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On existence, multiplicity, uniqueness and stability of positive solutions of a Leslie–Gower type diffusive predator–prey system

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Abstract. In this paper, we consider a Leslie–Gower type diffusive predator–prey system. By using topological degree theory, bifurcation theory, energy estimates and asymptotic behavior analysis, we prove the existence, multiplicity, uniqueness and stability of positive steady states solutions under certain conditions on parameters.

Keywords: Leslie–Gower type diffusive predator–prey system, positive steady state solutions, uniqueness, multiplicity, stability.

1 Introduction

In [1], Hsu and Huang considered the following predator–prey model:

$$\begin{aligned} &\frac{\mathrm{d}u}{\mathrm{d}t} = ru\left(1-\frac{u}{K}\right) - vf(u), \quad t>0,\\ &\frac{\mathrm{d}v}{\mathrm{d}t} = sv\left(1-\frac{hv}{u}\right), \quad t>0, \end{aligned} \tag{1}$$

where u and v, respectively, represent the populations of the prey and the predator, and r, s, K, h are positive constants. The prey grows logistically with carrying capacity K and intrinsic growth rate r in the absence of predation. The predator consumes the prey according to the functional response $f(u)$ and grows logistically with intrinsic growth rate s. The carrying capacity of the predator is proportional to the population size of the prey. The term hv/u is called the Leslie–Gower term. It measures the loss in the predator population due to rarity of its favorite food (see [2] for details). Problem (1) was studied extensively in recent years (see [2] for f is of Holling type I, see [3–6] for f is of Holling type II, see [7] for f is of Holling type III).

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If we incorporate the Beddington-DeAngelis functional response into model (1), then it becomes

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= u\left(a-u\right) - \frac{cuv}{1+pu+mv}, \quad t>0, \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= v\left(b-\frac{v}{u}\right), \quad t>0, \end{aligned} \tag{2}$$

after suitable rescaling, where a, b, c, p, m are positive constants. Problem (2) with delay was studied in [8], where the authors considered the local and global stability of the equilibria by constructing suitable Lyapunov functional.

For model (1), if the favorite food u of v is lacking severely, the predator v can switch over to other populations, but its growth will be limited by the fact that its most favorite food u is not available in abundance. By considering the above reasons, a Leslie–Gower predator–prey system with saturated functional response is proposed in [9] by using Holling type II functional response, i.e.,

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= u\left(a-u\right) - \frac{cuv}{1+pu}, \quad t>0, \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= v\left(b-\frac{v}{1+qu}\right), \quad t>0, \end{aligned} \tag{3}$$

where a, b, c, p, q are positive constants.

On the other hand, the spatial component of ecological interactions has been identified as an important factor in how ecological communities are shaped, and understanding the role space is challenging both theoretically and empirically [10]. Empirical evidence suggests that the spatial scale and structure of environment can influence population interactions [11]. Based on above considerations, in this paper, we study the following Leslie–Gower type diffusive predator–prey system:

$$\begin{aligned} u_t - \Delta u &= u\left(a-u-\frac{cv}{1+pu+mv}\right), \quad x\in\varOmega,\ t>0, \\ v_t - \Delta v &= v\left(b-\frac{v}{1+qu}\right), \quad x\in\varOmega,\ t>0, \end{aligned} \tag{4}$$

with homogeneous Dirichlet boundary condition and the initial values $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$, where $\varOmega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\varOmega$. u and v represent the population of the prey and the predator, respectively. The parameters a, b, c, p, q, m are assumed to be positive constants. The homogeneous Dirichlet boundary condition means that the habitat $\varOmega$, where the two species live, is surrounded by a hostile environment (see [12] for more detailed biological implication to this model). The initial values $u_0(x)$ and $v_0(x)$ are non-negative continuous functions and not identically zero. Problem (4) with homogeneous Neumann boundary condition was studied in [13] for $m = 0$, [14] for $m > 0$. However, there is no paper about the dynamics of (4) with homogeneous Dirichlet boundary condition, and we will consider this problem in this paper (see Theorem 1 and Remark 1).

We also consider the steady-state solution $(u(x), v(x))$ of (4), i.e., $(u(x), v(x))$ satisfies

$$\begin{aligned} &-\Delta u = u\left(a - u - \frac{cv}{1 + pu + mv}\right), \quad x \in \Omega, \\ &-\Delta v = v\left(b - \frac{v}{1 + qu}\right), \quad x \in \Omega, \\ &u = v = 0, \quad x \in \partial\Omega. \end{aligned} \tag{5}$$

The main concern of problem (5) in this paper is the existence, multiplicity, and uniqueness of positive solutions. Here, we say (u, v) is a positive solution of (5) if (u, v) is a solution such that both u and v are positive in Ω. The existence, multiplicity, uniqueness and stability of positive solutions to problem (5) with $m = 0$ are concerned in [15, 16]. The case of $m > 0$ will be considered in this paper (see Theorem 2 and Corollary 1 for the existence of positive solutions, see Theorem 5 for the multiplicity results, and see Theorems 6 and 7 for the uniqueness results).

The organization of the remaining part of the paper is as follows. In Section 2, we consider the dynamics of problem (4); In Section 3, we study the existence of positive solutions of problem (5), while the uniqueness is considered in Section 4. In the paper, we use $\|\cdot\|_X$ as the norm of Banach space X, $\langle\cdot,\cdot\rangle$ as the duality pair of a Banach space X and its dual space X^*. For a linear operator L, we use $\mathcal{N}(L)$ as the null space of L and $\mathcal{R}(L)$ as the range space of L, and we use $L[w]$ to denote the image of w under the linear mapping L. For a multilinear operator L, we use $L[w_1, w_2, \dots, w_k]$ to denote the image of $(w_1, w_2, \dots, w_k)$ under L, and when $w_1 = w_2 = \dots = w_k$, we use $L[w_1]^k$ instead of $L[w_1, w_1, \dots, w_1]$. For a nonlinear operator F, we use F_u as the partial derivative of F with respect to argument u.

2 Preliminaries and stability analysis

In this section, we consider the stability/unstability of the trivial and semi-trivial solutions of (5). First, we introduce some notations and basic facts, which are well-known (see, for example, [11, 17]). For any $q \in C(\overline{\Omega})$, the linear eigenvalue problem

$$\begin{aligned} &-\Delta w + q(x)w = \rho w, \quad x \in \Omega, \\ &w = 0, \quad x \in \partial\Omega, \end{aligned} \tag{6}$$

has an infinite sequence of eigenvalues $\rho_1 < \rho_2 \leqslant \rho_3 \leqslant \cdots$, which are bounded below. It is also known that the principal eigenvalue

$$\rho = \rho_1 = \rho_1\bigl(q(x)\bigr)$$

is simple, and all solutions of (6) with $\rho = \rho_1(q(x))$ are multiples of a particular eigenfunction, which does not change sign in Ω and its normal derivative does not vanish on the boundary $\partial\Omega$. Furthermore, ρ_1 is strictly increasing in the sense that for $q_1(x), q_2(x) \in$

$C(\bar{\Omega})$, $q_1(x) \leqslant q_2(x)$ and $q_1(x) \not\equiv q_2(x)$ implies that $\rho_1(q_1(x)) < \rho_1(q_2(x))$. In particular, we denote by $\lambda_1 = \rho_1(0)$ and its corresponding normalized positive eigenfunction $\omega(x)$ satisfies $\max_{x\in\Omega} \omega(x) = 1$.

