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Stability and nonlinear dynamics in a Solow model with pollution

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Abstract. In this paper, we introduce a time-to-build technology in a Solow model with pollution. We show that Hopf bifurcations occur as the delay passes through critical values. The direction and the stability criteria of the bifurcating periodic solutions are obtained by the normal form theory and the center manifold theorem. Numerical experiments confirm the analytical results with regard to the emergence of nonlinear dynamics.

Keywords: pollution, Solow, time delay.

1 Introduction

The issue of relationship between human economic activity and the environmental quality has long captured the interest of the scientific community. Focusing on economic literature, John and Pecchenino [1], in an overlapping generations context, introduce a model in which multiple equilibria may exist; Antoci [2] highlights the undesired effects of environmental deterioration on individuals' choices (the so called defensive expenditures); Antoci and Borghesi [3] investigate the possible consequences on environment due to interaction between rich and poor countries. In addition, several studies argue that interplays between economic growth and environment also lead to nonlinear dynamics in both variables (see [4]). In a seminal paper, Day [5] studies the chaotic dynamics arising from the negative effects of pollution on productivity; Zhang [6], Naimzada and Sodini [7] study specifications of the John and Pecchenino's model generating nonlinear dynamics; Antoci and Sodini [8], Antoci et al. [9] show the existence of different bifurcation scenarios in overlapping generations models with environment; Zhao and Zhang [10] describe a price competition model in which impacts of carbon emission trading may be engine of nonlinear dynamics, while Antoci et al. [9] analyse a continuous time model with optimizing agents able to show indeterminacy and persistent oscillations in environmental and economic variables.

On the other hand, there also exists a burgeoning literature aiming at studying nonlinear dynamics in ecological models (see [11] and the literature cited within) and economic models where delays in variables are introduced. Among others, Szydlowski and Krawiec [12] and Szydlowski et al. [13] use several modern mathematical methods of detecting cyclic behaviour in Kalecki's classical model; Maddalena and Fanelli [14] propose a mathematical model with time delay to describe the process of diffusion of a new technology; Zak [15] examines the effect of Kaleckian lags in the Solow and Cass–Koopmans growth models; Matsumoto and Szidarovszky [16, 17] develop growth models with unimodal production functions (with respect to the stock of capital) and time-to-build technology; Bianca et al. [18] investigate an endogenous labor shift model under a dual economy with time delay.

The objective of our paper is to highlight the relevance of temporal lags in a context in which economic activity depletes a free-access natural resource. Different from Matsumoto and Szidarovszky [16, 17], we consider a technology that does not introduce environmental quality as an input of the production function. In particular, the marginal productivity of capital is always positive and decreasing, there exists only one transmission channel (pollution) through which output affects the environment, and pollution does not produce any negative external effects on production. At this aim we consider the model proposed by Xepapadeas in the Handbook of Environmental economics [19, p. 1227, Eqs. (8), (9)]. Specifically, the author generalizes the neoclassical Solow growth model with a fixed savings ratio, considering that economic growth is accompanied by pollution accumulation. The mathematical structure and the results of the model are quite simple: a two dimensional dynamical system displays a unique non trivial steady state that captures the whole set of positive initial conditions. Furthermore, the time evolution of capital accumulation and pollution is monotone.

Despite the neoclassical regularity assumptions on production function and absence of optimizing behaviour of agents, when temporal lags in the production of capital goods are introduced, the results drastically change and (temporal or persistent) oscillations may characterize the dynamics of the model although the production function is always increasing in the capital stock. In particular we show that, regardless of parameter specification, there exists a threshold value in time lag below which the dynamics of the state variables are monotone or characterized by dumping oscillations. When the threshold value is reached, the dynamical system undergoes a supercritical or subcritical Hopf bifurcation. If the bifurcation is supercritical, an attracting invariant curve exists and persistent oscillations characterize the dynamics of the model. The paper is organised as follows. Section 2 builds on the model; Section 3 studies the emergence of the Hopf bifurcation; Section 4 uses the center manifold theory to investigate the direction of Hopf bifurcation and the stability of periodic solutions; Section 5 presents some simulations and economic interpretations; Section 6 concludes.

2 The model

We consider the Xepapadeas' model [19] with population normalized to one:

$$\begin{aligned} \dot{k} &= sk^{\alpha} - \delta k, \\ \dot{p} &= \phi k^{\alpha} - mp. \end{aligned} \tag{1}$$

The first equation describes the classical Solow model when Cobb–Douglas technology is assumed: k is the individual's capital stock, $\delta > 0$ denotes capital depreciation rate, s is the saving ratio and $\alpha \in (0, 1)$ is capital's share. Here p represents the stock of pollution, ϕ emissions per unit of output and $m > 0$ reflects exponential pollution decay. Xepapadeas proved this model to have a unique non-trivial steady state which is globally asymptotically stable for non negative initial conditions.

