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Fixed point results in b-metric spaces using families of control functions and their application to dynamic programming

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Abstract. We obtain some fixed point theorems for mappings acting in b-metric spaces. The results extend those obtained in [R.P. Agarwal, E. Karapınar, A.-F. Roldán-López-de-Hierro, On an extension of contractivity conditions via auxiliary functions, *Fixed Point Theory Appl.*, 2015:104, 2015] using families of control functions, here also through conditions that involve α-admissibility of type S. We furnish an illustrative example to demonstrate the validity of the hypotheses and the degree of usefulness of our results. As an application, the existence of solution for functional equations arising in dynamic programming is discussed, followed by suitable examples.

Keywords: admissibility types, b-metric spaces, control functions, dynamic programming.

1 Introduction

Many researchers extended Banach contraction principle by considering more general contractive mappings on various distance spaces – b-metric spaces [5, 8] are one of the

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examples. Alber and Guerre-Delabrière [3] were the first to introduce weak contractive conditions in the setup of Hilbert spaces. On the other hand, Khan, Swaleh and Sessa [9] introduced the concept of an altering distance function, which is a control function that alters the distance between two points in a metric space. After the appearance of Rhoades theorem (see [12]), many results have been obtained involving contractivity conditions in which some families of functions play a key role. In particular, Agarwal et al. [1] unified most of these results for mappings acting in metric spaces using certain families of altering distance functions.

The aim of the present manuscript is to study what kind of altering distance functions might we include in an efficient contractivity condition in the case when we are given mappings acting in a b-metric space. Moreover, the concept of α-admissibility is used, it was introduced by Samet et al. [13] and extended by Sintunavarat [14] as α-admissibility of type S. Using these concepts, we study sufficient conditions on the functions that appear in very complex contractivity conditions with α-admissibility of type S in the framework of b-metric spaces.

We furnish an illustrative example to demonstrate the validity of the hypotheses of our results and necessity of some assumptions. Our results generalize and improve several fixed point results in metric spaces and b-metric spaces. As an application, the existence of solution for functional equations arising in dynamic programming is discussed, followed by suitable examples.

2 Preliminaries

In this section, we will introduce some essential notations, definitions and preliminary results that will be used in the article. Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{R}_+$ and $\mathbb{R}$ the sets of positive integers, nonnegative real numbers and real numbers, respectively.

Recall that a function $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is called an altering distance function [9] if the following properties hold:

1. Λ is continuous and nondecreasing;
2. $\Lambda(t) = 0$ iff $t = 0$.

Definition 1. (See [8].) Let X be a nonempty set, and let $s \geqslant 1$ be a given real number. A function $d : X \times X \to \mathbb{R}_+$ satisfying the following conditions:

(B_1) $d(x, y) = 0$ if and only if $x = y$;
(B_2) $d(x, y) = d(y, x)$;
(B_3) $d(x, y) \leqslant s[d(x, z) + d(z, y)]$

for all $x, y, z \in X$ is called a b-metric on X. The pair (X, d) is called a b-metric space with coefficient $s \geqslant 1$.

Any metric space is a b-metric space with $s = 1$, and the class of b-metric spaces is effectively larger than that of metric spaces. The easiest example of a b-metric space, which is not a metric space, is $(\mathbb{R}, d)$, where $d(x, y) = (x - y)^2$. A more general example is the following:

Example 1. Let (X,d) be a metric space, and let the mapping $d_b : X \times X \to \mathbb{R}_+$ be defined by $d_b(x,y) = [d(x,y)]^p$ for all $x,y \in X$, where $p > 1$ is a fixed real number. Then (X,d_b) is a b-metric space with coefficient $s = 2^{p-1}$. The triangular inequality (B_3) can easily be checked using the convexity of function $\mathbb{R}_+ \ni x \mapsto x^p$.

The concepts of b-convergent sequence, b-Cauchy sequence, b-continuity and b-completeness in b-metric spaces are introduced in the same way as in metric spaces (see, e.g., [7]). In particular, a function $f : X \to Y$ between two b-metric spaces is called b-continuous at a point $x \in X$ if it is b-sequentially continuous at x, that is, if $\{x_n\}$ is b-convergent to x in X implies that $\{fx_n\}$ is b-convergent to fx in Y.

Each b-convergent sequence in a b-metric space has a unique limit, and it is also a b-Cauchy sequence. However, a b-metric itself might not be continuous. Hence, the following lemma about b-convergent sequences is required in the proof of our main results.

Lemma 1. (*See* [2].) *Let (X,d) be a b-metric space with coefficient $s \geqslant 1$, and let $\{x_n\}$ and $\{y_n\}$ be b-convergent to points $x,y \in X$, respectively. Then*

$$\frac{1}{s^2} d(x,y) \leqslant \liminf_{n\to\infty} d(x_n,y_n) \leqslant \limsup_{n\to\infty} d(x_n,y_n) \leqslant s^2 d(x,y).$$

If $x = y$, then $\lim_{n\to\infty} d(x_n,y_n) = 0$. Moreover, for each $z \in X$, we have

$$\frac{1}{s} d(x,z) \leqslant \liminf_{n\to\infty} d(x_n,z) \leqslant \limsup_{n\to\infty} d(x_n,z) \leqslant s d(x,z).$$

The concept of α-admissibility was first introduced by Samet et al. [13] and extended as admissibility of type S by Sintunavarat [14] in the framework of metric spaces and b-metric spaces, respectively.

Definition 2. (See [13, 14].) Let X be a nonempty set, let $\alpha : X \times X \to \mathbb{R}_+$ and $f : X \to X$ be two mappings, and $s \geqslant 1$ be a given real number. Then we say that

(i) the mapping f is α-admissible if

$$x,y \in X \quad \text{and} \quad \alpha(x,y) \geqslant 1 \quad \Longrightarrow \quad \alpha(fx,fy) \geqslant 1;$$

this is denoted as $f \in \mathcal{P}(X,\alpha)$;

(ii) the mapping f is α-admissible of type S if

$$x,y \in X \quad \text{and} \quad \alpha(x,y) \geqslant s \quad \Longrightarrow \quad \alpha(fx,fy) \geqslant s;$$

this is denoted as $f \in \mathcal{P}_s(X,\alpha)$;

(iii) f is a weak α-admissible mapping if

$$x \in X \quad \text{and} \quad \alpha(x,fx) \geqslant 1 \quad \Longrightarrow \quad \alpha(fx,ffx) \geqslant 1;$$

this is denoted as $f \in \mathcal{WP}(X,\alpha)$;

(iv) f is a weak α-admissible mapping of type S if

$$x \in X \quad \text{and} \quad \alpha(x, fx) \geqslant s \quad \Longrightarrow \quad \alpha(fx, ffx) \geqslant s;$$

this is denoted as $f \in \mathcal{WP}_s(X, \alpha)$.

Remark 1. (See [14].) It is easy to see that the following assertions hold:

1. α-admissibility $\Rightarrow$ weak α-admissibility, that is,

$$\mathcal{P}(X, \alpha) \subset \mathcal{WP}(X, \alpha).$$

2. α-admissibility of type $S \Rightarrow$ weak α-admissibility of type S, that is,

$$\mathcal{P}_s(X, \alpha) \subset \mathcal{WP}_s(X, \alpha).$$

None of these inclusions can be reversed. Moreover, $\mathcal{P}(X, \alpha) \neq \mathcal{P}_s(X, \alpha)$, that is, the classes of α-admissible mappings and α-admissible mappings of type S are, in general, independent.