We define $C_0(\overline{\Omega}) = \{u \in C(\overline{\Omega})\colon\ u = 0 \text{ on } \partial\Omega\}$ and consider the logistic type problem

$$\begin{aligned} &-\Delta w = aw - bw^2, \quad x \in \Omega,\\ &w = 0, \quad x \in \partial\Omega, \end{aligned} \tag{7}$$

where $a, b > 0$ are constants. It is well known that if $a \leqslant \lambda_1$, $w = 0$ is the unique non-negative solution of (7), while (7) has a unique positive solution if $a > \lambda_1$. We denote the positive solution by θ_a when $b = 1$, and then the positive solution for general $b > 0$ is θ_a/b. In addition, the mapping $a \mapsto \theta_a$ is strictly increasing, continuously differentiable from (λ_1, ∞) to $C^2(\Omega) \cap C_0(\overline{\Omega})$ and $\theta_a \to 0$ uniformly on $\overline{\Omega}$ as $a \to \lambda_1$. Moreover, $0<\theta_a<a$ in Ω.

The corresponding initial boundary value problem of (7) is

$$\begin{aligned} &w_t - \Delta w = aw - bw^2, \quad x \in \Omega,\ t > t_0 \geqslant 0,\\ &w = 0, \quad x \in \partial\Omega,\ t > t_0,\\ &w(x, t_0) \geqslant 0,\ \not\equiv 0, \quad x \in \Omega. \end{aligned} \tag{8}$$

We denote the unique positive solution of (8) by $w_a^b(x,t)$, $x \in \overline{\Omega}$ and $t \geqslant t_0$. It is easy to show that the trivial solution 0 is globally stable if $a \leqslant \lambda_1$, while when $a > \lambda_1$, the solution $w_a^b(\cdot, t)$ of (8) converges to θ_a/b uniformly on $\overline{\Omega}$ as $t \to \infty$.

Now, let $(u(x,t), v(x,t))$ be the solution of (4). Clearly, (u, v) exists globally and $u(x,t) > 0$, $v(x,t) > 0$ for $x \in \Omega$ and $t > 0$ by the maximum principle for parabolic equations. Furthermore, we can obtain the following asymptotic behavior of $(u(x,t), v(x,t))$, which imply the extinction of the prey or predator.

Theorem 1. *Let $(u(x,t), v(x,t))$ be the positive solution of* (4) *and denote $\rho_1(c\theta_b/(1 + m\theta_b))$ by $\chi(b)$ for fixed $b > \lambda_1$. Then*:

(i) $(0,0)$ *is globally asymptotic stable if $a \leqslant \lambda_1$ and $b \leqslant \lambda_1$, i.e., $(u(x,t), v(x,t)) \to (0,0)$ uniformly on $\overline{\Omega}$ as $t \to \infty$*; $(0,0)$ *is unstable if $a > \lambda_1$ or $b > \lambda_1$.*

(ii) *Assume $b > \lambda_1$.* $(0,\theta_b)$ *is locally asymptotic stable if $a < \chi(b)$*; $(0,\theta_b)$ *is globally asymptotic stable if $a \leqslant \lambda_1$, i.e., $(u(x,t), v(x,t)) \to (0,\theta_b)$ uniformly on $\overline{\Omega}$ as $t \to \infty$*; $(0,\theta_b)$ *is unstable if $a > \chi(b)$.*

(iii) *Assume $a > \lambda_1$.* $(\theta_a, 0)$ *is globally asymptotic stable if $b \leqslant \lambda_1$, i.e., $(u(x,t), v(x,t)) \to (\theta_a, 0)$ uniformly on $\overline{\Omega}$ as $t \to \infty$*; $(0,0)$ *is unstable if $b > \lambda_1$.*

Proof. We first consider the global stability results.

(i) Since $u_t - \Delta u \leqslant au - u^2$, we see that $0 \leqslant u(x,t) \leqslant w_a^1(x,t)$ for all $x \in \overline{\Omega}$ and $t \geqslant 0$ by using comparison principle for parabolic equations, where $w_a^1(x,t)$ is the unique positive solution of (8) with $b = 1$ and $w(x,0) = u_0(x)$. So, $u(x,t) \to 0$ uniformly on $\overline{\Omega}$ as $t \to \infty$. Then for any $\varepsilon > 0$, there exists a $T(\varepsilon) \gg 1$ such that $u(x,t) \leqslant \varepsilon$ for $x \in \overline{\Omega}$ and $t > T$. By the second equation of (4), we have, for $x \in \Omega$ and $t > T$,

$v_t - \Delta v \leqslant bv - v^2/(1+q\varepsilon)$, and $v(x,t) \to 0$ uniformly on $\overline{\Omega}$ as $t \to \infty$ by similar analysis as above.

(ii) The proof of $u(x,t) \to 0$ uniformly on $\overline{\Omega}$ as $t \to \infty$ is the same as (i). Furthermore, we obtain that for any $\varepsilon > 0$, there exists a $T(\varepsilon) \gg 1$ such that $v(x,t) \leqslant w_b^{1/(1+q\varepsilon)}(x)$ for $x \in \overline{\Omega}$ and $t \geqslant T$ by similar argument in (i). Since $w_b^{1/(1+q\varepsilon)} \to (1+q\varepsilon)\theta_b(x)$ uniformly on $\overline{\Omega}$ as $t \to \infty$, we get $\limsup_{t\to\infty} v(x,t) \leqslant \theta_b(x)$ uniformly on $\overline{\Omega}$ by arbitrariness of ε. We observe that $v_t \geqslant bv - v^2$, thus $v(x,t) \geqslant w_b^1(x,t)$ with $w_b^1(x,0) = v_0(x)$ for $x \in \overline{\Omega}$ and $t \geqslant 0$, which shows that $v(x,t) \geqslant w_b^1(x,t) \to \theta_b(x)$ uniformly on $\overline{\Omega}$ as $t \to \infty$. This implies that $\liminf_{t\to\infty} v(x,t) \geqslant \theta_b(x)$ uniformly on $\overline{\Omega}$. So, $v(x,t) \to \theta_b(x)$ uniformly on $\overline{\Omega}$ as $t \to \infty$. The verification of (iii) is similar to that of (ii).

Next, we consider the locally stable or unstable results. We only prove the case (ii) since the proofs of other two cases are similar. From the linearization principle, the stability of $(0,\theta_b)$ is determined by the following eigenvalue problem:

$$\begin{aligned} &-\Delta\phi + \left(\frac{c\theta_b}{1+m\theta_b} - a\right)\phi = \mu\phi, \quad x \in \Omega, \\ &-\Delta\psi - q\theta_b^2\phi + (2\theta_b - b)\psi = \mu\psi, \quad x \in \Omega, \\ &\phi = \psi = 0, \quad x \in \partial\Omega. \end{aligned} \tag{9}$$

Since (9) is not completely coupled, we only need to consider the following two eigenvalue problems:

$$\begin{aligned} &-\Delta\psi + (2\theta_b - b)\psi = \mu\psi, \quad x \in \Omega, \\ &\psi = 0, \quad x \in \partial\Omega, \end{aligned} \tag{10}$$

and

$$\begin{aligned} &-\Delta\phi + \left(\frac{c\theta_b}{1+m\theta_b} - a\right)\phi = \mu\phi, \quad x \in \Omega, \\ &\phi = 0, \quad x \in \partial\Omega. \end{aligned} \tag{11}$$

Then it follows from [18, p. 76] that the eigenvalues of (9) are the union of the eigenvalues of (10) and (11). Denote the principal eigenvalue of (10) and (11) by μ_* and μ^*, respectively. Then

$$\begin{aligned} &\mu_* = \rho_1(2\theta_b - b) > \rho_1(\theta_b - b) = 0, \\ &\mu^* = \rho_1\left(\frac{c\theta_b}{1+m\theta_b} - a\right) = \chi(b) - a. \end{aligned}$$

Combining the above results, one can see that if $a < \chi(b)$, then all eigenvalues of (9) are positive, and thus $(0,\theta_b)$ is locally asymptotic stable. On the other hand, if $a > \chi(b)$, then (9) has a negative eigenvalue, which implies the instability of $(0,\theta_b)$. The proof is finished. $\square$

Remark 1. Since all the trivial and semi-trivial solutions of (5) are unstable if $a > \chi(b)$ and $b > \lambda_1$, then system (4) is persistent (see [11] for details). Moreover, we will prove that (5) has at least one positive solution under the condition of $a > \chi(b)$ and $b > \lambda_1$ (see Theorem 2).