In this paper, building on a contribution by Zak [15], we modify system (1) with a time lag introduced in the Kaleckian spirit by assuming the same lag with regard to the role played by the capital stock in both the dynamics of capital and dynamics of pollution. This assumption leads to the following dynamical system:

$$\begin{aligned} \dot{k} &= sk_d^{\alpha} - \delta k_d, \\ \dot{p} &= \phi k_d^{\alpha} - mp, \end{aligned} \tag{2}$$

where $k_d := k_{t-\tau}$ and $\tau > 0$ represents time delay.

3 Local stability and Hopf bifurcation analysis

In this section, we will study stability of the stationary solutions and existence of local Hopf bifurcation of system (2). The first step is to find stationary solutions. Since time delay does not change the equilibria, system (2) has exactly the same equilibrium points of the corresponding steady states for zero delay. Hence, there exists a unique non-trivial steady state equilibrium (k_*, p_*) defined by $sk_*^{\alpha-1} = \delta$, $\phi k_*^{\alpha} = mp_*$. Using the transformation $x = k - k_*$, $y = p - p_*$, and then linearizing the resulting system at the origin yields

$$\begin{aligned} \dot{x} &= (\alpha - 1)\delta x_d, \\ \dot{y} &= -my + \alpha\phi k_*^{\alpha-1} x_d, \end{aligned} \tag{3}$$

where $x_d = k_{t-\tau} - k_*$. Writing system (3) in the form

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -m \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} (\alpha-1)\delta & 0 \\ \alpha\phi k_*^{\alpha-1} & 0 \end{bmatrix} \begin{bmatrix} x_d \\ y_d \end{bmatrix}, \tag{4}$$

we see that the characteristic equation resulting from (4) is given by

$$\begin{vmatrix} -\lambda + (\alpha-1)\delta e^{-\lambda\tau} & 0 \\ \alpha\phi k_*^{\alpha-1} e^{-\lambda\tau} & -m-\lambda \end{vmatrix} = (\lambda + m)\bigl[\lambda - (\alpha-1)\delta e^{-\lambda\tau}\bigr] = 0. \tag{5}$$

It is well known that the equilibrium is locally asymptotically stable if all roots of (5) have negative real part. When $\tau = 0$, the characteristic equation (5) reduces to $(\lambda + m) \times [\lambda - (\alpha - 1)] = 0$, whose solutions are $\lambda = -m < 0$ and $\lambda = \alpha - 1 < 0$. In this case, the equilibrium is locally asymptotically stable. Therefore, if instability occurs for a particular value of $\tau > 0$, then a characteristic root of

$$\lambda - (\alpha - 1)\delta \mathrm{e}^{-\lambda\tau} = 0 \tag{6}$$

must intersect the imaginary axis (Rouche's theorem, see, e.g., [20]).

Lemma 1. *Eq.* (6) *has a unique pair of simple purely imaginary roots* $\pm\mathrm{i}\omega_0$ *at the sequence of critical values* τ_j, *where*

$$\omega_0 = (1 - \alpha)\delta, \qquad \tau_j = \frac{\pi}{2(1 - \alpha)\delta} + 2\pi j \quad (j = 0, 1, 2, \dots).$$

Moreover, if $\lambda_j(\tau) = \nu_j(\tau) + \mathrm{i}\omega_j(\tau)$ *denote a root of Eq.* (6) *near* $\tau = \tau_j$ *such that* $\nu_j(\tau_j) = 0$, $\omega_j(\tau_j) = \omega_0$ $(j = 0, 1, 2, \dots)$*, then the following transversality condition is satisfied*:

$$\mathrm{Re}\big[\lambda'(\tau_j)\big] \equiv \frac{\mathrm{d}[\mathrm{Re}\,\lambda_j(\tau)]}{\mathrm{d}\tau}\bigg|_{\tau=\tau_j} > 0 \quad (j = 0, 1, 2, \dots).$$

Proof. Let $\lambda = \mathrm{i}\omega$ $(\omega > 0)$ be a root of Eq. (6). Then

$$\mathrm{i}\omega - (\alpha - 1)\delta(\cos\omega\tau - \mathrm{i}\sin\omega\tau) = 0.$$

Separating the real and imaginary parts gives

$$\omega = (1 - \alpha)\delta\sin\omega\tau, \qquad 0 = \cos\omega\tau.$$

So, the first part of the statement holds. Let $\lambda_j(\tau) = \nu_j(\tau) + \mathrm{i}\omega_j(\tau)$ denote a root of Eq. (6) near $\tau = \tau_j$ such that $\nu_j(\tau_j) = 0$, $\omega_j(\tau_j) = \omega_0$ $(j = 0, 1, 2, \dots)$. Differentiating Eq. (6) with respect to τ, we get

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} + (\alpha - 1)\delta\mathrm{e}^{-\lambda\tau}\left(\tau\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} + \lambda\right) = 0.$$

Hence, using (6), we arrive at

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1} = -\frac{1}{\lambda^2} - \frac{\tau}{\lambda}.$$

