

Adaptive hybrid function projective synchronization of chaotic systems with fully unknown periodical time-varying parameters*

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Abstract. In this paper, an adaptive learning control approach is presented for the hybrid functional projective synchronization (HFPS) of different chaotic systems with fully unknown periodical time-varying parameters. Differential-difference hybrid parametric learning laws and an adaptive learning control law are constructed via the Lyapunov–Krasovskii functional stability theory, which make the states of two different chaotic systems asymptotically synchronized in the sense of mean square norm. Moreover, the boundedness of the parameter estimates are also obtained. The Lorenz system and Chen system are illustrated to show the effectiveness of the hybrid functional projective synchronization scheme.

Keywords: adaptive learning control, chaotic system, hybrid functional projective synchronization, differential-difference mixed parametric learning law, Lyapunov–Krasovskii function.

1 Introduction

Many chaotic systems have been proposed to be analyzed over the past decades [1]. A basic characteristic of a chaotic system is its extreme sensitivity to initial conditions, small differences in the initial conditions can cause unexpected different system states. Recently, chaos synchronization has received increasing attention, especially in secure communication, physics, chemical reactor, bio-engineering, medical science and artificial neural networks. Up to now, different types of synchronization phenomena have been presented such as complete synchronization [2], generalized synchronization [3], phase synchronization [4], projective synchronization [5], etc. Besides, many control schemes such as the OGY method [6], learning control method [7], feedback method [8], backstepping method [9] and adaptive control method [10] have been employed to synchronize identical or non-identical chaotic systems with different conditions. Among all kinds of chaos synchronization schemes, projective synchronization, first introduced by Mainieri

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and Rehack [11], characterized by a scaling factor by which two systems synchronize proportionally, has been extensively investigated. In [12], a hybrid projective synchronization (HPS), in which the different state variables can synchronize up to different scaling factors, was numerically observed in coupled partially linear chaotic complex nonlinear systems without adding any control term.

However, most of the projective synchronization are studied with constant scaling factor. Du et al. [13], presented a new synchronization scheme, called function projective synchronization. Compared to projective synchronization, function projective synchronization means that two systems can synchronize proportionally by a functional scaling factor. In [14], a modified function projective synchronization between hyperchaotic Lorenz system and hyperchaotic Lu system was investigated by using adaptive method. By Lyapunov stability theory, the adaptive control law and the parameter update law were derived to make the state of two hyperchaotic systems modified function projective synchronized. However, in [14], the parameters update law was related to the unknown parameters, which will lead to infeasibility in engineering applications. The characteristics, unpredictability of the functional scaling factor varying with time, is such that function projective synchronization can improve the security when applied in secure communication, so the research on hybrid function projective synchronization has become a new branch in chaos synchronization fields.

On the other hand, chaotic systems are unavoidably exposed to environments which may cause their parameters to vary within certain ranges, such as environment temperature, voltage fluctuation, mutual interference among components, and so on. But parameters of some systems in practical circumstances cannot be exactly known in advance, and may drift around their nominal values. The effect of these uncertainties will destroy the synchronization and even break it. Therefore, there is important theoretical significance and practical application to study synchronization in such systems with unknown parameters [15–17]. In these research results, the common method used to solve the parametric uncertainties is the adaptive control scheme in which the unknown system parameters are updated adaptively according to certain rules. For example, in [15] and [16] it was assumed that the parameters of the driving system were totally uncertain or unknown to the response system, and the parameters of the response system can be different from those of the driving system. Some studies supposed that the parameters of the driving and the response systems were identical but there were also some parametric uncertainties or perturbations [17]. In [18], based on the Lyapunov stability theory and adaptive bounding technique, a robust adaptive control law and the parameters update law were derived to make the states of two different chaotic systems asymptotically synchronized. In the control strategy, the parameters did not need to know thoroughly if the time-varying parameters are bounded by the product of a known function of t and an unknown constant. Another key issue in the adaptive synchronization technique for chaotic system is the problem of parameter identification [19, 20]. In [19], authors have given a great important condition on the consistency of the estimated parameters, that is functions with unknown parameters should be linearly independent on the synchronized orbit. In [20], an adaptive modified function projective synchronization (AMFPS) was proposed for uncertain hyperchaotic systems with identical or non-identical dimension