3 Main results

Before discussing our main result, we will introduce the following four families of functions. They are defined in a similar way as in the paper [1], but adapted for the use in b-metric spaces. In what follows, $s \geqslant 1$ will be a given real number.

Let $\mathcal{G}_1$ be the family of all functions $\phi : \mathbb{R}_+^n \to \mathbb{R}_+$ with $n \geqslant 2$ such that, for all $r, z_3, z_4, \ldots, z_n \in \mathbb{R}_+$:

(G_{11}) ϕ is continuous in its first two arguments;
(G_{12}) $\phi(r, 0, z_3, z_4, \ldots, z_n) \leqslant r/(2s)$;
(G_{13}) $\phi(r, r, z_3, z_4, \ldots, z_n) \leqslant r$ if $r > 0$;
(G_{14}) $\phi(0, r, z_3, z_4, \ldots, z_n) \leqslant r$ if $r > 0$.

Let $\mathcal{G}_2$ be the family of all functions $\phi : \mathbb{R}_+^n \to \mathbb{R}_+$ with $n \geqslant 3$ such that, for all $q, r, z_4, z_5, \ldots, z_n \in \mathbb{R}_+$:

(G_{21}) ϕ is continuous in its first three arguments;
(G_{22}) $\phi(q, 0, r, z_4, z_5, \ldots, z_n) \leqslant \max\{q/(2s), r\}$;
(G_{23}) $\phi(q, q, r, z_4, z_5, \ldots, z_n) \leqslant \max\{q, r\}$;
(G_{24}) $\phi(r, 0, 0, z_4, z_5, \ldots, z_n) \leqslant r$ for all $r > 0$.

Let $\mathcal{G}_3$ be the family of all functions $\phi : \mathbb{R}_+^n \to \mathbb{R}_+$ with $n \geqslant 3$ and $s \geqslant 1$ such that, for all $q, r, z_4, z_5, \ldots, z_n \in \mathbb{R}_+$:

(G_{31}) ϕ is continuous in its first three arguments;
(G_{32}) $\phi(r, q, r, z_4, z_5, \ldots, z_n) \leqslant \max\{q, r\}$;
(G_{33}) $\phi(0, 0, r, z_4, z_5, \ldots, z_n) \leqslant r$ if $r > 0$;
(G_{34}) $\phi(0, r, 0, z_4, z_5, \ldots, z_n) \leqslant r$ if $r > 0$.

Let $\mathcal{G}_4$ be the family of all functions $\phi : \mathbb{R}_+^n \to \mathbb{R}_+$ with $n \geqslant 4$ such that, for all $q, r, t, z_5, z_6, \ldots, z_n \in \mathbb{R}_+$:

(G_{41}) ϕ is continuous in its first four arguments;
(G_{42}) $\phi(q, 0, r, t, z_5, z_6, \ldots, z_n) \leqslant \max\{q/(2s), r, t\}$;
(G_{43}) $\phi(r, r, r, 0, z_5, z_6, \ldots, z_n) \leqslant r$ if $r > 0$;
(G_{44}) $\phi(r, r, 0, 0, z_5, z_6, \ldots, z_n) \leqslant r$ if $r > 0$.

Some examples of functions belonging to these classes (in the case $s = 1$) are given in [1]. It is easy to modify these examples to the case when $s > 1$.

We will present fixed point results for mappings belonging to the class $\mathcal{WP}_s(X, \alpha)$. Throughout this paper, $\mathrm{Fix}(f)$ denotes the set of all fixed points of a self-mapping f on a nonempty set X, that is, $\mathrm{Fix}(f) = \{x \in X\colon fx = x\}$. Also, for all elements x and y in a b-metric space (X, d) with coefficient $s \geqslant 1$ and the given functions $\phi_i \in \mathcal{G}_i$, $i \in \{1, 2, 3, 4\}$, we will denote

$$\begin{aligned} M_s(x, y) = \max\bigl\{&d(x, y),\ d(x, fx),\ d(y, fy),\\ &\phi_1\bigl(d(x, fy), d(fx, y)\bigr),\ \phi_2\bigl(d(x, fy), d(fx, y), d(x, y)\bigr),\\ &\phi_3\bigl(d(x, fx), d(y, fy), d(x, y)\bigr),\\ &\phi_4\bigl(d(x, fy), d(fx, y), d(x, fx), d(y, fy)\bigr)\bigr\}. \end{aligned}$$

Definition 3. Let (X, d) be a b-metric space with coefficient $s \geqslant 1$, let $\alpha : X \times X \to \mathbb{R}_+$ be a given mapping, and let $\varLambda_1, \varLambda_2 : \mathbb{R}_+ \to \mathbb{R}_+$ be two altering distance functions. We say that a mapping $f : X \to X$ is an $(\alpha, \varLambda_1, \varLambda_2)_s$-contraction mapping if the following condition holds:

$$\begin{aligned} &x, y \in X \quad \text{with } \alpha(x, y) \geqslant s\\ &\quad\implies \quad \varLambda_1\bigl(s^4 d(fx, fy)\bigr) \leqslant \varLambda_1\bigl(M_s(x, y)\bigr) - \varLambda_2\bigl(M_s(x, y)\bigr) \end{aligned} \tag{1}$$

for some functions $\phi_i \in \mathcal{G}_i$, $i \in \{1, 2, 3, 4\}$. We denote by $\varDelta_s(X, \alpha, \varLambda_1, \varLambda_2)$ the collection of all $(\alpha, \varLambda_1, \varLambda_2)_s$-contraction mappings on a b-metric space (X, d).

Theorem 1. *Let (X, d) be a b-complete b-metric space with coefficient $s \geqslant 1$, let $\varLambda_1, \varLambda_2 : \mathbb{R}_+ \to \mathbb{R}_+$ be two altering distance functions, and let $\alpha : X \times X \to \mathbb{R}_+$ and $f : X \to X$ be given mappings. Suppose that the following conditions hold*:

(S_1) $f \in \varDelta_s(X, \alpha, \varLambda_1, \varLambda_2) \cap \mathcal{WP}_s(X, \alpha)$;
(S_2) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geqslant s$;*
(S_3) *α has a transitive property of type S, that is, for $x, y, z \in X$,*

$$\alpha(x, y) \geqslant s \quad \text{and} \quad \alpha(y, z) \geqslant s \quad \implies \quad \alpha(x, z) \geqslant s;$$

(S_4) *f is b-continuous.*

Then $\mathrm{Fix}(f) \neq \emptyset$.