On the other hand,

$$\begin{aligned}
\mathrm{sign}\left\{\frac{\mathrm{d}[\mathrm{Re}\,\lambda_j(\tau)]}{\mathrm{d}\tau}\bigg|_{\tau=\tau_j}\right\} &= \mathrm{sign}\left\{\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}\bigg|_{\tau=\tau_j}\right\} \\
&= \mathrm{sign}\left\{\mathrm{Re}\left(-\frac{1}{\lambda^2} - \frac{\tau}{\lambda}\right)\bigg|_{\tau=\tau_j}\right\} = \mathrm{sign}\left\{\frac{1}{\omega_0^2}\right\} = 1.
\end{aligned}$$

This completes the proof. □

Based on the above analysis, the root $\lambda_j(\tau)$ of Eq. (6) near τ_j crosses the imaginary axis from the left to the right as τ continuously varies from a number less than τ_j to one greater than τ_j by Rouche's theorem.

Lemma 2. *If $\tau \in [0, \tau_0)$, all roots of the characteristic equation* (5) *have negative real parts. If $\tau = \tau_0$, all roots of Eq.* (5)*, except $\pm\mathrm{i}\omega_0$, have negative real parts. If $\tau \in (\tau_j, \tau_{j+1})$ for $j = 0, 1, 2, \ldots$, Eq.* (5) *has $2(j+1)$ roots with positive real parts.*

Summing up all works above lead us to state the following theorem.

Theorem 1. (i) *The equilibrium (k_*, p_*) is asymptotically stable for $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0$.*

(ii) *System* (2) *undergoes a Hopf bifurcation occurs at the equilibrium (k_*, p_*) when $\tau = \tau_j$ for $j = 0, 1, 2, \ldots$.*

4 Stability and direction of the Hopf bifurcation

In the previous section, we obtain the conditions for Hopf bifurcations to occur at the critical value τ_j ($j = 0, 1, 2, \ldots$). In this section, based on the normal form method and the center manifold theorem introduced by Hassard et al. [21], we will study the direction of Hopf bifurcation and the stability of bifurcating periodic solutions from the positive equilibrium (k_*, p_*). These techniques allow us to have analytical results with respect to the type of the emerging bifurcation. For convenience, let $\tau = \tau_j + \mu$, where $\mu \in \mathbb{R}$. Then $\mu = 0$ is the Hopf bifurcation value for system (2). Set $u = (u_1, u_2)^{\mathrm{T}} = (x, y)^T \in \mathbb{R}^2$. Normalizing the delay τ by the time scaling $t \to t/\tau$, system (2) can be written as a functional differential equation in $C = C([-1, 0], \mathbb{R}^2)$ as

$$\dot{u} = L_\mu(u_t) + f(\mu, u_t), \tag{7}$$

where $L_\mu : C \to \mathbb{R}^2$ is a linear continuous operator and $f : \mathbb{R} \times C \to \mathbb{R}^2$ are defined as follows. For $\varphi = (\varphi_1, \varphi_2) \in C$,

$$L_\mu(\varphi) = (\tau_j + \mu) \begin{bmatrix} 0 & 0 \\ 0 & -m \end{bmatrix} \varphi(0) + (\tau_j + \mu) \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \varphi(-1) \tag{8}$$

with

$$a = (\alpha - 1)\delta, \qquad b = \alpha\phi k_*^{\alpha - 1} = \frac{\alpha\phi\delta}{s},$$

and

$$f(\mu, \varphi) = (\tau_j + \mu) \begin{bmatrix} f^{(1)} \\ f^{(2)} \end{bmatrix}$$

with

$$f^{(1)} = \frac{1}{2} P^*_{x_d x_d} \varphi_1(-1)^2 + \frac{1}{3!} P^*_{x_d x_d x_d} \varphi_1(-1)^3 + \cdots,$$
$$f^{(2)} = \frac{1}{2} Q^*_{x_d x_d} \varphi_1(-1)^2 + \frac{1}{3!} Q^*_{x_d x_d x_d} \varphi_1(-1)^3 + \cdots.$$

We have

$$\begin{aligned} P^*_{x_d x_d} &\equiv P_{x_d x_d}(k_*, p_*) = s\alpha(\alpha-1)k_*^{\alpha-2} < 0, \\ Q^*_{x_d x_d} &= \phi\alpha(\alpha-1)k_*^{\alpha-2} < 0, \\ P^*_{x_d x_d x_d} &\equiv P_{x_d x_d x_d}(k_*, p_*) = s\alpha(\alpha-1)(\alpha-2)k_*^{\alpha-3} > 0, \\ Q^*_{x_d x_d x_d} &\equiv Q_{x_d x_d x_d}(k_*, p_*) = \phi\alpha(\alpha-1)(\alpha-2)k_*^{\alpha-3} > 0. \end{aligned}$$