structures, a sufficient condition on the properties of AMFPS and the identification of the parameters were derived. However, in [19, 20], just the unknown constant parameters in the driving and response systems were concerned. The learning control methods [21, 22] have been applied to chaotic systems in the presence of time-varying uncertainties with the uniform periodicity and the pseudo-periodic problem. In [23], it was supposed that the parameters of the driving system are unknown time-varying periodic with a determined period, and the parameters of the response system are known, adaptive learning control scheme has been employed to realize generalized projective synchronization between two different chaotic systems. However, when there are unknown time-varying parameters in both the driving system and the response system, the problem of hybrid function projective synchronization is not solved, yet.

Motivated by the foregoing discussion, we will formulate the hybrid function projective synchronization problem of different chaotic systems both with unknown periodical time-varying parameters. According to Lyapunov–Krasovskii stability theory, differential-difference mixed-type parametric learning laws and an adaptive learning control law are constructed to make the states of two different chaotic systems asymptotically HFP synchronized, and also the boundedness of the parameter estimates are proven. At last, numerical simulation results are presented to verify the effectiveness of the proposed approach.

The rest of this paper is organized as follows. Section 2 gives the problem formulation. The learning control scheme is presented in Section 3. In Section 4, a simulation example is employed to conform the effectiveness of the proposed scheme. The conclusions are given in Section 5.

2 Problem description

Consider a class of chaotic systems with unknown time-varying parameters, described as follow:

$$\dot{x} = f(x) + F(x)\theta(t), \quad (1)$$

where $x \in R^n$ is the state vector, $f(x) : R^n \rightarrow R^n$ and $F(x) : R^n \rightarrow R^{n \times p}$ are continuous nonlinear vector function and nonlinear matrix function of system (1), respectively. $f(x)$ and $F(x)$ are local Lipschitz continuous functions with respect to x . $\theta(t) = (\Theta + \Phi(t)) \in R^p$ is the uncertain parameters vector. Here Θ is the nominal value of $\theta(t)$ and $\Phi(t)$ is the time-varying uncertainty or disturbance. Equation (1) is considered as the drive system.

The response system with a controller $u(x, y, t) \in R^n$ is introduced as follows:

$$\dot{y} = g(y) + G(y)\eta(t) + u(x, y, t), \quad (2)$$

where $y \in R^n$ is the state vector, $g(y) : R^n \rightarrow R^n$ and $G(y) : R^n \rightarrow R^{n \times p}$ are continuous nonlinear vector function and nonlinear matrix function of system (2), respectively. $g(y)$ and $G(y)$ are Lipschitz continuous functions with respect to y . $\eta(t) = (\Xi + \Psi(t)) \in R^p$ is the uncertain parameters vector. Here Ξ is the nominal value of $\eta(t)$ and $\Psi(t)$ is the time-varying uncertainty or disturbance, $u(x, y, t) \in R^n$ is the control vector.

Remark 1. The systems (1) and (2) studied in this paper depend linearly on the unknown parameters. The class of nonlinear dynamical systems include an extensive variety of chaotic systems such as Lorenz system, Rösler system, Duffing system, Chua's circuit, generalized Lorenz system, etc.

Remark 2. It should be emphasized that in [23], generalized projective synchronization of chaotic systems has been obtained via adaptive learning control approach, where the response system has no parameter uncertainty, and the problem of hybrid function projective synchronization has not concerned in [23], and the property of parameter identification has also not considered. Up to now, there is no effective approach for synchronization of systems (1) and (2) with fully unknown time-varying parameters. In this paper, we shall present an adaptive hybrid function projective synchronization scheme for systems (1) and (2), the characteristics of synchronization and the boundedness of estimated parameters will be obtained.

Next, we assume that $\theta(t) \in R^p$, $\eta(t) \in R^p$ for all $t \in R^+$ are the unknown continuous periodic time-varying function vector with a known period T , that is to say, $\theta(t) = \theta(t - T)$, $\eta(t) = \eta(t - T)$.