Proof. By the given condition (S_2) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geqslant s$. Define the sequence $\{x_n\}$ by $x_{n+1} = fx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If there is $n \in \mathbb{N} \cup \{0\}$ so that $x_n = x_{n+1}$, then we have $x_n \in \mathrm{Fix}(f)$, and hence, the conclusion holds. So we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. It follows that

$$d(x_n, x_{n+1}) > 0$$

for all $n \in \mathbb{N} \cup \{0\}$. Now, we need to prove that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2}$$

It follows by induction from $f \in \mathcal{WP}_s(X, \alpha)$ and $\alpha(x_0, fx_0) \geqslant s$ that

$$\alpha(x_n, x_{n+1}) \geqslant s \tag{3}$$

for all $n \in \mathbb{N} \cup \{0\}$. It follows from $f \in \varDelta_s(X, \alpha, \varLambda_1, \varLambda_2)$ that inequality (3) implies

$$\begin{aligned}\varLambda_1\big(d(fx_n, fx_{n+1})\big) &\leqslant \varLambda_1\big(s^4 d(fx_{n+1}, fx_n)\big)\\ &\leqslant \varLambda_1\big(M_s(x_n, x_{n+1})\big) - \varLambda_2\big(M_s(x_n, x_{n+1})\big)\end{aligned} \tag{4}$$

for all $n \in \mathbb{N} \cup \{0\}$. Note that for each $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}&M_s(x_n, x_{n+1})\\ &\quad= \max\big\{d(x_n, x_{n+1}),\ d(x_n, fx_n),\ d(x_{n+1}, fx_{n+1}),\\ &\qquad\quad \phi_1\big(d(x_n, fx_{n+1}), d(fx_n, x_{n+1})\big),\\ &\qquad\quad \phi_2\big(d(x_n, fx_{n+1}), d(fx_n, x_{n+1}), d(x_n, x_{n+1})\big),\\ &\qquad\quad \phi_3\big(d(x_n, fx_n), d(x_{n+1}, fx_{n+1}), d(x_n, x_{n+1})\big),\\ &\qquad\quad \phi_4\big(d(x_n, fx_{n+1}), d(fx_n, x_{n+1}), d(x_n, fx_n), d(x_{n+1}, fx_{n+1})\big)\big\}.\end{aligned}$$

Let $d_n = d(x_n, x_{n+1})$ for all $n \in \mathbb{N}_0$. Taking into account the properties of considered functions in $\bigcup_{i=1}^4 \mathcal{G}_i$, we deduce that

$$d(x_n, x_{n+1}) = d_n, \qquad d(x_n, fx_n) = d(x_n, x_{n+1}) = d_{n+1},$$
$$d(x_{n+1}, fx_{n+1}) = d(x_{n+1}, x_{n+2}) = d_{n+1},$$

$$\begin{aligned}&\phi_1\big(d(x_n, fx_{n+1}), d(fx_n, x_{n+1})\big)\\ &\quad= \phi_1\big(d(x_n, x_{n+2}), d(x_{n+1}, x_{n+1})\big) = \phi_1\big(d(x_n, x_{n+2}), 0\big)\\ &\quad\overset{G_{12}}{\leqslant} \frac{d(x_n, x_{n+2})}{2s} \leqslant \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}\\ &\quad\leqslant \max\big\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\big\} = \max\{d_n, d_{n+1}\},\end{aligned}$$

$$\begin{aligned}&\phi_2\big(d(x_n, fx_{n+1}), d(fx_n, x_{n+1}), d(x_n, x_{n+1})\big)\\ &\quad= \phi_2\big(d(x_n, x_{n+2}), 0, d(x_n, x_{n+1})\big) \overset{G_{22}}{\leqslant} \max\left\{\frac{d(x_n, x_{n+2})}{2s}, d(x_n, x_{n+1})\right\}\\ &\quad\leqslant \max\left\{\frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}, d(x_n, x_{n+1})\right\} \leqslant \max\{d_n, d_{n+1}\},\end{aligned}$$

$$\begin{aligned}
&\phi_3\big(d(x_n, fx_n), d(x_{n+1}, fx_{n+1}), d(x_n, x_{n+1})\big) \\
&\quad = \phi_3\big(d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})\big) \\
&\quad = \phi_3(d_n, d_{n+1}, d_n) \overset{\mathrm{G}_{32}}{\leqslant} \max\{d_n, d_{n+1}\}, \\
&\phi_4\big(d(x_n, fx_{n+1}), d(fx_n, x_{n+1}), d(x_n, fx_n), d(x_{n+1}, fx_{n+1})\big) \\
&\quad = \phi_4\big(d(x_n, x_{n+2}), 0, d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\big) \\
&\quad \overset{\mathrm{G}_{42}}{\leqslant} \max\left\{\frac{d(x_n, x_{n+2})}{2s}, d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\right\} \\
&\quad \leqslant \max\left\{\frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}, d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\right\} \\
&\quad \leqslant \max\big\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\big\} = \max\{d_n, d_{n+1}\}.
\end{aligned}$$

Hence,

$$M_s(x_n, x_{n+1}) = \max\{d_n, d_{n+1}\}.$$

If $M_s(x_{n^*}, x_{n^*+1}) = d_{n^*+1}$ for some $n^* \in \mathbb{N} \cup \{0\}$, then inequality (4) implies that

$$\Lambda_1\big(d(x_{n^*+1}, x_{n^*+2})\big) \leqslant \Lambda_1\big(d(x_{n^*+1}, x_{n^*+2})\big) - \Lambda_2\big(d(x_{n^*+1}, x_{n^*+2})\big),$$

hence $d(x_{n^*+1}, x_{n^*+2}) = 0$, which is a contradiction. Therefore, $M_s(x_n, x_{n+1}) = d_n = d(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$. From (4) we have

$$\Lambda_1\big(d(x_{n+1}, x_{n+2})\big) \leqslant \Lambda_1\big(d(x_n, x_{n+1})\big) - \Lambda_2\big(d(x_n, x_{n+1})\big) \tag{5}$$

for all $n \in \mathbb{N} \cup \{0\}$. Since Λ_1 is a nondecreasing mapping, $\{d(x_n, x_{n+1})\}$ is a decreasing sequence in $\mathbb{R}$. Since $\{d(x_n, x_{n+1})\}$ is bounded from below, there exists $r \geqslant 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r.$$

Letting $n \to \infty$ in (5), we get

$$\Lambda_1(r) \leqslant \Lambda_1(r) - \Lambda_2(r).$$

This is only possible if $\Lambda_2(r) = 0$ and thus $r = 0$. Hence, (2) is proved.

Next, we prove that $\{x_n\}$ is a b-Cauchy sequence in X. Assume to the contrary that there exists $\epsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k) > m(k) \geqslant k$ and

$$d(x_{m(k)}, x_{n(k)}) \geqslant \epsilon > d(x_{m(k)}, x_{n(k)-1}) \tag{6}$$

for all $k \in \mathbb{N}$. By (B_3) and (6) we get

$$\begin{aligned}
\epsilon \leqslant d(x_{m(k)}, x_{n(k)}) &\leqslant sd(x_{m(k)}, x_{n(k)-1}) + sd(x_{n(k)-1}, x_{n(k)}) \\
&< s\epsilon + sd(x_{n(k)-1}, x_{n(k)}).
\end{aligned} \tag{7}$$

Taking the upper limit in (7) as $k \to \infty$ and using (2), we get

$$\epsilon \leqslant \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)}) \leqslant s\epsilon. \tag{8}$$

Also from (B_3) we obtain

$$d(x_{m(k)}, x_{n(k)}) \leqslant s\big[d(x_{m(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)})\big] \tag{9}$$

and

$$d(x_{m(k)}, x_{n(k)+1}) \leqslant s\big[d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1})\big]. \tag{10}$$