By the Riesz representation theorem, there exists a bounded variation function $\eta(\theta, \mu)$ for $\theta \in [-1, 0]$ such that

$$L_\mu \varphi = \int_{-1}^{0} d\eta(\theta, \mu)\varphi(\theta)\,\mathrm{d}\theta \quad \text{for } \varphi \in C,$$

where

$$\eta(\theta, \mu) = \begin{cases} L_\mu(\varphi(-1), \mu), & \theta = -1, \\ 0, & \theta \in (-1, 0], \end{cases}$$

which can be satisfied by

$$\mathrm{d}\eta(\theta, \mu) = \left\{ (\tau_j + \mu) \begin{bmatrix} 0 & 0 \\ 0 & -m \end{bmatrix} \Lambda(\theta) + (\tau_j + \mu) \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \Lambda(\theta + 1) \right\} \mathrm{d}\theta$$

with Λ denoting the Dirac delta function, namely, $\Lambda(\theta) = 0$ if $\theta \neq 0$, $\Lambda(\theta) = 1$ if $\theta = 0$. For $\varphi \in C^1([-1, 0], \mathbb{R}^2)$, we define

$$A(\mu)(\varphi) = \begin{cases} \frac{\mathrm{d}\varphi(\theta)}{\mathrm{d}\theta}, & \theta \in [-1, 0), \\ \int_{-1}^{0} \mathrm{d}\eta(\theta, \mu)\varphi(\theta)\,\mathrm{d}\theta = L_\mu(\varphi), & \theta = 0, \end{cases}$$

and

$$R(\mu)(\varphi) = \begin{cases} 0, & \theta \in [-1, 0), \\ f(\mu, \varphi), & \theta = 0. \end{cases}$$

Then we can write (7) as the following ordinary differential equation:

$$\dot{u}_t = A(\mu)u_t + R(\mu)u_t, \tag{9}$$

where $u_t = u(t + \theta)$ for $\theta \in [-1, 0]$. For $\psi \in \tilde{C} = C([0, 1], (\mathbb{R}^2)^*)$, we define

$$L^*_\mu \psi = \int_{-1}^{0} \mathrm{d}\eta^{\mathrm{T}}(-r, \mu)\psi(-r)\,\mathrm{d}(-r).$$

The adjoint operator A^* of A is expressed as

$$A^*(\mu)\psi = \begin{cases} -\frac{\mathrm{d}\psi(r)}{\mathrm{d}r}, & r \in (0, 1], \\ \int_{-1}^{0} \mathrm{d}\eta^{\mathrm{T}}(r, \mu)\psi(-r), & r = 0. \end{cases}$$

We define the following bilinear inner product in $\tilde{C} \times C$ for $\psi \in \tilde{C}$ and $\varphi \in C$:

$$\langle \psi, \varphi \rangle = \bar{\psi}(0)\varphi(0) - \int\limits_{\theta=-1}^{0} \int\limits_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta)\,\mathrm{d}\eta(\theta)\varphi(\xi)\,\mathrm{d}\xi, \tag{10}$$

where $\eta(\theta) = \eta(\theta, 0)$. Since A and A^* are adjoint operators, then $\pm\mathrm{i}\omega_0\tau_j$ are eigenvalues of A and A^*. Let $q(\theta)$ and $q^*(r)$ be the eigenvectors of A and A^* associated with eigenvalues $\mathrm{i}\omega_0\tau_j$ and $-\mathrm{i}\omega_0\tau_j$, respectively. These eigenvectors satisfy $A(0)q(\theta) = \mathrm{i}\omega_0\tau_j q(\theta)$ and $A(0)q^*(r) = -\mathrm{i}\omega_0\tau_j q^*(r)$. Let $q(\theta) = q(0)\mathrm{e}^{\mathrm{i}\omega_0\tau_j\theta} = (1, \rho)^{\mathrm{T}}\mathrm{e}^{\mathrm{i}\omega_0\tau_j\theta}$, where ρ is a complex value. From (8), we derive

$$\begin{aligned} &\left\{\tau_j \begin{bmatrix} 0 & 0 \\ 0 & -m \end{bmatrix} - \mathrm{i}\omega_0\tau_j I + \tau_j \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \mathrm{e}^{-\mathrm{i}\omega_0\tau_j} \right\} q(0) \\ &\quad = \tau_j \begin{bmatrix} -\mathrm{i}\omega_0 + a\mathrm{e}^{-\mathrm{i}\omega_0\tau_j} & 0 \\ b\mathrm{e}^{-\mathrm{i}\omega_0\tau_j} & -\mathrm{i}\omega_0 - m \end{bmatrix} q(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned}$$

From which we have

$$q(0) = (1, \rho)^{\mathrm{T}} = \left(1, \frac{b\mathrm{e}^{-\mathrm{i}\omega_0\tau_j}}{\mathrm{i}\omega_0 + m}\right)^{\mathrm{T}}.$$