3 Existence of positive solutions

In this section, we will study the existence of positive solutions of problem (5) by calculating the fixed point indices in positive cones. Thanks to Theorem 1, in the following, without special statement, we always assume that $a > \lambda_1$ and $b > \lambda_1$. Next, we give some results for a priori estimate of the positive solutions to (5).

Proposition 1. *Let $a, b > \lambda_1$ and (u, v) be a positive solution of* (5). *Then*

$$u(x) < \theta_a(x) < a \quad \textit{and} \quad \theta_b(x) < v(x) < (1 + qa)\theta_b(x) < b(1 + qa) \quad \textit{for } x \in \Omega.$$

Proof. Since $-\Delta u < u(a-u)$ and $-\Delta v > v(b-v)$, it is obvious that $u(x) < \theta_a(x) < a$ and $v(x) > \theta_b(x)$ for $x \in \Omega$. Since v is a lower solution of the problem

$$-\Delta w = w\left(b - \frac{w}{1 + qa}\right), \quad x \in \Omega,\ w = 0,\ x \in \partial\Omega, \tag{12}$$

$v(x) < (1 + qa)\theta_b(x)$ for $x \in \Omega$ because the unique positive solution of (12) is $(1 + qa)\theta_b(x)$. □

For later purpose, we consider the following system:

$$\begin{aligned} &-\Delta u = u\left(a - u - \frac{tcv}{1 + pu + mv}\right), \quad x \in \Omega, \\ &-\Delta v = v\left(b - \frac{v}{1 + tqu}\right), \quad x \in \Omega, \\ &u = v = 0, \quad x \in \partial\Omega, \end{aligned} \tag{13}$$

where $t \in [0, 1]$.

In the following, we will calculate the fixed point indices in positive cones. First of all, we introduce some notations. We set $E = C_0(\overline{\Omega}) \times C_0(\overline{\Omega})$ and let W be a positive cone with natural order, namely, $W \subset E$ and $W = K \bigoplus K$, where $K = \{w \in C_0(\overline{\Omega})\colon w \geqslant 0 \text{ in } \Omega\}$. We also define $D = \{(u, v) \in W\colon\ u < a,\ v < b(1 + qa)\}$. For $t \in [0, 1]$, we construct a compact operator family A_t as

$$A_t(u, v) = (-\Delta + C)^{-1} \begin{pmatrix} u(a + C - u - \frac{tcv}{1+pu+mv}) \\ v(b + C - \frac{v}{1+tqu}) \end{pmatrix},$$

where C is a large positive constant to be determined later. For fixed $t \in [0, 1]$, it is clear that finding non-negative solutions of (13) becames equivalently to solving the fixed point equation of A_t in W. For simplicity, let $A = A_1$. We also observe that finding the positive solution to (5) is equivalent to finding the positive fixed point of A.

Similar to the proof of Proposition 1, one can show that any non-negative solution of (13) is in D. We can choose $C = b(c+1)(1+qa)$ such that

$$u\left(a + C - u - \frac{tcv}{1+pu+mv}\right) \geqslant 0 \quad \text{and} \quad v\left(b + C - \frac{v}{1+tqu}\right) \geqslant 0$$

hold for all $t \in [0,1]$ and $(u,v) \in \overline{D}$. As a consequence, $\deg_W(A_t, D)$ is well defined, and moreover, it is easily seen from the homotopy invariance of degree that $\deg_W(A_t, D)$ is independent of t. On the other hand, we know that $(0,0)$, $(\theta_a, 0)$, and $(0, \theta_b)$ are all the non-negative trivial and semi-trivial solutions of (5). Using the theory of fixed point index developed by Amann [19] and Dancer [20], as in the proof of [21, 22], it follows from simple analysis and computations that the following lemma is valued.

Lemma 1. *Let $a > \lambda_1$ and $b > \lambda_1$. Then*:
- (i) $\deg_W(A, D) = 1$;
- (ii) $\mathrm{index}_W(I - A, (0,0)) = \mathrm{index}_W(I - A, (\theta_a, 0)) = 0$;
- (iii) $\mathrm{index}_W(I - A, (0,\theta_b)) = 0$ *if* $a > \chi(b)$ *while* $\mathrm{index}_W(I - A, (0,\theta_b)) = 1$ *if* $a < \chi(b)$,

where $\chi(b)$ is defined in Theorem 4.

Applying Lemma 1, we yield the following existence result for the positive solution to (5).

Theorem 2. (i) If $a > \chi(b)$ and $b > \lambda_1$, then problem (5) has at least one positive solution.
(ii) If problem (5) has a positive solution, then there exists a positive constant $\iota(b) \in (\lambda_1, \chi(b))$ such that $a > \iota(b)$ and $b > \lambda_1$.

Proof. (i) By Lemma 1, we have

$$\begin{aligned}
&\deg_W(A, D) - \mathrm{index}_W\bigl(I - A, (0,0)\bigr) \\
&\quad - \mathrm{index}_W\bigl(I - A, (\theta_a, 0)\bigr) - \mathrm{index}_W\bigl(I - A, (0, \theta_b)\bigr) \\
&= 1
\end{aligned}$$

under the condition $a > \chi(b)$ and $b > \lambda_1$. So, (5) has at least one positive solution.

(ii) Assume (u,v) is a positive solution of (5), then it follows from Krein–Rutman theorem and Propositions 1 that

$$a = \rho_1\left(u + \frac{cv}{1+pu+mv}\right) > \rho_1\left(\frac{c\theta_b}{1+pa+m\theta_b}\right)$$

and

$$b = \rho_1\left(\frac{v}{1+qu}\right) > \lambda_1.$$

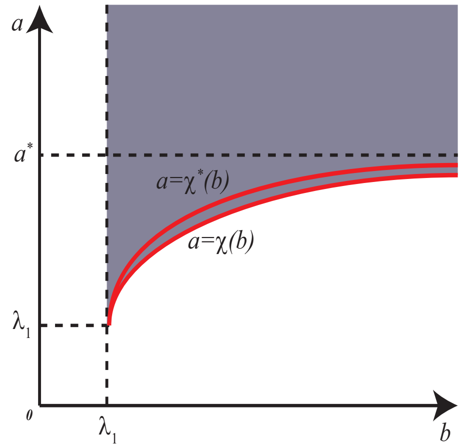

Fig. 1. The region of existence of positive solutions of problem (5) in (a,b)-plane, where $\chi^*(b) = \lambda_1 + cb/(1+mb)$ and $a^* = \lambda_1 + c/m$.