Taking the upper limit as $k \to \infty$ in (9) and (10), from (2) and (8) we get

$$\epsilon \leqslant s\Big(\limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)+1})\Big) \quad \text{and} \quad \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) \leqslant s^2\epsilon,$$

i.e.,

$$\frac{\epsilon}{s} \leqslant \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) \leqslant s^2\epsilon. \tag{11}$$

Similarly, we can show that

$$\frac{\epsilon}{s} \leqslant \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)}) \leqslant s^2\epsilon. \tag{12}$$

Finally, we obtain

$$\begin{aligned}
&d(x_{m(k)+1}, x_{n(k)+1})\\
&\quad\leqslant s\big[d(x_{m(k)+1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)+1})\big]\\
&\quad\leqslant sd(x_{m(k)+1}, x_{m(k)}) + s^2\big[d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1})\big].
\end{aligned}$$

By taking the upper limit as $k \to \infty$ in the above inequality, we have

$$\limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) \leqslant s^3\epsilon. \tag{13}$$

Using (B_3) again, we have

$$\begin{aligned}
&d(x_{m(k)}, x_{n(k)})\\
&\quad\leqslant s\big[d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{n(k)})\big]\\
&\quad\leqslant sd(x_{m(k)}, x_{m(k)+1}) + s^2\big[d(x_{m(k)+1}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)})\big].
\end{aligned} \tag{14}$$

Taking the upper limit as $k \to \infty$ in (14), from (2) and (8) we get

$$\frac{\epsilon}{s^2} \leqslant \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}). \tag{15}$$

From (13) and (15) we get

$$\frac{\epsilon}{s^2} \leqslant \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) \leqslant s^3\epsilon.$$

Using the transitivity property of type S of α, we get $\alpha(x_{m(k)}, x_{n(k)}) \geqslant s$. Since $f \in \Delta_s(X, \alpha, \Lambda_1, \Lambda_2)$, this implies

$$\begin{aligned}
&\Lambda_1\bigl(s^4\, d(x_{m(k)+1}, x_{n(k)+1})\bigr) \\
&\quad = \Lambda_1\bigl(s^4 d(fx_{m(k)}, fx_{n(k)})\bigr) \\
&\quad \leqslant \Lambda_1\bigl(M_s(x_{m(k)}, x_{n(k)})\bigr) - \Lambda_2\bigl(M_s(x_{m(k)}, x_{n(k)})\bigr),
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
M_s(x_{m(k)}, x_{n(k)}) = \max\bigl\{ & d(x_{m(k)}, x_{n(k)}),\ d(x_{m(k)}, fx_{m(k)}),\ d(x_{n(k)}, fx_{n(k)}), \\
& \phi_1\bigl(d(x_{m(k)}, fx_{n(k)}), d(fx_{m(k)}, x_{n(k)})\bigr), \\
& \phi_2\bigl(d(x_{m(k)}, fx_{n(k)}), d(fx_{m(k)}, x_{n(k)}), d(x_{m(k)}, x_{n(k)})\bigr), \\
& \phi_3\bigl(d(x_{m(k)}, fx_{m(k)}), d(x_{n(k)}, fx_{n(k)}), d(x_{m(k)}, x_{n(k)})\bigr), \\
& \phi_4\bigl(d(x_{m(k)}, fx_{n(k)}), d(fx_{m(k)}, x_{n(k)}), \\
& \quad d(x_{m(k)}, fx_{m(k)}), d(x_{n(k)}, fx_{n(k)})\bigr)\bigr\}.
\end{aligned}$$

Taking the upper limit as $k \to \infty$ in the above inequality and using (2), (8), (11) and (12), we get

$$\limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)}) \leqslant s\epsilon, \qquad \limsup_{k\to\infty} d(x_{m(k)}, x_{m(k)+1}) = 0,$$
$$\limsup_{k\to\infty} d(x_{n(k)}, x_{n(k)+1}) = 0,$$

$$\begin{aligned}
&\limsup_{k\to\infty} \phi_1\bigl(d(x_{m(k)}, fx_{n(k)}), d(fx_{m(k)}, x_{n(k)})\bigr) \\
&\quad \overset{\mathrm{G}_{11}}{\leqslant} \phi_1\Bigl(\limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}),\ \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)})\Bigr) \\
&\quad = \phi_1\bigl(s^2\epsilon, s^2\epsilon\bigr) \overset{\mathrm{G}_{13}}{\leqslant} s^2\epsilon;
\end{aligned}$$

$$\begin{aligned}
&\limsup_{k\to\infty} \phi_2\bigl(d(x_{m(k)}, fx_{n(k)}), d(fx_{m(k)}, x_{n(k)}), d(x_{m(k)}, x_{n(k)})\bigr) \\
&\quad \overset{\mathrm{G}_{21}}{\leqslant} \phi_2\Bigl(\limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}),\ \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)}), \\
&\qquad\quad \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)})\Bigr) \\
&\quad \leqslant \phi_2\bigl(s^2\epsilon, s^2\epsilon, s\epsilon\bigr) \overset{\mathrm{G}_{23}}{\leqslant} \max\{s^2\epsilon, s\epsilon\};
\end{aligned}$$

$$\begin{aligned}
&\limsup_{k\to\infty} \phi_3\bigl(d(x_{m(k)}, fx_{m(k)}), d(x_{n(k)}, fx_{n(k)}), d(x_{m(k)}, x_{n(k)})\bigr) \\
&\quad \overset{\mathrm{G}_{31}}{\leqslant} \phi_3\Bigl(\limsup_{k\to\infty} d(x_{m(k)}, x_{m(k)+1}),\ \limsup_{k\to\infty} d(x_{n(k)}, x_{n(k)+1}), \\
&\qquad\quad \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)})\Bigr) \\
&\quad = \phi_3(0, 0, s\epsilon) \overset{\mathrm{G}_{33}}{\leqslant} s\epsilon,
\end{aligned}$$

$$\begin{aligned}
&\limsup_{k\to\infty} \phi_4\big(d(x_{m(k)}, fx_{n(k)}), d(fx_{m(k)}, x_{n(k)}), d(x_{m(k)}, fx_{m(k)}),\\
&\qquad d(x_{n(k)}, fx_{n(k)})\big)\\
&\overset{\mathrm{G}_{41}}{\leqslant} \phi_4\Big(\limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}),\ \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)}),\\
&\qquad \limsup_{k\to\infty} d(x_{m(k)}, x_{m(k)+1}),\ \limsup_{k\to\infty} d(x_{n(k)}, x_{n(k)+1})\Big)\\
&\leqslant \phi_4(s^2\epsilon, s^2\epsilon, 0, 0) \overset{\mathrm{G}_{44}}{\leqslant} s^2\epsilon.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\epsilon = \max\left\{\epsilon, \frac{\frac{\epsilon}{s} + \frac{\epsilon}{s}}{2s}\right\} &\leqslant \limsup_{k\to\infty} M_s(x_{m(k)}, x_{n(k)})\\
&\leqslant \max\{s\epsilon, s^2\epsilon, s^2\epsilon, \epsilon, s^2\epsilon\} = s^2\epsilon.
\end{aligned}$$

Similarly, we can show that

$$\epsilon \leqslant \liminf_{k\to\infty} M_s(x_{m(k)}, x_{n(k)}) \leqslant s^2\epsilon.$$

Taking the upper limit as $k \to \infty$ in (16), we have

$$\begin{aligned}
\Lambda_1(s^2\epsilon) = \Lambda_1\left(s^4\left(\frac{\epsilon}{s^2}\right)\right) &\leqslant \Lambda_1\Big(s^4 \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1})\Big)\\
&\leqslant \Lambda_1\Big(\limsup_{k\to\infty} M_s(x_{m(k)}, x_{n(k)})\Big) - \Lambda_2\Big(\liminf_{k\to\infty} M_s(x_{m(k)}, x_{n(k)})\Big)\\
&\leqslant \Lambda_1(s^2\epsilon) - \Lambda_2(\epsilon).
\end{aligned}$$

This implies that $\Lambda_2(\epsilon) = 0$ and then $\epsilon = 0$, which is a contradiction. Therefore, $\{x_n\}$ is a b-Cauchy sequence.