On the other hand, suppose that $q^*(r) = q^*(0)\mathrm{e}^{\mathrm{i}\omega_0\tau_j r} = D(1, \sigma)\mathrm{e}^{\mathrm{i}\omega_0\tau_j r}$ is the eigenvector of A^* corresponding to $-\mathrm{i}\omega_0\tau_j$, where σ and D are complex values. In this case, we obtain

$$\begin{aligned} &\left\{\tau_j \begin{bmatrix} 0 & 0 \\ 0 & -m \end{bmatrix}^{\mathrm{T}} + \mathrm{i}\omega_0\tau_j I + \tau_j \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}^{\mathrm{T}} \mathrm{e}^{-\mathrm{i}\omega_0\tau_j} \right\} q^*(0) \\ &\quad = \tau_j \begin{bmatrix} \mathrm{i}\omega_0 + a\mathrm{e}^{-\mathrm{i}\omega_0\tau_j} & b\mathrm{e}^{-\mathrm{i}\omega_0\tau_j} \\ 0 & \mathrm{i}\omega_0 - m \end{bmatrix} q^*(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned}$$

Therefore, we can choose

$$\sigma = -\frac{b\mathrm{e}^{-\mathrm{i}\omega_0\tau_j}}{\mathrm{i}\omega_0 + a\mathrm{e}^{-\mathrm{i}\omega_0\tau_j}}.$$

In order to assure $\langle q^*, q \rangle = 1$, we need to determine the value of D. From (10), we have

$$\begin{aligned} \langle q^*, q \rangle &= \bar{D}(1, \bar{\sigma})(1, \rho)^{\mathrm{T}} - \int\limits_{\theta=-1}^{0} \int\limits_{\xi=0}^{\theta} (1, \bar{\sigma})\mathrm{e}^{-\mathrm{i}(\xi-\theta)\omega_0\tau_j}\,\mathrm{d}\eta(\theta)(1, \rho)^{\mathrm{T}}\mathrm{e}^{\mathrm{i}\xi\omega_0\tau_j}\,\mathrm{d}\xi \\ &= \bar{D}\left[1 + \rho\bar{\sigma} - \int\limits_{\theta=-1}^{0} (1, \bar{\sigma})\theta\mathrm{e}^{\mathrm{i}\omega_0\tau_j\theta}\,\mathrm{d}\eta\,(\theta)(1, \rho)^{\mathrm{T}}\right] \\ &= \bar{D}\left[1 + \rho\bar{\sigma} - \mathrm{e}^{-\mathrm{i}\omega_0\tau_j\theta}(1, \bar{\sigma})\tau_j \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} (1, \rho)^{\mathrm{T}}\right] \\ &= \bar{D}\big[1 + \rho\bar{\sigma} + \tau_j(b\bar{\sigma} + a)\mathrm{e}^{-\mathrm{i}\omega_0\tau_j}\big]. \end{aligned}$$

Consequently, using

$$D = \frac{1}{1 + \bar{\rho}\sigma + \tau_j(b\sigma + a)\mathrm{e}^{\mathrm{i}\omega_0\tau_j}},$$

we get $\langle q^*, q\rangle = 1$. Furthermore, since $\langle \psi, A\varphi\rangle = \langle A^*\psi, \varphi\rangle$, we obtain $-\mathrm{i}\omega_0\langle q^*, \bar{q}\rangle = \langle q^*, A\bar{q}\rangle = \langle A^* q^*, \bar{q}\rangle = \langle -\mathrm{i}\omega_0 q^*, \bar{q}\rangle = \mathrm{i}\omega_0\langle q^*, \bar{q}\rangle$. Hence, $\langle q^*, \bar{q}\rangle = 0$. Now, we apply the idea of Hassard et al. [21] to compute the coordinates to describe the center manifold at the critical point. For u_t, a solution of (9) at $\mu = 0$, we define

$$z = \langle q^*, u_t\rangle \quad \text{and} \quad W(t,\theta) = u_t(\theta) - 2\,\mathrm{Re}\big[zq(\theta)\big], \tag{11}$$

where z and $\bar{z}$ are the local coordinates for the center manifold in the direction of q^* and $\bar{q}^*$. On the center manifold, we get

$$W(t,\theta) = W(z,\bar{z},\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots .$$

Note that W is real if u_t is real. We consider only real solutions. For the solution u_t, since $\mu = 0$, we have

$$\begin{aligned} \dot{z} &= \langle q^*, \dot{u}_t\rangle = \mathrm{i}\omega_0\tau_j z + \big\langle \bar{q}^*(\theta), f\big(0, W(z,\bar{z},\theta)2\,\mathrm{Re}\big[zq(\theta)\big]\big)\big\rangle \\ &= \mathrm{i}\omega_0\tau_j z + \bar{q}^*(0) f\big(0, W(z,\bar{z},0)2\,\mathrm{Re}\big[zq(0)\big]\big). \end{aligned}$$

We rewrite this as

$$\dot{z} = \mathrm{i}\omega_0\tau_j z + g(z,\bar{z}),$$

where

$$g(z,\bar{z}) = \bar{q}^*(0) f_0(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots . \tag{12}$$