Similar to the proof of [23, Lemma 1.2], we can see $\rho_1(c\theta_b/(1+pa+m\theta_b))$ is strictly decreasing with respect to a, and

$$\lim_{a\to 0}\rho_1\left(\frac{c\theta_b}{1+pa+m\theta_b}\right) = \chi(b) \quad \text{and} \quad \lim_{a\to\infty}\rho_1\left(\frac{c\theta_b}{1+pa+m\theta_b}\right) = \lambda_1.$$

Then there exists a unique $\iota(b) \in (\lambda_1, \chi(b))$ such that $\iota(b) = \rho_1(c\theta_b/(1+p\iota(b)+m\theta_b))$ and $a > \rho_1(c\theta_b/(1+pa+m\theta_b))$ is equivalent to $a > \iota(b)$. The proof is finished. □

Remark 2. (i) Since $\chi(b)$ is strictly increasing in $b \in (\lambda_1, \infty)$ and $\theta_b < b$, we have $\chi(b) < \lambda_1 + cb/(1+mb)$. Then it follows from Theorem 2 that if $a \geqslant \lambda_1 + cb/(1+mb)$ and $b > \lambda_1$, then (5) has at least one positive solution.

(ii) By the monotone of $\chi(b)$, $\lim_{b\to\lambda_1}\chi(b) = \lambda_1$, and $\lim_{b\to\infty}\chi(b) = \lambda_1 + c/m$, we can characterize the coexistence region of problem (5) in (a,b)-plane (see Fig. 1).

(iv) If $b(pc-m)(1+qa) \leqslant 1$, then one can show that $u + cv/(1+pu+mv)$ is strictly increasing in $u \in [0,\infty)$ for fixed v by virtue of $v < b(1+qa)$. So, if (u,v) is a positive solution of (5), it follows from the Krein–Rutman theorem that

$$a = \rho_1\left(u + \frac{cv}{1+pu+mv}\right) > \rho_1\left(\frac{cv}{1+mv}\right) > \chi(b).$$

Then it follows from (i) of Theorem 2 that if $b(pc-m)(1+qa) \leqslant 1$, problem (5) has at least one positive solution if and only if $a > \chi(b)$ and $b > \lambda_1$. Moreover, we will prove if $b(pc-m)(1+qa) \leqslant 1$, problem (5) with $N = 1$ has a unique positive solution if and only if $a > \chi(b)$ and $b > \lambda_1$ (see Theorem 7).

Finally, the following global bifurcation result is obtained by Theorem 2 (see Fig. 2).

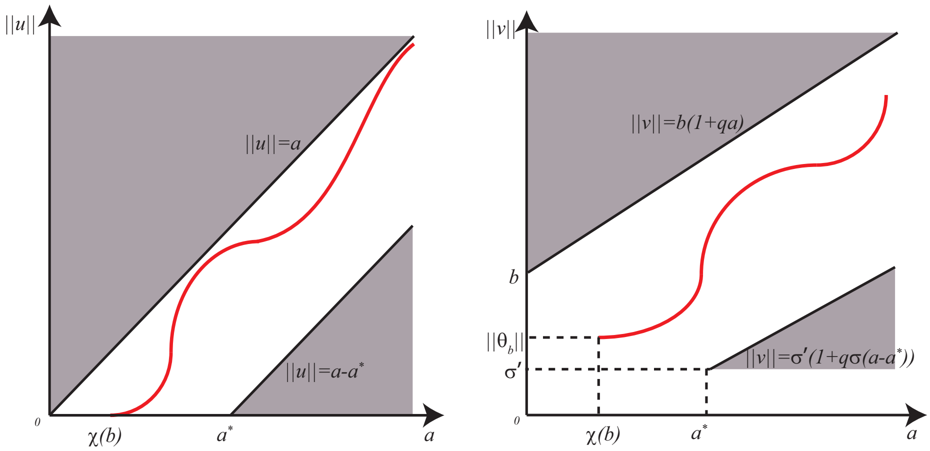

Fig. 2. Possible global bifurcation diagrams of (5), where $a^* = \lambda_1 + c/m$, $\|\cdot\| = \|\cdot\|_{L^\infty(\Omega)}$.

Corollary 1. *If we regard a as the main continuation parameter of* (5)*, then there exists an unbounded component $\Sigma \subset \mathbb{R} \times C_0(\overline{\Omega}) \times C_0(\overline{\Omega})$ of the set of positive solutions of* (5) *such that*

$$\big(\chi(b), 0, \theta_b\big) \in \overline{\Sigma} \quad \textit{and} \quad P_a\Sigma \supset \big(\chi(b), \infty\big),$$

where P_a stands for the projection operator into the a-component of the tern. Moreover, for any positive solution $(u(x), v(x))$ of (5)*, it holds $u(x) \to \infty$ and $v(x) \to \infty$ uniformly on any compact subset of Ω as $a \to \infty$.*

Proof. We only need to prove $u(x) \to \infty$ and $v(x) \to \infty$ uniformly on any compact subset of Ω as $a \to \infty$ since the other conclusions are obvious by Theorem 2. Without loss of generality, we assume that $a > \lambda_1 + c/m$. It follows from the first equation of (5) that

$$-\Delta u > u\left(a - \frac{c}{m} - u\right), \quad x \in \Omega,\ u = 0,\ x \in \partial\Omega.$$

Then $u(x) > (a - \lambda_1 - c/m)\omega(x)$ for $x \in \overline{\Omega}$ by comparison principle for elliptic equations. Let Ω' be any compact subset of Ω and assume $\min_{\overline{\Omega'}} \omega(x) = \sigma$. Then $u(x) > \sigma(a - \lambda_1 - c/m)$ in Ω', which tends to infinity as $a \to \infty$ uniformly in $x \in \Omega'$. Furthermore, by the second equation of (5), we have

$$\begin{aligned}
&-\Delta v = v\left(b - \frac{v}{1 + q\sigma(a - \lambda_1 - c/m)}\right), \quad x \in \Omega',\\
&v > 0, \quad x \in \partial\Omega'.
\end{aligned}$$

Then $v(x) > (1 + q\sigma(a - \lambda_1 - c/m))\theta_b^{\Omega'}(x)$ for $x \in \overline{\Omega'}$ by comparison principle for elliptic equations, where $\theta_b^{\Omega'}(x)$ is the unique positive solution of

$$\begin{aligned}
&-\Delta w = w(b - w), \quad x \in \Omega',\\
&w = 0, \quad x \in \partial\Omega'.
\end{aligned}$$

Let Ω'' be any compact subset of Ω' and assume $\min_{\overline{\Omega''}} \theta_b^{\Omega'}(x) = \sigma'$. Then $v(x) > \sigma'(1 + q\sigma(a - \lambda_1 - c/m))$ in Ω'', which tends to infinity as $a \to \infty$ uniformly in $x \in \Omega''$. □

4 Backward bifurcation and multiplicity of positive solutions

In this section, we will study the multiplicity of positive solutions of (5) through bifurcation method, and we will see if backward bifurcation happens, then (5) has at least two positive solutions (see Fig. 3 and Theorem 5). In the following, we will use a as a bifurcation parameter, and consider the bifurcation of positive solutions from the branch of semi-trivial solutions $\{(a, 0, \theta_b)\colon a > \iota(b)\}$, where $\iota(b) \in (\lambda_1, \chi(b))$ is defined in Theorem 2 and $\chi(b)$ is defined in Theorem 4. By linearizing (5) at $(0, \theta_b)$, we obtain the following eigenvalue problem:

$$\begin{aligned} &-\Delta\phi + \left(\frac{c\theta_b}{1 + m\theta_b} - a\right)\phi = \mu\phi, \quad x \in \Omega,\\ &-\Delta\psi - q\theta_b^2\phi + (2\theta_b - b)\psi = \mu\psi, \quad x \in \Omega,\\ &\phi = \psi = 0, \quad x \in \partial\Omega. \end{aligned} \tag{14}$$

A necessary condition for bifurcation is that the principle eigenvalue of (14) is zero, which occurs if $a = \chi(b)$. Let Φ with $\int_\Omega \Phi^2 dx = 1$ be the positive eigenfunction corresponding to $\chi(b)$. Denote $X = C^2(\Omega) \cap C_0(\overline{\Omega})$ and $Y = C(\overline{\Omega})$. Since $\rho_1(2\theta_b - b) > 0$, the operator $-\Delta + 2\theta_b - b : X \mapsto Y$ is invertible, and $(-\Delta + 2\theta_b - b)^{-1}$ maps positive functions to positive functions by maximum principle. Let $\Psi = (-\Delta + 2\theta_b - b)^{-1}(q\theta_b^2\Phi)$, then both Φ and Ψ are positive in Ω, and (Φ, Ψ) is a solution of (14) with $\mu = 0$.