Remark 3. Since $\theta(t) = \Theta + \Phi(t)$, $\eta(t) = \Xi + \Psi(t)$, obviously $\Phi(t)$ and $\Psi(t)$ are unknown continuous periodic time-varying functions vector with a known period T .

We define a synchronizing error as

$$e(t) = x - H(t)y, \quad (3)$$

where $H(t) = \text{diag}\{h_1(t), h_2(t), \dots, h_n(t)\}$ is a scaling function matrix, $h_i(t)$ is a continuous differentiable bounded function and $h_i(t) \neq 0$ for all $t \in R$.

Therefore, the goal of control is to design and implement an appropriate controller $u(x, y, t)$ such that the response system (2) could be asymptotically synchronized with the drive system (1) in the L_T^2 norm sense as follows:

$$\lim_{t \rightarrow \infty} \sqrt{\int_{t-T}^t \|e(\tau)\|^2 d\tau} = 0. \quad (4)$$

Remark 4. As well known, if equality (4) is achieved with $H(t) = \alpha(t)I$, $\alpha(t)$ is a scalar continuous differentiable bounded function of t , then a function projective synchronization occurs between system (1) and (2).

3 Design of the learning controller

The dynamic equation of synchronization error (3) can be easily obtained from Eqs. (1) and (2), which is described as follows:

$$\begin{aligned} \dot{e} &= \dot{x} - \dot{H}(t)y - H(t)\dot{y} \\ &= f(x) + F(x)\Theta + F(x)\Phi(t) - \dot{H}(t)y - H(t)g(y) - H(t)G(y)\Xi \\ &\quad - H(t)G(y)\Psi(t) - H(t)u(x, y, t). \end{aligned} \quad (5)$$

According to Eq. (5), we design the controller in the following form:

$$u(x, y, t) = H^{-1}(t) [Ke + f(x) + F(x)\hat{\Theta}(t) + F(x)\hat{\Phi}(t) - \dot{H}(t)y - H(t)g(y) - H(t)G(y)\hat{\Xi}(t) - H(t)G(y)\hat{\Psi}(t)], \quad (6)$$

where $K \in R^{n \times n}$ is a designed positive-definite matrix, $\hat{\Theta}(t)$, $\hat{\Phi}(t)$, $\hat{\Xi}(t)$ and $\hat{\Psi}(t)$ are estimations to Θ , $\Phi(t)$, Ξ and $\Psi(t)$, respectively.

For the constant parameter vector Θ and Ξ , the differential type adaptive learning law are designed as

$$\dot{\hat{\Theta}}(t) = \Gamma_1 F^T(x)e, \quad (7)$$

$$\dot{\hat{\Xi}}(t) = -\Gamma_4 G^T(y)H(t)e, \quad (8)$$

where $\Gamma_1 \in R^{p \times p}$, $\Gamma_4 \in R^{p \times p}$ are positive diagonal learning gain matrices.

For the period time-varying parameter vectors $\Phi(t)$ and $\Psi(t)$, the difference type adaptive learning law are employed

$$\hat{\Phi}(t) = \begin{cases} \hat{\Phi}(t-T) + \Gamma_2 F^T(x)e, & t \in [T, \infty), \\ \Gamma_0(t)F^T(x)e, & t \in [0, T), \\ 0, & t \in [-T, 0), \end{cases} \quad (9)$$

and

$$\hat{\Psi}(t) = \begin{cases} \hat{\Psi}(t-T) - \Gamma_5 G^T(y)H(t)e, & t \in [T, \infty), \\ -\Gamma_3(t)G^T(y)H(t)e, & t \in [0, T), \\ 0, & t \in [-T, 0), \end{cases} \quad (10)$$