Noticing from (11) that $u_t(\theta) = (u_{1t}(\theta), u_{2t}(\theta)) = W(t,\theta) + zq(\theta) + \bar{z}\bar{q}(\theta)$ and recalling $q(\theta) = (1,\rho)^{\mathrm{T}}\mathrm{e}^{\mathrm{i}\omega_0\tau_j\theta}$, we obtain

$$\begin{aligned} u_{1t}(0) &= z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \cdots , \\ u_{2t}(0) &= \rho z + \overline{\rho z} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + \cdots , \\ u_{1t}(-1) &= \mathrm{e}^{-\mathrm{i}\omega_0\tau_j}z + \mathrm{e}^{\mathrm{i}\omega_0\tau_j}\bar{z} \\ &\quad + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} + W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + \cdots , \\ u_{2t}(-1) &= \mathrm{e}^{-i\omega_0\tau_j}\rho z + \mathrm{e}^{\mathrm{i}\omega_0\tau_j}\overline{\rho z} \\ &\quad + W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\bar{z} + W_{02}^{(2)}(-1)\frac{\bar{z}^2}{2} + \cdots . \end{aligned}$$

From the definition of $f(\mu, u_t)$, we see

$$g(z,\bar{z}) = \bar{q}^*(0) f_0(z,\bar{z}) = \tau_j\bar{D}\big(f_0^{(1)} + \bar{\sigma}f_0^{(2)}\big). \tag{13}$$

Expanding and comparing the coefficients with those in (12), we obtain

$$
\begin{aligned}
g_{20} &= \tau_j \bar{D}\big[P^*_{x_d x_d}\mathrm{e}^{-2\mathrm{i}\omega_0\tau_j} + \bar{\sigma}Q^*_{x_d x_d}\mathrm{e}^{-2\mathrm{i}\omega_0\tau_j}\big],\\
g_{02} &= \tau_j \bar{D}\big[P^*_{x_d x_d}\mathrm{e}^{2\mathrm{i}\omega_0\tau_j} + \bar{\sigma}Q^*_{x_d x_d}\mathrm{e}^{2\mathrm{i}\omega_0\tau_j}\big],\\
g_{11} &= \tau_j \bar{D}P^*_{x_d x_d} + \bar{\sigma}Q^*_{x_d x_d}
\end{aligned}
$$

and

$$
\begin{aligned}
g_{21} = {}& W^{(1)}_{20}(-1)\left[\mathrm{e}^{\mathrm{i}\omega_0\tau_j}\big(P^*_{x_d x_d} + \bar{\sigma}Q^*_{x_d x_d}\big) + \frac{\bar{\rho}(1+\bar{\sigma})}{2}\right]\\
&+ W^{(1)}_{11}(-1)\big[2\mathrm{e}^{-\mathrm{i}\omega_0\tau_j}\big(P^*_{x_d x_d} + \bar{\sigma}Q^*_{x_d x_d}\big) + 2\rho(1+\bar{\sigma})\big]\\
&+ W^{(2)}_{11}(0)\big[2\mathrm{e}^{-\mathrm{i}\omega_0\tau_j}(1+\bar{\sigma})\big] + W^{(2)}_{20}(0)\big[\mathrm{e}^{\mathrm{i}\omega_0\tau_j}(1+\bar{\sigma})\big].
\end{aligned}
$$

In order to determine g_{21}, we still need to compute $W_{20}(\theta)$ and $W_{11}(\theta)$. From (9) and (11), we get

$$
\begin{aligned}
\dot{W} = \dot{u}_t - \dot{z}q - \dot{\bar{z}}\bar{q} &= \begin{cases} AW - 2\,\mathrm{Re}\{\bar{q}^*(0)f_0 q(\theta)\}, & \theta \in [-1,0),\\ AW - 2\,\mathrm{Re}\{\bar{q}^*(0)f_0 q(0)\} + f_0, & \theta = 0,\end{cases}\\
&\equiv AW + H(z,\bar{z},\theta),
\end{aligned} \tag{14}
$$

where

$$
H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots. \tag{15}
$$

By calculating the derivative of W, we find

$$
\dot{W} = \dot{W}_z\dot{z} + W_{\bar{z}}\dot{\bar{z}}. \tag{16}
$$

Comparing the coefficients of z^2, $z\bar{z}$ and $\bar{z}$ of (16) with those of (14) gives

$$
(A - 2\mathrm{i}\omega_0\tau_j I)W_{20}(\theta) = -H_{20}(\theta), \qquad AW_{11}(\theta) = -H_{11}(\theta), \tag{17}
$$

By (13) and (14), we know that for $\theta \in [-1,0)$,

$$
H(z,\bar{z},\theta) = -\bar{q}^*(0)f_0 q(\theta) - q^*(0)\bar{f}_0\bar{q}(\theta) = -gq(\theta) - \bar{g}\bar{q}(\theta). \tag{18}
$$

Hence, comparing the coefficients of (18) with (15) yields

$$
H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \qquad H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta).
$$