By b-completeness of the b-metric space (X, d) there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$. By b-continuity of f we get $x_{n+1} = fx_n \to fx$ as $n \to \infty$, and since the limit of a sequence is unique, we deduce that $fx = x$. This shows that $\mathrm{Fix}(f) \neq \emptyset$. □

Now we present a result that does not use continuity of the given mapping.

Theorem 2. *Let (X, d) be a b-complete b-metric space with coefficient $s \geqslant 1$, let $\Lambda_1, \Lambda_2 : \mathbb{R}_+ \to \mathbb{R}_+$ be two altering distance functions, and let $\alpha : X \times X \to \mathbb{R}_+$ and $f : X \to X$ be given mappings. Suppose that the following conditions hold*:

- (S_1) *$f \in \Delta_s(X, \alpha, \Lambda_1, \Lambda_2) \cap \mathcal{WP}_s(X, \alpha)$;*
- (S_2) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geqslant s$;*
- (S_3) *α has a transitive property of type S;*
- ($\widetilde{\mathrm{S}_4}$) *(X, d) is α_s-regular, that is, if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geqslant s$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geqslant s$ for all $n \in \mathbb{N}$.*

Then $\mathrm{Fix}(f) \neq \emptyset$.

Proof. As in the proof of Theorem 1, we obtain a b-Cauchy sequence $\{x_n\}$ in the b-complete b-metric space (X,d) satisfying $\alpha(x_n,x_{n+1}) \geqslant s$ for all $n \in \mathbb{N}$. Hence, there exists $x \in X$ such that

$$\lim_{n\to\infty} d(x_n,x) = 0. \tag{17}$$

By the α_s-regularity of X we have $\alpha(x_n,x) \geqslant s$ for all $n \in \mathbb{N}$. It follows from $f \in \Delta_s(X,\alpha,\Lambda_1,\Lambda_2)$ that

$$\Lambda_1(s^4 d(fx_n,fx)) \leqslant \Lambda_1\bigl(M_s(x_n,x)\bigr) - \Lambda_2\bigl(M_s(x_n,x)\bigr), \tag{18}$$

where

$$\begin{aligned} M_s(x_n,x) = \max\bigl\{ & d(x_n,x),\ d(x_n,fx_n),\ d(x,fx),\ \phi_1\bigl(d(x_n,fx),d(fx_n,x)\bigr),\\ & \phi_2\bigl(d(x_n,fx),d(fx_n,x),d(x_n,x)\bigr),\\ & \phi_3\bigl(d(x_n,fx_n),d(x,fx),d(x_n,x)\bigr),\\ & \phi_4\bigl(d(x_n,fx),d(fx_n,x),d(x_n,fx_n),d(x,fx)\bigr)\bigr\}. \end{aligned}$$

Taking the upper limit as $n \to \infty$ in (18) and using Lemma 1, we get

$$\begin{aligned} \Lambda_1\bigl(d(x,fx)\bigr) &\leqslant \Lambda_1\bigl(s^3 d(x,fx)\bigr) = \Lambda_1\left(s^4 \frac{1}{s} d(x,fx)\right)\\ &\leqslant \Lambda_1\Bigl(s^4 \limsup_{n\to\infty} d(x_{n+1},fx)\Bigr)\\ &\leqslant \Lambda_1\Bigl(\limsup_{n\to\infty} M_s\bigl(d(x_n,x)\bigr)\Bigr) - \Lambda_2\Bigl(\liminf_{n\to\infty} M_s\bigl(d(x_n,x)\bigr)\Bigr)\\ &\leqslant \Lambda_1\bigl(d(x,fx)\bigr) - \Lambda_2\bigl(d(x,fx)\bigr), \end{aligned}$$

which implies that $\Lambda_2(d(x,fx)) = 0$. It follows that $d(x,fx) = 0$, equivalently, $fx = x$ and thus $\mathrm{Fix}(f) \neq \emptyset$. This completes the proof. $\square$

The following example will demonstrate the use of our results.

Example 2. Let $X = \mathbb{R}_+$ and $d : X \times X \to \mathbb{R}_+$ be defined by

$$d(x,y) = (x-y)^2$$

for all $x,y \in X$. Then (X,d) is a b-complete b-metric space with coefficient $s = 2$. Define mappings $f : X \to X$ and $\alpha : X \times X \to [0,\infty)$ by

$$fx = \begin{cases} 0, & x \in [0,1),\\ 0.15, & x = 1,\\ \ln(3x-1) & \text{otherwise} \end{cases} \quad \text{and} \quad \alpha(x,y) = \begin{cases} 2, & 0 \leqslant y \leqslant x \leqslant 1,\\ 1 & \text{otherwise.} \end{cases}$$

Let $\Lambda_1, \Lambda_2 \in \mathbb{R}_+ \to \mathbb{R}_+$ and $\phi_i \in \mathcal{G}_i$, $i \in \{1,2,3,4\}$, be the functions given by

$$\Lambda_1(t) = \frac{3t}{2}, \quad \Lambda_2(t) = \frac{3t}{4}$$

for all $t \in \mathbb{R}_+$,

$$\phi_1(q,r) = \max\left\{r, \frac{q+r}{4}\right\}, \qquad \phi_2(q,r,t) = \frac{qr}{1+t},$$
$$\phi_3(q,r,t) = \max\{q,r,t\}, \qquad \phi_4(q,r,t,u) = \frac{1+q+r}{1+t+u}t$$

for all $q,r,t,u \geqslant 0$. We have to prove that $f \in \varDelta_s(X,\alpha,\varLambda_1,\varLambda_2)$.

Let $x,y \in X$ be such that $\alpha(x,y) \geqslant s = 2$. Then $0 \leqslant y \leqslant x \leqslant 1$, and relation (1) is nontrivial only when $x = 1$ and $y \in [0,1)$. In this case, $d(fx,fy) = d(0.15,0) = 0.0225$ and

$$\begin{aligned}M_s(x,y) = \max\biggl\{&(1-y)^2,\ 0.85^2,\ y^2, \max\biggl\{(0.15-y)^2, \frac{1+(0.15-y)^2}{4}\biggr\},\\ &\frac{(y-0.15)^2}{1+(1-y)^2}, \max\bigl\{0.85^2, y^2, (1-y)^2\bigr\}, \frac{1+1+(0.15-y)^2}{1+0.85^2+y^2}0.85^2\biggr\}\\ \geqslant{}& 0.7225.\end{aligned}$$

Hence,

$$\begin{aligned}\varLambda_1\bigl(s^4 d(fx,fy)\bigr) &= \frac{3}{2}\bigl(16\cdot 0.15^2\bigr) = 0.54 < 0.541875 = \frac{3}{4}0.7225\\ &\leqslant \varLambda_1\bigl(M_s(x,y)\bigr) - \varLambda_2\bigl(M_s(x,y)\bigr).\end{aligned}$$

This implies that (1) holds and thus $f \in \varDelta_s(X,\alpha,\varLambda_1,\varLambda_2)$.