With the functions Φ, Ψ and Banach spaces X, Y defined above, we have the following result regarding the bifurcation of positive solutions of (5) from $(a, 0, \theta_b)$ at $a = \chi(b)$.

Theorem 3. *Assume $a > \iota(b)$ and $b > \lambda_1$. Then $a = \chi(b)$ is a bifurcation value of (5), where positive solutions bifurcate from the line of semi-trivial solutions $\Gamma_0 = \{(a, 0, \theta_b)\colon a > \iota(b)\}$; near $(\chi(b), 0, \theta_b)$, all the positive solution of (5) lie on a smooth curve $\Gamma_1 = \{(a(s), u(s), v(s))\colon s \in (0, \delta)\}$ for some $\delta > 0$ such that*

$$\begin{aligned} a(s) &= \chi(b) + sa_1 + sa_2(s),\\ u(s) &= s\Phi + su_1(s, x),\\ v(s) &= \theta_b + s\Psi + sv_1(s, x), \end{aligned} \tag{15}$$

where $s \mapsto (a_2(s), u_1(s), v_1(s))$ is a smooth function from $(0, \delta)$ to $\mathbb{R} \times X \times X$ such that $a_2(0) = 0$, $u_1(0, x) = v_1(0, x) = 0$ and

$$a_1 = \int\limits_\Omega \left(1 - \frac{cp\theta_b}{(1 + m\theta_b)^2}\right)\Phi^3 \,\mathrm{d}x + c\int\limits_\Omega \frac{\Phi^2\Psi}{(1 + m\theta_b)^2}\,\mathrm{d}x. \tag{16}$$

Moreover, $\chi(b)$ is the unique value for which positive solutions bifurcate from Γ_0.

Proof. Define a nonlinear mapping $F : \mathbb{R} \times X \times X \to Y \times Y$ by

$$F(a,u,v) = \begin{pmatrix} \Delta u + u(a - u - \frac{cv}{1+pu+mv}) \\ \Delta v + v(b - \frac{v}{1+qu}) \end{pmatrix}.$$

By straightforward calculations, we obtain

$$F_{(u,v)}(a,u,v)[\xi,\eta] = \begin{pmatrix} \Delta\xi + a\xi - 2u\xi - \frac{cv+cmv^2}{(1+pu+mv)^2}\xi - \frac{cu+cpu^2}{(1+pu+mv)^2}\eta \\ \Delta\eta + \frac{qv^2}{(1+qu)^2}\xi + b\eta - \frac{2v}{1+qu}\eta \end{pmatrix},$$

$$F_a(a,u,v) = \begin{pmatrix} u \\ 0 \end{pmatrix}, \qquad F_{a(u,v)}(a,u,v)[\xi,\eta] = \begin{pmatrix} \xi \\ 0 \end{pmatrix},$$

$$F_{(u,v)(u,v)}(a,u,v)[\xi,\eta]^2 = \begin{pmatrix} A_1\xi^2 + A_2\xi\eta + A_3\eta^2 \\ B_1\xi^2 + B_2\xi\eta + B_3\eta^2 \end{pmatrix},$$

where

$$A_1 = \frac{2cp(v+mv^2)}{(1+pu+mv)^3} - 2, \qquad A_2 = -\frac{2c(1+pu+mv+2mpuv)}{(1+pu+mv)^3},$$

$$A_3 = \frac{2cm(u+pu^2)}{(1+pu+mv)^3},$$

$$B_1 = -\frac{2q^2v^2}{(1+qu)^3}, \qquad B_2 = \frac{4qv}{(1+qu)^2}, \qquad B_3 = -\frac{2}{1+qu}.$$

At $(a,u,v) = (\chi(b), 0, \theta_b)$, it is easy to see that the kernel $\mathcal{N}F_{(u,v)}(\chi(b),0,\theta_b) = \operatorname{span}\{(\Phi,\Psi)\}$ and the range space $\mathcal{R}F_{(u,v)}(\chi(b),0,\theta_b) = \{(f,g) \in Y \times Y\colon \int_\Omega f(x) \times \Phi(x)\,\mathrm{d}x = 0\}$. Moreover, $F_{a(u,v)}(\chi(b),0,\theta_b)[\Phi,\Psi] = (\Phi,0) \notin \mathcal{R}F_{(u,v)}(\chi(b),0,\theta_b)$ since $\int_\Omega \Phi^2 dx = 1 \neq 0$. Thus we apply [24, Thm. 1.7] to conclude that the set of positive solutions of (5) near $(\chi(b),0,\theta_b)$ is a smooth curve Γ_1 satisfying (15). Moreover, $a_1 = a'(0)$ can be calculated by (see [25])

$$\begin{aligned} a'(0) &= -\frac{\int_\Omega F_{(u,v)(u,v)}(\chi(b),0,\theta_b)[\Phi,\Psi]^2\Phi\,\mathrm{d}x}{2\int_\Omega F_{a(u,v)}(\chi(b),0,\theta_b)[\Phi,\Psi]\Phi\,\mathrm{d}x} \\ &= \frac{2\int_\Omega (1 - \frac{cp\theta_b}{(1+m\theta_b)^2})\Phi^3\,\mathrm{d}x + 2c\int_\Omega \frac{\Phi^2\Psi}{(1+m\theta_b)^2}\,\mathrm{d}x}{2\int_\Omega \Phi^2\,\mathrm{d}x} \\ &= \int_\Omega \left(1 - \frac{cp\theta_b}{(1+m\theta_b)^2}\right)\Phi^3\,\mathrm{d}x + c\int_\Omega \frac{\Phi^2\Psi}{(1+m\theta_b)^2}\,\mathrm{d}x. \end{aligned}$$

Since the proof of $\chi(b)$ is the unique value for which positive solutions bifurcate from Γ_0 is basically same as [26, Prop. 2.2], we omit its proof. The proof is finished. $\square$

Next, we discuss the stability of the positive solutions obtained from Theorem 3.

Theorem 4. *Assume the conditions in Theorem* 3 *hold, and let* a_1 *be defined as in* (16). *If* $a_1 \neq 0$*, then there exists* $\tilde{\delta} \in (0, \delta]$ *such that for* $s \in (0, \tilde{\delta})$*, the positive solution* $(a(s), u(s), v(s))$ *bifurcating from* $(\chi(b), 0, \theta_b)$ *is not degenerate, where* δ *is the constant in Theorem* 3. *Moreover,* $(u(s), v(s))$ *is unstable if* $a_1 < 0$*, and it is stable if* $a_1 > 0$.