where $\Gamma_2, \Gamma_5 \in R^{p \times p}$ are positive diagonal learning gain matrices, and $\Gamma_0(t), \Gamma_3(t) \in R^{p \times p}$ are continuous, positive, diagonal gain matrices for the first period $[0, T)$ satisfying $\Gamma_0(0) = \Gamma_3(0) = 0$, $\Gamma_0(T) = \Gamma_2$, $\Gamma_3(T) = \Gamma_5$, and each element of $\Gamma_0(t)$ and $\Gamma_3(t)$ is chosen to be strictly increasing. The purpose of choosing $\Gamma_0(t)$ and $\Gamma_3(t)$ is to ensure the continuity of time-varying parameter updating $\hat{\Phi}(t)$ and $\hat{\Psi}(t)$ for $t \in [0, \infty)$, then, it is ensured that the boundedness of Lyapunov–Krasovskii functional in the first period $[0, T)$. To verify this, it only needs to prove the continuity of $\hat{\Phi}(t)$ and $\hat{\Psi}(t)$ at the instants $t = iT$, $i = 1, 2, \dots$. Continuous property of $\hat{\Phi}(t)$ and $\hat{\Psi}(t)$ for the first period $[0, T)$ is obvious. Hence, we only need to focus on $t \in [iT, (i+1)T]$, $i = 1, 2, \dots$.

Lemma 1. $\hat{\Phi}(t)$ and $\hat{\Psi}(t) \forall t \in [0, \infty)$ are of continuity.

Proof. Similar to [23], we can easily prove the continuous property of $\hat{\Phi}(t)$ and $\hat{\Psi}(t)$ for all $t \in [0, \infty)$.

Substituting Eq. (6) into (5), thereby, we can get

$$\dot{e} = -Ke + F(x)\tilde{\Theta}(t) + F(x)\tilde{\Phi}(t) - H(t)G(y)\tilde{\Xi}(t) - H(t)G(y)\tilde{\Psi}(t), \quad (11)$$

where $\tilde{\Theta}(t) = \Theta - \hat{\Theta}(t)$, $\tilde{\Phi}(t) = \Phi(t) - \hat{\Phi}(t)$, $\tilde{\Xi}(t) = \Xi - \hat{\Xi}(t)$, $\tilde{\Psi}(t) = \Psi(t) - \hat{\Psi}(t)$. \square

4 Convergence analysis

The convergence property of the proposed adaptive learning control method is summarized in the following theorem.

Theorem 1. *For a given scaling function matrix $H(t)$ and any initial conditions $x(0)$ and $y(0)$, there are the adaptive learning control law (6) and parameter learning laws (7)–(10) that ensure the asymptotical convergence of the synchronizing error $e(t)$ in the L_T^2 norm sense, that is to say, $\lim_{t \rightarrow \infty} \sqrt{\int_{t-T}^t \|e(\tau)\|^2 d\tau} = 0$. Meanwhile, the constant parameter estimates $\hat{\Theta}(t)$, $\hat{\Xi}(t)$ are bounded and the time-varying parameter estimates errors are bounded in the sense of L_T^2 norm, that is*

$$\int_{t-T}^t \tilde{\Phi}^T(\tau) \tilde{\Phi}(\tau) d\tau \leq m_1, \quad \int_{t-T}^t \tilde{\Psi}^T(\tau) \tilde{\Psi}(\tau) d\tau \leq m_2,$$

where m_1, m_2 are some positive constants.

Proof. To facilitate the convergence analysis, we denote $V(t, e(t), \tilde{\Theta}(t), \tilde{\Phi}(t), \tilde{\Xi}(t), \tilde{\Psi}(t)) = V(t)$.

Choose a Lyapunov–Krasovskii functional as follows:

$$V(t) = \begin{cases} \frac{1}{2} e^T e + \frac{1}{2} \tilde{\Theta}^T(t) \Gamma_1^{-1} \tilde{\Theta}(t) + \frac{1}{2} \tilde{\Xi}^T(t) \Gamma_4^{-1} \tilde{\Xi}(t) \\ + \frac{1}{2} \int_0^t \tilde{\Phi}^T(\tau) \Gamma_2^{-1} \tilde{\Phi}(\tau) d\tau + \frac{1}{2} \int_0^t \tilde{\Psi}^T(\tau) \Gamma_5^{-1} \tilde{\Psi}(\tau) d\tau, & t \in [0, T), \\ \frac{1}{2} e^T e + \frac{1}{2} \tilde{\Theta}^T(t) \Gamma_1^{-1} \tilde{\Theta}(t) + \frac{1}{2} \tilde{\Xi}^T(t) \Gamma_4^{-1} \tilde{\Xi}(t) \\ + \frac{1}{2} \int_{t-T}^t \tilde{\Phi}^T(\tau) \Gamma_2^{-1} \tilde{\Phi}(\tau) d\tau + \frac{1}{2} \int_{t-T}^t \tilde{\Psi}^T(\tau) \Gamma_5^{-1} \tilde{\Psi}(\tau) d\tau, & t \in [T, \infty). \end{cases} \quad (12)$$