It is easy to see that $f \in \mathcal{WP}_s(X,\alpha)$. Indeed, if $x \in X$ is such that $\alpha(x,fx) \geqslant 2$, then $x \in [0,1]$. But then $0 \leqslant ffx \leqslant fx \leqslant 1$ and hence $\alpha(fx,ffx) \geqslant 2$.

Also, we can easily see that (X,d) is α_s-regular, and there is $x_0 = 1$ such that $\alpha(x_0,fx_0) = \alpha(1,0.15) = 2$.

Therefore, all the conditions of Theorem 2 are satisfied. Then we can conclude that $\mathrm{Fix}(f) \neq \emptyset$ (indeed, $0 \in \mathrm{Fix}(f)$).

Observe that, for $x,y > 1$, condition (1) might not hold; hence, using of the function α, is necessary.

Finally, we use Remark 1 to establish the following results for the class $\mathcal{P}_s(X,\alpha)$.

Corollary 1. *Let (X,d) be a b-complete b-metric space with coefficient $s \geqslant 1$, let $\varLambda_1,\varLambda_2 : \mathbb{R}_+ \to \mathbb{R}_+$ be altering distance functions, and let $\alpha : X\times X \to \mathbb{R}_+$ and $f : X \to X$ be given mappings. Suppose that the following conditions hold*:

- ($\widetilde{\mathrm{S}_1}$) $f \in \varDelta_s(X,\alpha,\varLambda_1,\varLambda_2) \cap \mathcal{P}_s(X,\alpha)$;
- (S_2) *there exists $x_0 \in X$ such that $\alpha(x_0,fx_0) \geqslant s$*;
- (S_3) *α has a transitive property of type S*;
- (S_4) *f is b-continuous.*

Then $\mathrm{Fix}(f) \neq \emptyset$.

Corollary 2. *Let (X,d) be a b-complete b-metric space with coefficient $s \geqslant 1$, let $\varLambda_1,\varLambda_2 : \mathbb{R}_+ \to \mathbb{R}_+$ be two altering distance functions, and let $\alpha : X\times X \to \mathbb{R}_+$ and $f : X \to X$ be given mappings. Suppose that the following conditions hold*:

($\widetilde{\mathrm{S}_1}$) $f \in \Delta_s(X, \alpha, \Lambda_1, \Lambda_2) \cap \mathcal{P}_s(X, \alpha)$;
(S_2) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, fx_0) \geqslant s$;
(S_3) α *has a transitive property type* S;
($\widetilde{\mathrm{S}_4}$) (X, d) *is* α_s*-regular.*

Then $\mathrm{Fix}(f) \neq \emptyset$*.*

4 Existence theorems for functional equations arising in dynamic programming

The existence, uniqueness, and iterative approximations of solutions for several classes of functional equations arising in dynamic programming were studied by a lot of researchers. The basic form of the functional equation of dynamic programming is given by Bellman and Lee [6]. Thereafter a lot of work has been done in this direction, and existence and uniqueness results have been obtained for solutions and common solutions of some functional equations, as well as systems of functional equations in dynamic programming with the use of fixed point results. For details, see, e.g., [4, 10, 11] and the references therein.

We will apply our results to the following dynamic programming problem.

Let $\mathcal{X} = B(W)$ be the set of all bounded real-valued functions on a nonempty set W. According to the pointwise addition of functions, multiplication by scalar, and with the norm $\|\cdot\|_\infty$ given by

$$\|u\|_\infty = \sup_{x \in W} \bigl|u(x)\bigr|$$

for all $u \in \mathcal{X}$, we have that $(\mathcal{X}, \|\cdot\|_\infty)$ is a Banach space and the respective convergence is uniform. The respective distance will be denoted by d_∞. If we consider a Cauchy sequence $\{u_n\}$ in $\mathcal{X}$, then it converges uniformly to a function, say h^*, which is bounded, i.e, $u^* \in \mathcal{X}$.

Moreover, we can define a b-metric d_b by

$$d_b(u, v) = \bigl[d_\infty(u, v)\bigr]^p$$

for all $u, v \in B(W)$ and some $p > 1$. Since $(B(W), d_\infty)$ is complete, we deduce that $(B(W), d_b)$ is a b-complete b-metric space with $s = 2^{p-1}$.

In this section, we study the existence of a solution of the following functional equation arising in dynamic programming:

$$q(x) = \sup_{y \in D}\bigl\{g(x, y) + G\bigl(x, y, q\bigl(\tau(x, y)\bigr)\bigr)\bigr\}, \quad x \in W, \tag{19}$$

for the given functions $g : W \times D \to \mathbb{R}$, $G : W \times D \times \mathbb{R} \to \mathbb{R}$ and $\tau : W \times D \to W$, where D is a so-called decision space. Consider the operator $f : \mathcal{X} \to \mathcal{X}$ given by

$$fu(x) = \sup_{y \in D}\bigl\{g(x, y) + G\bigl(x, y, u\bigl(\tau(x, y)\bigr)\bigr)\bigr\} \tag{20}$$

for $u \in \mathcal{X}$, $x \in W$; this mapping is well defined if the functions g and G are bounded. Also, for some $p \geqslant 1$, $s = 2^{p-1}$ and $\phi_i \in \mathcal{G}_i$, $i \in \{1,2,3,4\}$, denote

$$\begin{aligned} M_s\big(u(x), v(x)\big) = \max\big\{ & \big|u(x)-v(x)\big|^p,\ \big|u(x)-fu(x)\big|^p,\ \big|v(x)-fv(x)\big|^p, \\ & \phi_1\big(\big|u(x)-fv(x)\big|^p, \big|fu(x)-v(x)\big|^p\big), \\ & \phi_2\big(\big|u(x)-fv(x)\big|^p, \big|fu(x)-v(x)\big|^p, \big|u(x)-v(x)\big|^p\big), \\ & \phi_3\big(\big|u(x)-fu(x)\big|^p, \big|v(x)-fv(x)\big|^p, \big|u(x)-v(x)\big|^p\big), \\ & \phi_4\big(\big|u(x)-fv(x)\big|^p, \big|fu(x)-v(x)\big|^p, \\ & \quad \big|u(x)-fu(x)\big|^p, \big|v(x)-fv(x)\big|^p\big)\big\} \end{aligned}$$

for all $u, v \in \mathcal{X}$, $x \in W$ and

$$M_s(u,v) = \sup_{x \in W} M_s\big(u(x), v(x)\big).$$

Theorem 3. *Let $f : \mathcal{X} \to \mathcal{X}$ be given by* (20). *Suppose that the following hypotheses hold*:

(D$_1$) *$g : W \times D \to \mathbb{R}$ and $G(\cdot,\cdot,0) : W \times D \to \mathbb{R}$ are bounded functions*;

(D$_2$) *there exists $u_0 \in \mathcal{X}$ such that*

$$u_0(x) \leqslant \sup_{y \in D}\big\{g(x,y) + G\big(x,y,u_0\big(\tau(x,y)\big)\big)\big\}$$

for all $x \in W$;

(D$_3$) *there exists $\lambda \geqslant 0$ such that, for all $x \in W$, $y \in D$ and $t, r \in \mathbb{R}$,*

$$\big|G(x,y,t) - G(x,y,r)\big| \leqslant \lambda |t-r|;$$

(D$_4$) *the function G is non-decreasing in the third variable and satisfies*

$$\big|G\big(x,y,u(x)\big) - G\big(x,y,v(x)\big)\big|^p \leqslant \frac{1}{2^{4p-2}} M_s\big(u(x), v(x)\big)$$

for some $p > 1$ and all $x \in W$, $y \in D$, $u, v \in \mathcal{X}$.