Proof. In order to study the stability of the bifurcating positive solution $(u(s), v(s))$ of (5), we consider the following eigenvalue problem:

$$L(s)\begin{pmatrix}\xi(s)\\ \eta(s)\end{pmatrix} = \mu(s)\begin{pmatrix}\xi(s)\\ \eta(s)\end{pmatrix}, \quad x \in \Omega,$$
$$\xi(s) = \eta(s) = 0, \quad x \in \partial\Omega,$$

where

$$\begin{aligned} L(s) &= -F_{(u,v)}\big(a(s), u(s), v(s)\big) \\ &= \begin{pmatrix} -\Delta - a(s) + 2u(s) + \frac{cv(s)+cmv(s)^2}{(1+pu(s)+mv(s))^2} & \frac{cu(s)+cpu(s)^2}{(1+pu(s)+mv(s))^2} \\ -\frac{qv(s)^2}{(1+qu(s))^2} & -\Delta - b + \frac{2v(s)}{1+qu(s)} \end{pmatrix}. \end{aligned}$$

Furthermore,

$$\lim_{s\to 0} L(s) = \begin{pmatrix} -\Delta - \chi(b) + \frac{c\theta_b}{1+m\theta_b} & 0 \\ -q\theta_b^2 & -\Delta - b + 2\theta_b \end{pmatrix} := L_0.$$

Since $\chi(b) = \rho_1 c\theta_b/(1+m\theta_b)$ and $\rho_1(-b+2\theta_b) > 0$, then 0 is the first eigenvalue of L_0 with corresponding eigenfunction (Φ, Ψ). Moreover, the real part of all other eigenvalues of L_0 are positive and are apart from 0. By perturbation theory of linear operators (see [27]), we know that for $s > 0$ small, $L(s)$ has a unique eigenvalue $\mu(s)$ such that $\lim_{s\to 0}\mu(s) = 0$ and all other eigenvalues of $L(s)$ have positive real part and are part from 0.

Now we determine the sign of $\mu(s)$ for small $s > 0$. Consider the following eigenvalue problem

$$-F_{(u,v)}(a, 0, \theta_b)\begin{pmatrix}\phi(a)\\ \psi(a)\end{pmatrix} = \gamma(a)\begin{pmatrix}\phi(a)\\ \psi(a)\end{pmatrix}, \quad x \in \Omega,$$
$$\phi(a) = \psi(a) = 0, \quad x \in \partial\Omega.$$

Then $\phi(a)$ satisfies

$$\begin{aligned} &-\Delta\phi(a) + \frac{c\theta_b}{1+m\theta_b}\phi(a) - a\phi(a) = \gamma(a)\phi(a), \quad x \in \Omega, \\ &\phi(a) = 0, \quad x \in \partial\Omega. \end{aligned} \tag{17}$$

Since $\gamma(\chi(b)) = 0$ and $\phi(\chi(b)) = \Phi$, then by differentiating (17) with respect to a at $a = \chi(b)$, we obtain that

$$\begin{aligned} &-\Delta\varphi + \frac{c\theta_b}{1+m\theta_b}\varphi - \chi(b)\varphi - \Phi = \gamma'\big(\chi(b)\big)\Phi, \quad x \in \Omega, \\ &\varphi = 0, \quad x \in \partial\Omega, \end{aligned} \tag{18}$$

where $\varphi = \phi'(\chi(b))$. Multiplying both sided of (18) with Φ and integrating it over Ω, we obtain

$$\begin{aligned}\gamma'\bigl(\chi(b)\bigr)\int_\Omega \Phi^2\,\mathrm{d}x &= -\int_\Omega \Phi^2\,\mathrm{d}x + \int_\Omega \biggl(-\Delta\varphi\Phi + \frac{c\theta_b}{1+m\theta_b}\varphi\Phi - \chi(b)\varphi\Phi\biggr)\,\mathrm{d}x \\ &= -\int_\Omega \Phi^2\,\mathrm{d}x + \int_\Omega \varphi\biggl(-\Delta\Phi + \frac{c\theta_b}{1+m\theta_b}\Phi - \chi(b)\Phi\biggr)\,\mathrm{d}x \\ &= -\int_\Omega \Phi^2\,\mathrm{d}x,\end{aligned}$$

i.e., $\gamma'(\chi(b)) = -1$. Since $a_1 \neq 0$, then it follows from [28, Corol. 1.13] that $\mu(s) \neq 0$ for $s > 0$ small and

$$\lim_{s\to 0}\frac{\mu(s)}{s} = -\gamma'\bigl(\chi(b)\bigr)a'(0) = a_1. \tag{19}$$

Since all the other eigenvalues of $L(s)$ have positive real parts, then the conclusion follows from (19). □

Based on the above preparations, we give the multiplicity result on positive solutions of (5) as follows.

Theorem 5. *Assume the conditions of Theorem 3 are satisfied, and let a_1 be defined as in (16). If $a_1 < 0$, then there exists a positive constant $\varepsilon_b \in (0, \chi(b) - \iota(b))$ such that (5) has at least two positive solutions if $\chi(b) - \varepsilon_b < a < \chi(b)$, and it has at least one positive solution if $a \geqslant \chi(b) - \varepsilon_b$.*

Proof. From Theorem 3, (5) has a curve $\Gamma_1 = \{(a(s), u(s), v(s))\colon s \in (0,\delta)\}$ of positive solutions near $(\chi(b), 0, \theta_b)$. Since $a_1 < 0$, $a(s) < \chi(b)$ for $s > 0$ small. Assume on the contrary that (5) has a unique positive solution $(\hat u, \hat v)$ when $a < \chi(b)$ but near $\chi(b)$. Then it is obvious that $(\hat u, \hat v)$ is the positive solution bifurcating from $(\chi(b), 0, \theta_b)$, which was obtained from Theorem 3. That is $(\hat u, \hat v) = (u(s), v(s))$, which is not degenerate by Theorem 4. Thus $I - A_{(u,v)}(\hat u, \hat v) : \overline{W_{(\hat u,\hat v)}} \to \overline{W_{(\hat u,\hat v)}}$ is invertible, where A is the operator defined in Section 3. Since $(\hat u, \hat v)$ is an isolate interior point of D, $\mathrm{index}_W(A, (\hat u, \hat v)) = \pm 1$. Notice that $\lambda_1 < a < \chi(b)$ for $s > 0$ small and $b > \lambda_1$. It follows from Lemma 1 that

$$\begin{aligned}1 &= \deg_W(A, D) \\ &= \mathrm{index}_W\bigl(A, (0,0)\bigr) + \mathrm{index}_W\bigl(A, (\theta_a, 0)\bigr) + \mathrm{index}_W\bigl(A, (0, \theta_b)\bigr) \\ &\quad + \mathrm{index}_W\bigl(A, (\hat u, \hat v)\bigr) \\ &= 0 + 0 + 1 \pm 1,\end{aligned}$$

which is a contradiction. Thus if $a < \chi(b)$ and near $\chi(b)$, then there exists at least two positive solutions of (5).