By substituting these relations into (17), we can derive the following equations:

$$
\begin{aligned}
\dot{W}_{20}(\theta) &= 2\mathrm{i}\omega_0\tau_j I W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta)\\
&\quad = 2\mathrm{i}\omega_0\tau_j I W_{20}(\theta) + g_{20}q(0)\mathrm{e}^{\mathrm{i}\omega_0\tau_j\theta} + \bar{g}_{02}\bar{q}(0)\mathrm{e}^{-\mathrm{i}\omega_0\tau_j\theta},\\
\dot{W}_{11}(\theta) &= g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta) = g_{11}q(0)\mathrm{e}^{\mathrm{i}\omega_0\tau_j\theta} + \bar{g}_{11}\bar{q}(0)\mathrm{e}^{-\mathrm{i}\omega_0\tau_j\theta}.
\end{aligned}
$$

Solving for $W_{20}(\theta)$ and $W_{11}(\theta)$, we obtain

$$W_{20}(\theta) = \frac{\mathrm{i}g_{20}}{\omega_0\tau_j} q(0)\mathrm{e}^{\mathrm{i}\omega_0\tau_j\theta} + \frac{\mathrm{i}\bar{g}_{02}}{3\omega_0\tau_j}\bar{q}(0)\mathrm{e}^{-\mathrm{i}\omega_0\tau_j\theta} + E_1\mathrm{e}^{2\mathrm{i}\omega_0\tau_j\theta},$$
$$W_{11}(\theta) = -\frac{\mathrm{i}g_{11}}{\omega_0\tau_j} q(0)\mathrm{e}^{\mathrm{i}\omega_0\tau_j\theta} + \frac{\mathrm{i}\bar{g}_{11}}{\omega_0\tau_j}\bar{q}(0)\mathrm{e}^{-\mathrm{i}\omega_0\tau_j\theta} + E_2,$$

where $E_1 = (E_1^{(1)}, E_1^{(2)}) \in \mathbb{R}^2$ and $E_2 = (E_2^{(1)}, E_2^{(2)}) \in \mathbb{R}^2$ are constant vectors. E_1 and E_2 can be determined as follows. From (14), we know that

$$H(z, \bar{z}, 0) = -2\,\mathrm{Re}\bigl\{\bar{q}^*(0) f_0 q(0)\bigr\} + f_0,$$

so that we find

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(\theta) + B_1,$$
$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + B_2.$$

Here B_1, B_2 are known vectors. Then, from (17) and the definition of A, we have

$$\left[2\mathrm{i}\omega_0\tau_j I - \int_{-1}^{0} \mathrm{e}^{2\mathrm{i}\omega_0\tau_j\theta}\,\mathrm{d}\eta(\theta)\right] E_1 = B_1,$$
$$\left[-\int_{-1}^{0} \mathrm{e}^{2\mathrm{i}\omega_0\tau_j\theta}\,\mathrm{d}\eta(\theta)\right] E_2 = B_2. \tag{19}$$

Solving (19), we can obtain E_1 and E_2. Based on the above analysis, the values of g_{ij} are computed. Hence, we can calculate all of the following quantities which are required for the stability analysis of Hopf bifurcation:

$$c_1(0) = \frac{\mathrm{i}}{2\omega_0\tau_j}\left[g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right] + \frac{g_{21}}{2},$$
$$\mu_2 = -\frac{\mathrm{Re}[c_1(0)]}{\mathrm{Re}[\tau_j\lambda'(\tau_j)]}, \qquad \beta_2 = 2\,\mathrm{Re}\bigl[c_1(0)\bigr],$$
$$T_2 = -\frac{\mathrm{Im}[c_1(0)] + \mu_2\,\mathrm{Im}[\lambda'(\tau_j)]}{\omega_0\tau_j}.$$

It is well known (see [21]) that μ_2 determines the direction of the Hopf bifurcation: if $\mu_2 > 0$ (resp. $\mu_2 < 0$), then the Hopf bifurcation is supercritical (resp. subcritical) and the bifurcating periodic solutions exist for $\tau > \tau_j$ (resp. $\tau < \tau_j$). β_2 determines the stability of the bifurcating periodic solutions. The bifurcating periodic solutions are orbitally asymptotically stable (resp. unstable) if $\beta_2 < 0$ (resp. $\beta_2 > 0$), and T_2 determines the period of the bifurcating periodic solutions: the period increases (resp. decreases) if $T_2 > 0$ (resp. $T_2 < 0$). From the discussion in the previous section we know that $\mathrm{Re}[\lambda'(\tau_j)] > 0$. We, therefore, have the following result.

Theorem 2. *The direction of the Hopf bifurcation of system* (1) *at the equilibrium* (k_*, p_*) *when* $\tau = \tau_j$ *is supercritical* (*resp. subcritical*) *and the bifurcating periodic solutions are orbitally asymptotically stable* (*resp. unstable*) *if* $\mathrm{Re}[c_1(0)] < 0$ (*resp.* $\mathrm{Re}[c_1(0)] > 0$).