Then the functional equation (19) *has a bounded solution.*

Proof. First of all, we prove that fu is a bounded function on W, that is, $fu \in \mathcal{X}$ and the operator f is well defined. Indeed, let $u \in \mathcal{X}$ be arbitrary. As u is bounded, there exists $\lambda_1 > 0$ such that

$$\big|u(x)\big| \leqslant \lambda_1$$

for all $x \in W$. By hypothesis (D$_1$) there exist $\lambda_2, \lambda_3 > 0$ such that, for all $x \in W$ and all $y \in D$,

$$\big|G(x,y,0)\big| \leqslant \lambda_2 \quad \text{and} \quad g(x,y) \leqslant \lambda_3.$$

Now by hypothesis (D_3), for all $x \in W$ and all $y \in D$,

$$\begin{aligned}
&\left|g(x,y) + G\bigl(x,y,u\bigl(\tau(x,y)\bigr)\bigr)\right| \\
&\quad \leqslant \left|g(x,y)\right| + \left|G\bigl(x,y,u\bigl(\tau(x,y)\bigr)\bigr)\right| \\
&\quad \leqslant \left|g(x,y)\right| + \left|G\bigl(x,y,u\bigl(\tau(x,y)\bigr)\bigr) - G\bigl(x,y,0\bigr)\right| + \left|G(x,y,0)\right| \\
&\quad \leqslant \lambda_3 + \lambda\left|u\bigl(\tau(x,y)\right| + \lambda_2 \leqslant \lambda_3 + \lambda\lambda_1 + \lambda_2.
\end{aligned}$$

As a result, for all $x \in W$, we have that

$$\left|fu(x)\right| = \sup_{y\in D}\left|g(x,y) + G\bigl(x,y,u\bigl(\tau(x,y)\bigr)\bigr)\right| \leqslant \lambda_3 + \lambda\lambda_1 + \lambda_2.$$

This implies that fu is a bounded function on W, that is, the operator f is well defined.

Define a function $\alpha : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ by

$$\alpha(u,v) = \begin{cases} s & \text{if } u(x) \leqslant v(x) \text{ for all } x \in W, \\ \eta & \text{otherwise,} \end{cases} \tag{21}$$

where $s = 2^{p-1}$ and $\eta \in (0,s)$. It is easy to see that α has a transitive property. It follows from (D_1) that $f \in \mathcal{P}_s(\mathcal{X},\alpha)$ and so $f \in \mathcal{WP}_s(\mathcal{X},\alpha)$. From ($D_2$) and (21) we get that $\alpha(u_0, fu_0) \geqslant s$ for some $u_0 \in \mathcal{X}$. To prove condition (S_4) in Theorem 2, let $\{u_n\}$ be an increasing sequence in $\mathcal{X}$; then by (21), $\alpha(u_n,u_{n+1}) \geqslant s$ for all $n \in \mathbb{N}$. If $u_n \to u \in X$ as $n \to \infty$, then we get in a standard way that $u_n(x) \leqslant u(x)$ for any $x \in W$. Therefore, by (21), $\alpha(u_n,u) \geqslant s$ for all $n \in \mathbb{N}$. Thus, condition (S_4) holds.

Next, we show that $f \in \Delta_s(X,\alpha,\Lambda_1,\Lambda_2)$. Let $u,v \in \mathcal{X}$ be such that $\alpha(u,v) \geqslant s$, that is, $u(x) \leqslant v(x)$ for all $x \in W$; then from (D_4) we have

$$\begin{aligned}
\left|fu(x) - fv(x)\right|^p &= \Bigl|\sup_{y\in D}\bigl\{g(x,y) + G\bigl(x,y,u\bigl(\tau(x,y)\bigr)\bigr)\bigr\} \\
&\qquad - \sup_{y\in D}\bigl\{g(x,y) + G\bigl(x,y,v\bigl(\tau(x,y)\bigr)\bigr)\bigr\}\Bigr|^p \\
&\leqslant \sup_{y\in D}\left|G\bigl(x,y,u\bigl(\tau(x,y)\bigr)\bigr) - G\bigl(x,y,v\bigl(\tau(x,y)\bigr)\bigr)\right|^p \\
&\leqslant \frac{1}{2^{4p-2}} M\bigl(u(x),v(x)\bigr)
\end{aligned}$$

implying that

$$s^4 d_b(fu,fv) = s^4 \sup_{x\in W}\left|fu(x) - fv(x)\right|^p \leqslant \frac{1}{4}M_s(u,v).$$

Now consider control functions $\Lambda_1, \Lambda_2 : \mathbb{R}_+ \to \mathbb{R}_+$ given as

$$\Lambda_1(t) = t, \quad \Lambda_2(t) = \frac{3}{4}t, \quad t \in \mathbb{R}_+.$$

Notice that the last inequality does not depend on $x \in W$, and therefore we obtain

$$\Lambda_1\bigl(s^4 d_b(fu,fv)\bigr) \leqslant \Lambda_1\bigl(M_s(u,v)\bigr) - \Lambda_2\bigl(M_s(u,v)\bigr), \quad u,v \in B(W).$$

Thus, all the conditions of Theorem 2 are fulfilled, and there exists a fixed point of f, i.e., a bounded solution $u^* \in \mathcal{X}$ such that $fu^* = u^*$. In other words, for all $x \in W$,

$$u^*(x) = fu^*(x) = \sup_{y \in D} \bigl\{ g(x,y) + G\bigl(x, y, u^*\bigl(\tau(x,y)\bigr)\bigr) \bigr\}.$$

This completes the proof. $\square$

We state the following consequence of Theorem 3.

Corollary 3. *Suppose the following assumptions*:

(D$_1$) *$g : W \times D \to \mathbb{R}$ and $G(\cdot,\cdot,0) : W \times D \to \mathbb{R}$ are bounded functions*;
(D$_2$) *there exists $u_0 \in \mathcal{X}$ such that*

$$u_0(x) \leqslant \sup_{y \in D} \bigl\{ g(x,y) + G\bigl(x, y, u_0\bigl(\tau(x,y)\bigr)\bigr) \bigr\}$$

for all $x \in W$;
($\widetilde{\mathrm{D}_3}$) *the function $G : W \times D \times \mathbb{R} \to \mathbb{R}$ is non-decreasing in the third variable and satisfies*

$$\bigl|G(x,y,t) - G(x,y,r)\bigr| \leqslant \frac{2^{2/p}}{16} |t - r|$$

for some $p > 1$ and all $x \in W$, $y \in D$ and $t, r \in \mathbb{R}$.