By global bifurcation Theorem (see [29, Thm. 7.2.2]), the curve Γ_1 of bifurcating positive solutions is contained in a connected component Σ of the set of positive solutions of (5). Moreover, $\overline{\Sigma}$ contains another semi-trivial solutionon $\{(a, 0, \theta_b)\colon a > \iota(b)\}$, or $\overline{\Sigma}$

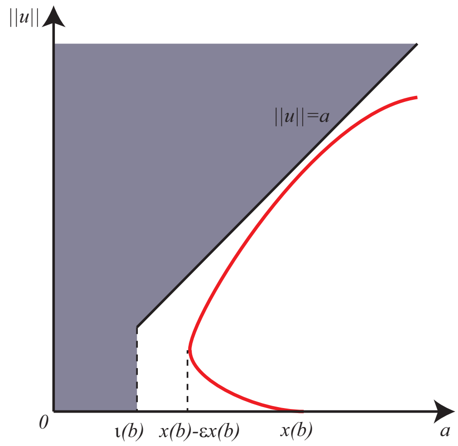

Fig. 3. Backward bifurcation of u when $a_1 < 0$, where $||u|| = \|u\|_{L^\infty(\Omega)}$.

contains semi-trivial solution $(a, \theta_a, 0)$, or Σ is unbounded. By Theorem 3, the first alternative is impossible since $\chi(b)$ is the unique bifurcation value for positive solutions of (5) bifurcate from $\{(a, 0, \theta_b)\colon a > \iota(b)\}$. It follows from Theorem 1 that the second alternative is also impossible since $(\theta_a, 0)$ is not degenerate for all $a > \lambda_1$ since $b > \lambda_1$. Thus Σ must be unbounded. Furthermore, it follows from Proposition 1 that $u(s, x) < a$ for $x \in \overline{\Omega}$ and it follows from Theorem 2 that there are no positive solutions when $a \leqslant \iota(b)$. Thus there exists $\varepsilon_b \in (0, \chi(b) - \iota(b))$ such that the projection of $\overline{\Sigma}$ on the a-axis is $[\chi(b) - \varepsilon_b, \infty)$. In particular, (5) has at least two positive solutions if $\chi(b) - \varepsilon_b < a < \chi(b)$, and it has at least one positive solution if $a \geqslant \chi(b) - \varepsilon_b$ (see Fig. 3). □

Remark 3. Since Φ and Ψ are independent of p, then $\lim_{p\to\infty} a_1 = -\infty$. So, $a_1 < 0$ can be achieved if p is large enough.

5 Uniqueness of positive solutions

In this section, we study the uniqueness and stability of positive solutions to (5). First, we consider the case that c/m is small enough. To this end, we need to consider the asymptotic behavior of positive solutions of (5) when $c/m \to 0$, which is given by the following lemma.

Lemma 2. *Assume $a > \lambda_1$ and $b > \lambda_1$.*

(i) *Suppose that (u_i, v_i) is a positive solution of (5) with $c = c_i$ and $m = m_i$, and $c_i/m_i \to 0$ as $i \to \infty$, then (u_i, v_i) converges to (θ_a, v^*) uniformly as $i \to \infty$, where v^* is the unique positive solution of*

$$-\Delta z = z\left(b - \frac{z}{1 + q\theta_a}\right), \quad x \in \Omega,$$
$$w = 0, \quad x \in \partial\Omega.$$

(ii) *There exists a positive constant* δ *small enough such that any positive solution of* (5) *is non-degenerate and linearly stable if* $c/m \leqslant \delta$.

Proof. Since the proof of (ii) is similar to [16, Thm. 1.10] or [30, Thm. 4], we only give the proof of (i). It is clear that (θ_b, v^*) is the unique positive solution of the following problem:

$$\begin{aligned} &-\Delta w = w(a-w), \quad x \in \Omega,\\ &-\Delta z = z\left(b - \frac{z}{1+qw}\right), \quad x \in \Omega,\\ &w = z = 0, \quad x \in \partial\Omega. \end{aligned} \tag{20}$$

Similar to the proof of Proposition 1, we have $\|u_i\|_{L^\infty(\Omega)} \leqslant a$ and $\|v_i\|_{L^\infty(\Omega)} \leqslant b(1+qa)$ and the upper bounds are both independent of i (thus independent of c_i and m_i), then $\|u_i\|_{C^{2+\alpha}(\Omega)}$ and $\|v_i\|_{C^{2+\alpha}(\Omega)}$, $0<\alpha<1$, are uniformly bounded by standard regularity theory of elliptic equations. Then there exits a subsequence of $\{(u_i, v_i)\}$, relabeled by itself, and two non-negative functions $w, z \in C^{2+\beta}(\overline{\Omega})$ with $0<\beta<\alpha$ such that $u_i \to w$ and $v_i \to z$ in $C^{2+\beta}(\overline{\Omega})$ as $i \to \infty$. Then (w, z) must be a solution of (20). From the strong maximum principle, we know each of w and z is either >0 in Ω or $\equiv 0$ in Ω. So, if we can show that $w>0$ and $z>0$ in Ω, then the proof is complete as the positive solution of (20) is unique, hence, it must be the limit of the sequence of $\{(u_i, v_i)\}$.

To the contrary, we assume that $w \equiv 0$, then $\|u_i\|_{L^\infty(\Omega)} \to 0$ as $i \to \infty$. Let $\overline{u}_i = u_i/\|u_i\|_{L^\infty(\Omega)}$, then $\overline{u}_i$ satisfies

$$\begin{aligned} &-\Delta\overline{u}_i = \overline{u}_i\left(a - u_i - \frac{c_i v_i}{1+pu_i+m_i v_i}\right), \quad x \in \Omega,\\ &\overline{u}_i = 0, \quad x \in \partial\Omega. \end{aligned}$$

Similar to the above arguments, $\|\overline{u}_i\|_{C^{2+\alpha}(\Omega)}$ is uniformly bounded, thus there exists a subsequence of $\{\overline{u}_i\}$, relabeled by itself, and a non-negative function $\overline{u} \in C^{2+\beta}(\overline{\Omega})$ with $0<\beta<\alpha<1$ such that $\overline{u}_i \to \overline{u}$ in $C^{2+\beta}(\overline{\Omega})$ as $i \to \infty$. Obviously, $\|\overline{u}\|_{L^\infty(\Omega)} = 1$ and $\overline{u}$ satisfies

$$\begin{aligned} &-\Delta\overline{u} = a\overline{u}, \quad x \in \Omega,\\ &\overline{u} = 0, \quad x \in \partial\Omega. \end{aligned}$$

Therefore, $a = \lambda_1$ must hold, which contradicts to the assumption that $a > \lambda_1$.

On the other hand, if we assume $z \equiv 0$, the same arguments as above show that $b = \lambda_1$, which again contradicts to the assumption that $b > \lambda_1$. □

Now we can state the first result about the uniqueness of positive solutions of (5) by Lemma 1, Theorem 2 and Lemma 2.

Theorem 6. *Suppose* $a > \lambda_1$ *and* $b > \lambda_1$, *and* $p, q > 0$ *are fixed constants, there exists a constant* $\delta > 0$ *such that* (5) *has a unique positive solution, which is locally asymptotic stable when* $0 < c/m \leqslant \delta$.

Proof. The existence of a positive solution easily follows from Theorem 2 since $a > \lambda_1$ is fixed and $\chi(b) \to \lambda_1$ as $c/m \to 0$. Hence, we only need to show the uniqueness and local stability. Recall $A = A_1$ is the operator defined in Section 3 and D is the region that positive solutions lie in. By compactness, A has at most finitely many positive fixed point in the region D. We denote all the positive fixed points of A in D by (u_i, v_i) for $i = 1, 2, \dots, l$. From (ii) of Lemma 2, we have $\mathrm{index}_W(A, (u_i, v_i)) = 1$ for each $i \in \{1, 2, \dots, l\}$. According to the additive property of Leray–Schauder degree and Lemma 1, we obtain

$$\begin{aligned} 1 &= \deg_W(A, D) \\ &= \mathrm{index}_W\big(A, (0,0)\big) + \mathrm{index}_W\big(A, (\theta_a, 0)\big) + \mathrm{index}_W\big(A, (0, \theta_b)\big) \\ &\quad + \sum_{i=1}^{l} \mathrm{index}_W\big(A, (u_i, v_i)\big) \\ &= 0 + 0 + 0 + l = l. \end{aligned}$$

Hence, $l \equiv 1$, which asserts the uniqueness. The local stability have been proved in Lemma 2. $\square$

According to Theorem 6, (5) has a unique positive solution if c/m is small, which is characterized by δ. However, it is hard to judge the quantity of δ. Based on this reason, next, we will give some specific conditions to ensure the uniqueness of positive solutions of (5) for $N = 1$. Consider the following problem:

$$\begin{aligned} &-u'' = u\left(a - u - \frac{cv}{1 + pu + mv}\right), \quad 0 < x < L, \\ &-v'' = v\left(b - \frac{v}{1 + qu}\right), \quad 0 < x < L, \\ &u(0) = u(L) = v(0) = v(L) = 0, \end{aligned} \tag{P_c}$$

where L is a positive constant and $'' := \mathrm{d}/\mathrm{d}x^2 = \Delta$.