Firstly, we prove that the finiteness of $V(t)$ for the first period $[0, T)$. According to the system dynamics (1) and (2) and the proposed control laws (6)–(10), it can be seen that the right-hand side of Eq. (5) is continuous with respect to all arguments. In light of the existence theorem of differential equation, synchronizing error equation (5) has a solution in an interval $[0, T_1) \subset [0, T)$, with $0 < T_1 \leq T$. Therefore, the boundedness of $V(t)$ over $[0, T_1)$ can be guaranteed and we only need to focus on the boundedness of $V(t)$ on the interval $[T_1, T)$.

For any $t \in [T_1, T)$, the time derivative of $V(t)$ for $t \in [T_1, T)$, is given by

$$\begin{aligned} \dot{V}(t) &= e^T \dot{e} + \tilde{\Theta}^T(t) \Gamma_1^{-1} \dot{\tilde{\Theta}}(t) + \tilde{\Xi}^T(t) \Gamma_4^{-1} \dot{\tilde{\Xi}}(t) \\ &+ \frac{1}{2} \tilde{\Phi}^T(t) \Gamma_2^{-1} \dot{\tilde{\Phi}}(t) + \frac{1}{2} \tilde{\Psi}^T(t) \Gamma_5^{-1} \dot{\tilde{\Psi}}(t), \quad t \in [T_1, T). \end{aligned} \quad (13)$$

Taking Eq. (11) into the first term on the right-hand side of $\dot{V}(t)$ in Eq. (13), we obtain

$$\begin{aligned} e^T \dot{e} &= -e^T K e + e^T F(x) \tilde{\Theta}(t) + e^T F(x) \tilde{\Phi}(t) \\ &- e^T H(t) G(y) \tilde{\Xi}(t) - e^T H(t) G(y) \tilde{\Psi}(t). \end{aligned} \quad (14)$$

Using the constant parameter updating laws (7) and (8), we have

$$\tilde{\Theta}^T(t)\Gamma_1^{-1}\dot{\tilde{\Theta}}(t) = -\tilde{\Theta}^T(t)\Gamma_1^{-1}\dot{\tilde{\Theta}}(t) = -\tilde{\Theta}^T(t)F^T(x)e, \quad (15)$$

$$\tilde{\Xi}^T(t)\Gamma_4^{-1}\dot{\tilde{\Xi}}(t) = -\tilde{\Xi}^T(t)\Gamma_4^{-1}\dot{\tilde{\Xi}}(t) = \tilde{\Xi}^T(t)G^T(y)H(t)e. \quad (16)$$

Then, let us focus on the fourth and fifth terms on the right-hand side of (13). Since $\Gamma_0(t)$ and $\Gamma_3(t)$ are positive-definite diagonal matrices, and each diagonal element is strictly increasing in $[0, T)$, therefore, $\Gamma_2^{-1} \leq \Gamma_0^{-1}(t) < \infty$, $\Gamma_5^{-1} \leq \Gamma_3^{-1}(t) < \infty$ are ensured on the interval $[T_1, T)$, then,

$$\begin{aligned} & \frac{1}{2}\tilde{\Phi}^T(t)\Gamma_2^{-1}\tilde{\Phi}(t) \\ & \leq \frac{1}{2}\tilde{\Phi}^T(t)\Gamma_0^{-1}(t)\tilde{\Phi}(t) \\ & = \frac{1}{2}[\tilde{\Phi}^T(t)\Gamma_0^{-1}(t)(\tilde{\Phi}(t) - \hat{\Phi}(t)) - 2\hat{\Phi}^T(t)\Gamma_0^{-1}(t)\tilde{\Phi}(t) + \hat{\Phi}^T(t)\Gamma_0^{-1}(t)\tilde{\Phi}(t)] \\ & \leq \frac{1}{2}[\tilde{\Phi}^T(t)\Gamma_0^{-1}(t)\tilde{\Phi}(t) - 2e^T F(x)\tilde{\Phi}(t)], \end{aligned} \quad (17)$$