Then the functional equation (19) *has a solution $u^* \in B(W)$.*

Proof. We will verify the conditions of Theorem 3. If we take $\lambda = 2^{2/p}/16 > 0$, then we have (D$_3$) of Theorem 3. Moreover, for all $x \in W$, all $y \in D$ and all $u, v \in B(W)$,

$$\bigl|u(x) - v(x)\bigr| \leqslant \sup_{x \in W} \bigl|u(x) - v(x)\bigr| = d_\infty(u,v).$$

Then

$$\bigl|G\bigl(x,y,u(x)\bigr) - G\bigl(x,y,v(x)\bigr)\bigr| \leqslant \frac{2^{2/p}}{16} \bigl|u(x) - v(x)\bigr|$$

implies that

$$\begin{aligned} &\bigl|G\bigl(x,y,u(x)\bigr) - G\bigl(x,y,v(x)\bigr)\bigr|^p \\ &\quad\leqslant \frac{1}{2^{4p-2}} \bigl|u(x) - v(x)\bigr|^p \leqslant \frac{1}{2^{4p-2}} d_b(u,v) \leqslant \frac{1}{2^{4p-2}} M_s(u,v), \end{aligned}$$

wherefrom condition (D$_4$) of Theorem 3 follows. Hence, Theorem 3 guarantees that functional equation (19) has a solution $u^* \in B(W)$. $\square$

Example 3. (Modified according to [4, Ex. 22], adapted for the use in b-metric spaces.) Consider the functional equation

$$u(x) = \sup_{y \in \mathbb{R}} \biggl\{ \arctan\bigl(x + 2|y|\bigr) + \ln\biggl(1 + x + \frac{1}{1 + |y|} + \frac{1}{8} \biggl| u\biggl(\frac{x}{1 + x + |y|}\biggr)\biggr|\biggr)\biggr\} \quad (22)$$

for $x \in [0,1]$. On comparing equation (22) with equation (19), we see that we have taken $W = [0,1]$, $D = \mathbb{R}$, $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is defined by $g(x,y) = \arctan(x + 2|y|)$, $\tau : [0,1] \times \mathbb{R} \to \mathbb{R}$ is defined by $\tau(x,y) = x/(1 + x + |y|)$, $G : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by $G(x,y,t) = \ln(1 + x + 1/|y| + |t|/8)$.

After calculations, we have $|g(x,y)| \leqslant \pi/2$ and $G(x,y,0) = \ln(1+x+1/(1+|y|)) \leqslant \ln 3$ for all $x \in [0,1]$ and all $y \in \mathbb{R}$. Hence, assumption (D_1) of Corollary 3 is satisfied. Assumption (D_2) is satisfied for $u_0(x) \equiv 1$.

Moreover, for all $x \in [0,1]$ and all $y,t,r \in \mathbb{R}$ with $|t| > |r|$, it follows that

$$\begin{aligned}
&\bigl|G(x,y,t) - G(x,y,r)\bigr| \\
&\quad = \left|\ln\left(1 + x + \frac{1}{1+|y|} + \frac{1}{8}|t|\right) - \ln\left(1 + x + \frac{1}{1+|y|} + \frac{1}{8}|r|\right)\right| \\
&\quad = \left|\ln \frac{1 + x + \frac{1}{1+|y|} + \frac{1}{8}|t|}{1 + x + \frac{1}{1+|y|} + \frac{1}{8}|r|}\right| = \left|\ln \frac{1 + x + \frac{1}{1+|y|} + \frac{1}{8}(|r| + (|t| - |r|))}{1 + x + \frac{1}{1+|y|} + \frac{1}{8}|r|}\right| \\
&\quad = \left|\ln\left(1 + \frac{1}{8}\frac{|t| - |r|}{1 + x + \frac{1}{1+|y|} + \frac{1}{8}|r|}\right)\right| \leqslant \left|\ln\left(1 + \frac{1}{8}\bigl(|t| - |r|\bigr)\right)\right| \\
&\quad = \left|\ln\left(1 + \frac{1}{8}\bigl(|t| - |r|\bigr)\right)\right| = \ln\left(1 + \frac{1}{8}\bigl||t| - |r|\bigr|\right) \\
&\quad \leqslant \ln\left(1 + \frac{1}{8}|t - r|\right) \leqslant \frac{1}{8}|t - r|,
\end{aligned}$$

which is assumption ($\widetilde{\mathrm{D}_3}$) of Corollary 3 for $p = 2$. Therefore, functional equation (22) has a solution $u^* \in B([0,1])$.

Example 4. Let $W = [1,\infty]$, $D = \mathbb{R}_+$. Consider the functional equation

$$u(x) = \sup_{y \in \mathbb{R}} \left\{1 + \frac{1}{x + 2y^2} + \frac{1}{1 + x^2 + y^2} + \frac{2^{2/3}}{16} \sin\bigl(u\bigl(2x^2 y\bigr)\bigr)\right\} \tag{23}$$

for $x \in W$. On comparing equation (23) with equation (19), we see that we have $g : [1,\infty] \times \mathbb{R}_+ \to \mathbb{R}_+$ defined by $g(x,y) = 1 + 1/(x + 2y^2)$, $\tau : [1,\infty] \times \mathbb{R}_+ \to \mathbb{R}_+$ defined by $\tau(x,y) = x/(1 + x + |y|)$, and $G : [1,\infty] \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ defined by $G(x,y,t) = 1/(1 + x^2 + y^2) + 2^{2/3}/16 \sin t$.

After calculations, we have $|g(x,y)| < 4/3$ and $G(x,y,0) = 1/(1 + x^2 + y^2) < 1/2$ for all $x \in [1,\infty]$ and all $y \in \mathbb{R}_+$. Hence, assumption (D_1) of Corollary 3 is satisfied. Assumption (D_2) is satisfied for $u_0(x) \equiv 1$.

Moreover, for all $x \in [1,\infty]$ and all $y,t,r \in \mathbb{R}$ with $|t| > |r|$, it follows that

$$\bigl|G(x,y,t) - G(x,y,r)\bigr| \leqslant \frac{2^{2/3}}{16}|\sin t - \sin r| \leqslant \frac{2^{2/3}}{16}|t - r|,$$

which is assumption ($\widetilde{\mathrm{D}_3}$) of Corollary 3 for $p = 3$. Therefore, functional equation (22) has a solution $u^* \in B([1,\infty])$.

5 Conclusion

Condition (1) considered in this paper is a generalized weakly contraction condition that includes several types of conditions based on various forms of control functions, and the obtained fixed point results include several results known thus far. In particular, it is shown that the term $(d(x, fy) + d(fx, y))/(2s)$ usually appearing in contraction conditions can be replaced by a more general term $\phi_1(d(x, fy), d(fx, y))$

Also, our results extend Alber and Guerre-Delabrière [3], Rhoades [12] and Agarwal et al. [1] fixed point results from metric to the setup of b-metric spaces. Furthermore, as it has been observed in some studies, fixed point results in b-metric spaces endowed with partial order, graph, binary relation or cyclic mappings can be derived from results under some suitable (weak) α-admissible conditions of type S. We have applied our results to get existence of solution for functional equations arising in dynamic programming.

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