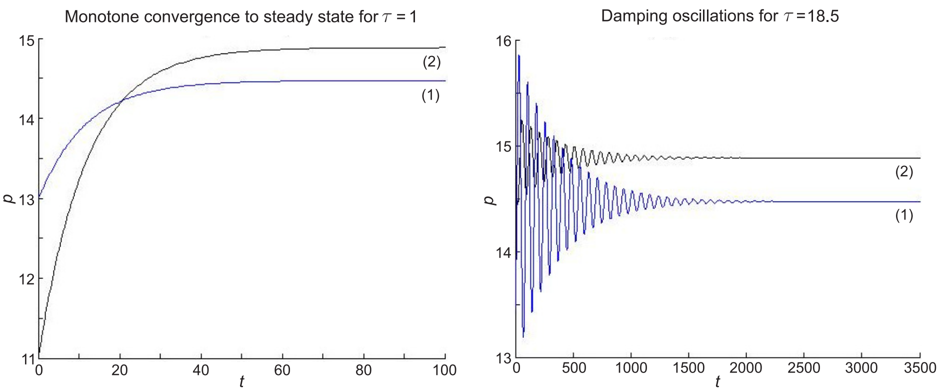
Remark 1. The expression of $\mathrm{Re}[c_1(0)]$ is quite cumbersome to be dealt with in a neat analytical form due to several nonlinearities. Nonetheless, by using software for algebraic manipulation it is possible to check its sign (given a certain parameter set) and thus to have analytical findings with regard to the type of the emerging bifurcation (see [11] for a similar approach). After having produced several numerical experiments, we conjecture that the Hopf bifurcation is always supercritical.

5 Numerical examples

In order to clarify the analytical results shown in the previous section, we now present some numerical simulations at different values of τ and we assume that parameters are set on a yearly basis. We fix $\alpha = 0.34$, $\delta = 0.12$, $\phi = 0.6$, $m = 0.1$, $s = 0.7$, and let τ vary. The steady state of the model is $(k_*, p_*) \simeq (14.47, 14.88)$ and, for $\tau < \tau_0 \simeq 19.83$, it is stable. In addition, since $\mathrm{Re}[c_1(0)] < 0$, the Hopf bifurcation is supercritical. For a relative low value of τ, the dynamics of k and p are monotone and converging to (k_*, p_*) (see Fig. 1a).

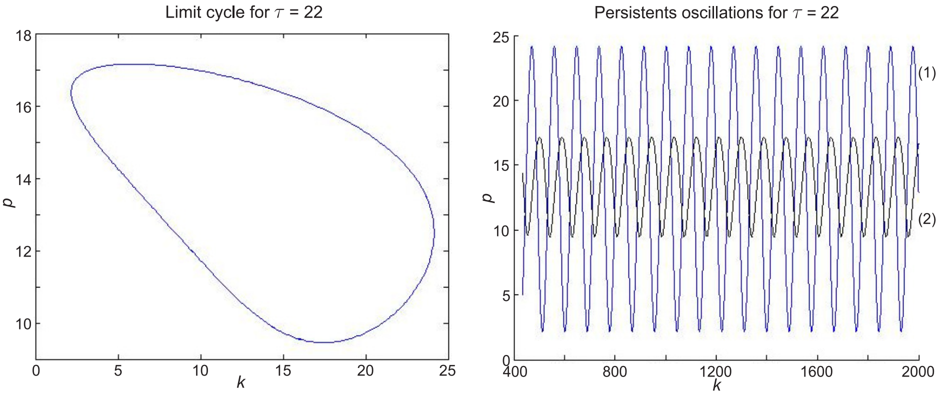
If we let τ increase, the dynamics become oscillating but converging to the stationary solution of the model (see Fig. 1b).

When τ crosses τ_0, a limit cycle surrounds the steady state and the model shows persistent oscillations (see Figs. 2a and 2b).



(a) (b)

Fig. 1. (a) Starting from initial conditions below (k_*, p_*) trajectories converge monotonically to steady state ($\tau = 1$, $k_0 = 13$, $p_0 = 11$); (b) Damped oscillations towards the corresponding equilibrium value of steady state ($\tau = 18.5$, $k_0 = 13$, $p_0 = 15$). Line (1) depicts the time evolution of the capital accumulation, line (2) depicts the evolution of pollution level.



(a) (b)

Fig. 2. (a) Limit cycle around (k_*, p_*) $(\tau = 22)$; (b) Corresponding trajectories ($k_0 = 10$, $p_0 = 10$). Line (1) depicts the time evolution of the capital accumulation, line (2) depicts the evolution of pollution level.

6 Conclusions

The aim of this paper is to introduce a time-to-build technology in an economic model with pollution, where the production function is strictly increasing in the capital stock. Even if we have not considered a negative feedback of pollution in the production function, we have shown that the existence of temporal lags in a context in which economic activity depletes a free-access natural resource can be a source of cyclical dynamics both in economic and environmental variables.

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