$$\begin{aligned} & \frac{1}{2}\tilde{\Psi}^T(t)\Gamma_5^{-1}\tilde{\Psi}(t) \\ & \leq \frac{1}{2}\tilde{\Psi}^T(t)\Gamma_3^{-1}(t)\tilde{\Psi}(t) \\ & = \frac{1}{2}[\tilde{\Psi}^T(t)\Gamma_3^{-1}(t)(\tilde{\Psi}(t) - \hat{\Psi}(t)) - 2\hat{\Psi}^T(t)\Gamma_3^{-1}(t)\tilde{\Psi}(t) + \hat{\Psi}^T(t)\Gamma_3^{-1}(t)\tilde{\Psi}(t)] \\ & \leq \frac{1}{2}[\tilde{\Psi}^T(t)\Gamma_3^{-1}(t)\tilde{\Psi}(t) + 2e^T H(t)G(y)\tilde{\Psi}(t)]. \end{aligned} \quad (18)$$

Substituting Eqs. (14)–(18) into Eq. (13), yields

$$\begin{aligned} \dot{V}(t) & \leq -e^T K e + \frac{1}{2}\tilde{\Phi}^T(t)\Gamma_0^{-1}(t)\tilde{\Phi}(t) + \frac{1}{2}\tilde{\Psi}^T(t)\Gamma_3^{-1}(t)\tilde{\Psi}(t) \\ & \leq \frac{1}{2}\tilde{\Phi}^T(t)\Gamma_0^{-1}(t)\tilde{\Phi}(t) + \frac{1}{2}\tilde{\Psi}^T(t)\Gamma_3^{-1}(t)\tilde{\Psi}(t). \end{aligned} \quad (19)$$

Note that $\tilde{\Phi}(t)$ and $\tilde{\Psi}(t)$ are continuous periodic functions, thus they are bounded. The boundedness of $\tilde{\Phi}(t)$ and $\tilde{\Psi}(t)$ lead to the boundedness of $\dot{V}(t)$. As $V(T_1)$ is bounded, the finiteness of $V(t) \forall t \in [T_1, T)$ is easily obtained.

Onwards, we will prove the asymptotical convergence of $e(t)$ in L_T^2 norm sense.

Firstly, let us compute the difference of $V(t)$ over one period for $t \in [T, \infty)$, that is

$$\Delta V(t) = V(t) - V(t - T) = \int_{t-T}^t \dot{V}(\tau) d\tau$$

$$\begin{aligned}
&= \int_{t-T}^t \left\{ e^T \dot{e} + \tilde{\Theta}^T(\tau) \Gamma_1^{-1} \dot{\tilde{\Theta}}(\tau) + \tilde{\Xi}^T(\tau) \Gamma_4^{-1} \dot{\tilde{\Xi}}(\tau) \right. \\
&\quad + \frac{1}{2} [\tilde{\Phi}^T(\tau) \Gamma_2^{-1} \tilde{\Phi}(\tau) - \tilde{\Phi}^T(\tau - T) \Gamma_2^{-1} \tilde{\Phi}(\tau - T)] \\
&\quad \left. + \frac{1}{2} [\tilde{\Psi}^T(\tau) \Gamma_5^{-1} \tilde{\Psi}(\tau) - \tilde{\Psi}^T(\tau - T) \Gamma_5^{-1} \tilde{\Psi}(\tau - T)] \right\} d\tau. \quad (20)
\end{aligned}$$

