{aligned}$$

This assures that Λ is lower semicontinuous in z and hence in Y. □

Lemma 4. *Let (Y,p) be a partial metric space. Then*

(i) *$\{z_m\}$ is a Cauchy sequence in (Y,p) if and only if it is a Cauchy sequence in the metric space (Y,d_p).*
(ii) *A partial metric space (Y,p) is complete if and only if the metric space (Y,d_p) is complete.*

Now, we give our main result in the setting of ordered partial metric spaces. We recall that if (Y,p) is a partial metric space and $(Y,\preccurlyeq)$ is a partially ordered set, then we say that $(Y,p,\preccurlyeq)$ is an ordered partial metric space.

Theorem 5. *Let $(Y, p, \preccurlyeq)$ be a complete ordered partial metric space, and let $g : Y \to Y$ be a nondecreasing mapping. Suppose that there exists a function $S \in \mathcal{S}$ such that*

$$S\big(p(gz, gw), p(z, w)\big) \geqslant 0 \tag{9}$$

for all $z, w \in Y$, $z \preccurlyeq w$. If there exists a point $z_0 \in Y$ such that $z_0 \preccurlyeq gz_0$ and g is continuous in the metric space (Y, d_p), then g has a fixed point $\zeta \in Y$ such that $p(\zeta, \zeta) = 0$.

Proof. We notice that from (8) it follows

$$p(z, w) = \frac{d_p(z, w) + p(z, z) + p(w, w)}{2} \tag{10}$$

for all $z, w \in Y$. Now, we consider Y equipped with the metric $d = 2^{-1}d_p$. Since (Y, p) is complete, by Lemma 4 we deduce that the metric space (Y, d) is complete. Thus, from (9) and (10), we obtain that for the mapping g, the following condition holds:

$$S\big(G\big(d(gz, gw), \varLambda(gz), \varLambda(gw)\big), G\big(d(z, w), \varLambda(z), \varLambda(w)\big)\big) \geqslant 0$$

for all $z, w \in Y$, $z \preccurlyeq w$, where $G \in \mathcal{G}$ is defined by $G(\delta, \theta, \lambda) = \delta + \theta + \lambda$ for all $\delta, \theta, \lambda \in [0, +\infty[$. Hence, the mapping $g : (X, d) \to (X, d)$ satisfies all the conditions of Theorem 1, and so, g has a fixed point such that $p(\zeta, \zeta) = 2\varLambda(\zeta) = 0$. $\square$

We notice that the Matthews fixed point theorem in the setting of ordered partial metric spaces follows from Theorem 5. It is sufficient to choose the function $S \in \mathcal{S}$ defined by $(\alpha, \beta) = \sigma\beta - \alpha$ for all $\alpha, \beta \in [0, +\infty[$ with $\sigma \in [0, 1[$.

Theorem 6. *Let $(Y, p, \preccurlyeq)$ be a complete ordered partial metric space, and let $g : Y \to Y$ be a nondecreasing mapping. Suppose that there exists a function $S \in \mathcal{S}$ such that*

$$S\big(p(gz, gw), p(z, w)\big) \geqslant 0 \tag{11}$$

for all $z, w \in Y$, $z \preccurlyeq w$. If there exists a point $z_0 \in Y$ such that $z_0 \preccurlyeq gz_0$ and Y is regular, then g has a fixed point ζ such that $p(\zeta, \zeta) = 0$.

Proof. We consider Y equipped with the metric $d = 2^{-1}d_p$. Since (Y, p) is complete, by Lemma 4 we deduce that the metric space (Y, d) is complete. By applying Lemma 3, we have that the function $\varLambda : Y \to [0, +\infty[$ defined by $\varLambda(z) = 2^{-1}p(z, z)$ is lower semicontinuous in (Y, d). Thus, from (11) and (10) we obtain that the mapping g satisfies the following condition:

$$S\big(G\big(d(gz, gw), \varLambda(gz), \varLambda(gw)\big), G\big(d(z, w), \varLambda(z), \varLambda(w)\big)\big) \geqslant 0$$

for all $z, w \in Y$, $z \preccurlyeq w$, where $G \in \mathcal{G}$ is defined by $G(\delta, \theta, \lambda) = \delta + \theta + \lambda$ for all $\delta, \theta, \lambda \in [0, +\infty[$. Hence, the mapping $g : (X, d) \to (X, d)$ satisfies all the conditions of Theorem 2. By Theorem 2 we can affirm that g has a fixed point such that $p(\zeta, \zeta) = 2\varLambda(\zeta) = 0$. $\square$

Remark 3. If in Theorems 5 and 6, we assume that the ordered metric space $(Y, p, \preccurlyeq)$ has property (H) (see p. 2), then the fixed point of g is unique.

References

1. H. Argoubi, B. Samet, C. Vetro, Nonlinear contractions involving simulation functions in a metric space with a partial order, *J. Nonlinear Sci. Appl.*, **8**(6):1082–1094, 2015.
2. D.W. Boyd, J.S.W. Wong, On nonlinear contractions, *Proc. Am. Math. Soc.*, **20**:458–464, 1969.
3. M. Jleli, B. Samet, C. Vetro, Fixed point theory in partial metric spaces via φ-fixed point's concept in metric spaces, *J. Inequal. Appl.*, **2014**:426, 2014.
4. F. Khojasteh, S. Shukla, S. Radenović, A new approach to the study of fixed point theorems via simulation functions, *Filomat*, **29**(6):1189–1194, 2015.
5. S.G. Matthews, Partial metric topology, *Ann. N. Y. Acad. Sci.*, **728**:183–197, 1994.
6. A. Nastasi, P. Vetro, Fixed point results on metric and partial metric spaces via simulation functions, *J. Nonlinear Sci. Appl.*, **8**(6):1059–1069, 2015.
7. J.J. Nieto, R. Rodriguez-Lopez, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, *Order*, **22**(3):223–239, 2005.
8. S.J. O'Neill, Partial metrics, valuations and domain theory, *Ann. N. Y. Acad. Sci.*, **806**(1):304–315, 1996.
9. D. Paesano, P. Vetro, Suzuki's type characterizations of completeness for partial metric spaces and fixed points for partially ordered metric spaces, *Topology Appl.*, **159**(3):911–920, 2012.
10. A.C.M. Ran, M.C.B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, *Proc. Am. Math. Soc.*, **132**:1435–1443, 2004.
11. S. Reich, Fixed points of contractive functions, *Boll. Unione Mat. Ital.*, **5**:26–42, 1972.
12. B.E. Rhoades, Some theorems on weakly contractive maps, *Nonlinear Anal., Theory Methods Appl.*, **47**(4):2683–2693, 2001.
13. B. Samet, C. Vetro, F. Vetro, From metric spaces to partial metric spaces, *Fixed Point Theory Appl.*, **2013**:5, 2013.
14. C. Vetro, F. Vetro, Metric or partial metric spaces endowed with a finite number of graphs: A tool to obtain fixed point results, *Topology Appl.*, **164**:125–137, 2014.