First let us introduce several lemmas, which will be used to get the uniqueness.

Lemma 3. *Assume $b(pc - m)(1 + qa) \leqslant 1$. Let (u_0, v_0) be an arbitrary positive solution of (P_c), then the linearized system of (P_c) at (u_0, v_0) has only the trivial solution $(0, 0)$. Hence, any positive solution of (P_c) is not degenerate.*

Proof. The linearized system of (P_c) at (u_0, v_0) is

$$\begin{aligned} &-\phi'' + L_1\phi = M_1\psi, \quad 0 < x < L, \\ &-\psi'' + L_2\psi = M_2\phi, \quad 0 < x < L, \\ &\phi(0) = \phi(L) = \psi(0) = \psi(L) = 0, \end{aligned} \tag{21}$$

where

$$L_1\phi = 2u_0 + \frac{cv_0(1 + mv_0)}{(1 + pu_0 + mv_0)^2} - a, \qquad L_2\psi = \frac{2v_0}{1 + qu_0} - b,$$

$$M_1 = -\frac{cu_0}{(1+pu_0+mv_0)^2} < 0, \qquad M_2 = \frac{qv_0^2}{(1+qu_0)^2} > 0.$$

Since (u_0, v_0) is a positive solution of (P_c), it follows from the Krein–Rutman theorem that

$$\rho_1\left(u_0 + \frac{cv_0}{1+pu_0+mv_0} - a\right) = 0 \quad \text{and} \quad \rho_1\left(\frac{v_0}{1+qu_0} - b\right) = 0.$$

Clearly, $\rho_1(L_2) > \rho_1(v_0/(1+qu_0) - b) = 0$. Since $v_0 < b(1+qa)$, we have $pcv_0 < 1 + pu_0 + mv_0$ if $b(pc-m)(1+qa) \leqslant 1$, which means

$$2u_0 + \frac{cv_0(1+mv_0)}{(1+pu_0+mv_0)^2} > u_0 + \frac{cv_0}{1+pu_0+mv_0}.$$

So, $\rho_1(L_2) > 0$. Then the operator $\Delta_1 u := -u'' + L_1 u : \mathcal{X} \to \mathcal{Y}$ and $\Delta_2 u := -u'' + L_2 u : \mathcal{X} \to \mathcal{Y}$ is invertible, where $\mathcal{X} = C_0^2[0,L] = \{u \in C^2[0,L]\colon u(0) = u(L) = 0\}$ and $\mathcal{Y} = C[0,L]$. Let $P = \{u \in \mathcal{X}\colon u \geqslant 0 \text{ in } \Omega\}$ be the usual cone of positive functions in $\mathcal{X}$, and let Δ_1^{-1} and Δ_2^{-1} be the inverse operator of Δ_1 and Δ_2, respectively. It is obvious that Δ_1^{-1} and Δ_2^{-1} are compact and strictly order-preserving operators with respect to P. Moreover, $\Delta_i^{-1}(P) \setminus \{0\} \subset \operatorname{int} P$ for $i = 1, 2$. In terms of Δ_1 and Δ_2, (21) can be written as

$$\Delta_1 \phi = M_1 \psi, \quad \Delta_2 \psi = M_2 \phi, \quad \phi, \psi \in \mathcal{X}.$$

In this setting we can show that the only solution of above problem is $\phi = \psi = 0$ using a similar proof as in [31, 32], which completes the proof. $\square$

A perturbation argument can be used to show that if (P_c) has exactly one positive solution, which is assumed to be non-degenerate, then $(P_{c+\epsilon})$ has also exactly one positive solution provided ϵ is small enough. For that purpose, we state the following lemma. Since the roof is basically same as [31, Lemma 5.4], we omit its proof.

Lemma 4. *Assume $b(pc-m)(1+qa) \leqslant 1$ and (P_c) has exactly one positive solution (u_0, v_0), which is not degenerate. Then there exists $\epsilon_0 > 0$ such that for every $\epsilon \in (-\epsilon_0, \epsilon_0)$, problem $(P_{c+\epsilon})$ has exactly one positive solution $(u(\epsilon), v(\epsilon))$. Moreover, $(u(0), v(0)) = (u_0, v_0)$ and the mapping $\epsilon \mapsto (u(\epsilon), v(\epsilon))$, from $(-\epsilon_0, \epsilon_0)$ to $P \times P$, is C^1.*

Since (P_0) has exactly one positive solution (θ_a, v^*), where v^* is given in Lemma 2, by using (iii) of Remark 2, Lemma 3 and Lemma 4, we can state the following uniqueness result. Again the proof is similar to [31, Thm. 5.1], thus we omit its proof.

Theorem 7. (*See Fig.* 4.) *Assume $b(pc-m)(1+qa) \leqslant 1$ and $q(pc-m)(1+\lambda_1^2) < 1$. Then problem (P_c) has a unique positive solution if and only if $a > \chi(b)$ and $b > \lambda_1$.*

Remark 4. The condition $q(pc-m)(1+\lambda_1^2) < 1$ is to ensure $\{a\colon b(pc-m) \times (1+qa) \leqslant 1\} \cap \{a\colon a > \chi(b)\} \neq \emptyset$ when $pc > m$.

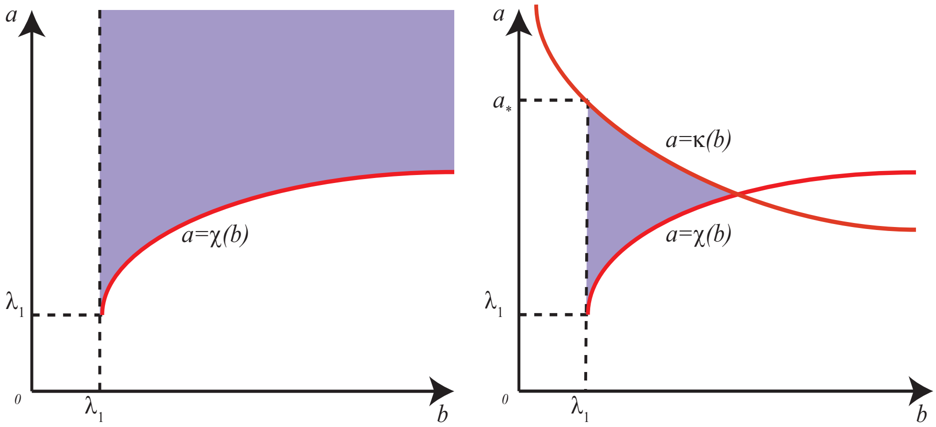

Fig. 4. The region of uniqueness of positive solutions of (P_c) in (a, b)-plane, the left is $pc \leqslant m$, the right is $pc > m$, where $\kappa(b) = 1/(q(pc - m)b) - 1/q$ and $a_* = \kappa(\lambda_1)$.

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