Using the difference type adaptive learning laws (9) and (10), the periodic property $\Phi(t) = \Phi(t - T)$, $\Psi(t) = \Psi(t - T)$ and the relationship

$$(A - B)^T \Gamma (A - B) - (A - C)^T \Gamma (A - C) = (C - B)^T \Gamma [2(A - B) + (B - C)],$$

where $A, B, C \in R^p$, $\Gamma \in R^{p \times p}$, the following equations can be obtained:

$$\begin{aligned}
&\frac{1}{2} [\tilde{\Phi}^T(t) \Gamma_2^{-1} \tilde{\Phi}(t) - \tilde{\Phi}^T(t - T) \Gamma_2^{-1} \tilde{\Phi}(t - T)] \\
&= \frac{1}{2} (\hat{\Phi}(t - T) - \hat{\Phi}(t))^T \Gamma_2^{-1} [2(\Phi(t) - \hat{\Phi}(t)) + (\hat{\Phi}(t) - \hat{\Phi}(t - T))] \\
&= -e^T F(x) \tilde{\Phi}(t) - \frac{1}{2} e^T F(x) \Gamma_2 F^T(x) e \quad (21)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2} [\tilde{\Psi}^T(t) \Gamma_5^{-1} \tilde{\Psi}(t) - \tilde{\Psi}^T(t - T) \Gamma_5^{-1} \tilde{\Psi}(t - T)] \\
&= \frac{1}{2} [\hat{\Psi}(t - T) - \hat{\Psi}(t)]^T \Gamma_5^{-1} [2(\Psi(t) - \hat{\Psi}(t)) + (\hat{\Psi}(t) - \hat{\Psi}(t - T))] \\
&= e^T H(t) G(y) \tilde{\Psi}(t) - \frac{1}{2} e^T H(t) G(y) \Gamma_5 G^T(y) H(t) e. \quad (22)
\end{aligned}$$

Substituting Eqs. (14)–(16), (21) and (22) into Eq. (20), we can obtain

$$\begin{aligned}
\Delta V(t) &= \int_{t-T}^t \left[-e^T K e - \frac{1}{2} e^T F(x) \Gamma_2 F^T(x) e - \frac{1}{2} e^T H G(y) \Gamma_5 G^T(y) H e \right] d\tau \\
&\leq \int_{t-T}^t -e^T K e d\tau \leq -\lambda_{\min} K \int_{t-T}^t e^T e d\tau, \quad (23)
\end{aligned}$$

where $\lambda_{\min}(K)$ is the minimum eigenvalue of the matrix K .

For any $t \in [iT, (i+1)T)$, denoting $t_0(t) = t - iT$, $i = 1, 2, \dots$, it is easily obtained that $t_0(t) \in [0, T)$. Applying Eq. (20) repeatedly, we have

$$V(t) = V(t_0(t)) + \sum_{j=0}^{i-1} \Delta V(t - jT). \quad (24)$$

Since $t_0(t) \in [0, T)$, according to Eqs. (23) and (24), we have

$$V(t) \leq V(t_0(t)) - \lambda_{\min}(K) \sum_{j=0}^{i-1} \int_{t-(j+1)T}^{t-jT} e^T e \, d\tau. \quad (25)$$

Noting the positiveness of $V(t)$, and bounded $V(t_0(t))$ for all $t_0(t) \in [0, T)$, by the convergence theorem of the sum of positive terms series, it can be obtained that the sum of series

$$\sum_{j=0}^{\infty} \int_{t-(j+1)T}^{t-jT} e^T e \, d\tau = m_0$$

is bounded, and also we can get the synchronizing error $e(t)$ converges to zero asymptotically in L_T^2 norm sense, that is to say, we have

$$\lim_{t \rightarrow \infty} \sqrt{\int_{t-T}^t \|e(\tau)\|^2 \, d\tau} = 0.$$

Moreover, from (25), the following inequality can be gotten

$$V(t) \leq V(t_0(t)) \leq m_3 \quad \forall t \in [0, \infty),$$

where m_3 is a positive constant.

Noting the definition of $V(t)$ in (12), one can get that $\hat{\Theta}(t)$, $\hat{\Xi}(t)$ are bounded and $\tilde{\Phi}$, $\tilde{\Psi}$ are bounded in the sense of L_T^2 norm sense. This completes the proof. \square

5 Numerical simulations

In this section, we employ an example to illustrate the effectiveness of the proposed novel adaptive synchronization method.

Consider the problem of the hybrid function projective synchronization between Lorenz system

$$\begin{aligned} \dot{x}_1 &= \theta_1(t)(x_2 - x_1), \\ \dot{x}_2 &= -x_1 x_3 - x_2 + \theta_2(t)x_1, \\ \dot{x}_3 &= x_1 x_2 - \theta_3(t)x_3 \end{aligned} \quad (26)$$

and Chen system

$$\begin{aligned} \dot{y}_1 &= \eta_1(t)(y_2 - y_1) + u_1, \\ \dot{y}_2 &= (\eta_2(t) - \eta_1(t))y_1 - y_1 y_3 + \eta_2(t)y_2 + u_2, \\ \dot{y}_3 &= y_1 y_2 - \eta_3(t)y_3 + u_3, \end{aligned} \quad (27)$$

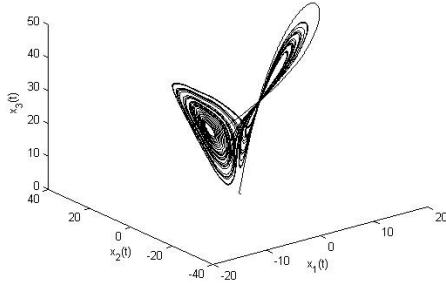


Fig. 1. Chaotic behavior of Lorenz system with $\theta(t)$.

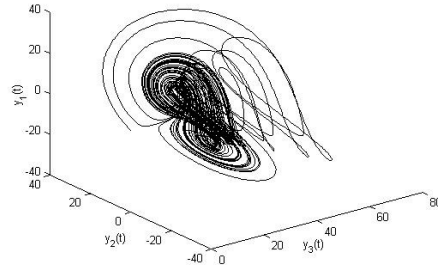


Fig. 2. Chaotic behavior of Chen system with $\eta(t)$.

where the unknown time-varying parameters $\theta(t) = [10 + 0.2 \sin t, 28 + 0.2 \cos t, 8/3 - 0.2 \sin t]^T$ and $\eta(t) = [35 + 0.1 \sin t, 28 + 0.1 \cos t, 3 - 0.1 \sin t]^T$ are continuous periodic function vectors with known periods $T = 2\pi$.

Comparing systems (26) and (27) with Eqs. (1) and (2), we obtain

$$f(x) = \begin{pmatrix} 0 \\ -x_2 - x_1 x_3 \\ x_1 x_2 \end{pmatrix}, \quad F(x) = \begin{bmatrix} x_2 - x_1 & 0 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & -x_3 \end{bmatrix},$$

$$g(y) = \begin{pmatrix} 0 \\ -y_1 y_3 \\ y_1 y_2 \end{pmatrix}, \quad G(y) = \begin{bmatrix} y_2 - y_1 & 0 & 0 \\ -y_1 & y_1 + y_2 & 0 \\ 0 & 0 & -y_3 \end{bmatrix}.$$

The chaotic behavior of the Lorenz system with $\theta(t) = [10 + 0.2 \sin t, 28 + 0.2 \cos t, 8/3 - 0.2 \sin t]^T$ is shown in Fig. 1. The chaotic behavior of the Chen system with $\eta(t) = [35 + 0.1 \sin t, 28 + 0.1 \cos t, 3 - 0.1 \sin t]^T$ is shown in Fig. 2. The nominal values of the parameter vectors $\theta(t)$ and $\eta(t)$ are $\Theta = [10, 28, 8/3]$ and $\Xi = [35, 28, 3]$, respectively. During the simulation, the initially estimated values of the unknown parameters Θ and Ξ are chosen as $\hat{\Theta}(0) = [-5, -3, 0]^T$ and $\hat{\Xi}(0) = [-2, 0, 3]^T$, respectively. The initial states of the drive system and response system are chosen as $x(0) = [0, 0.1, 1]^T$, $y(0) = [0.1, 0.1, 0.2]^T$, respectively. Taking the scaling function matrix to be

$$H(t) = \text{diag}\{4 + 0.6 \sin(2\pi t/99), 5 + \cos(2\pi t/99), 5 + \sin(2\pi t/99)\},$$

and we choose gain matrices as

$$K = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix}, \quad \Gamma_0(t) = \frac{t}{T} \Gamma_2,$$

$$\Gamma_1 = \begin{pmatrix} 25 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 0.11 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0.55 & 0 & 0 \\ 0 & 1.1 & 0 \\ 0 & 0 & 0.09 \end{pmatrix}, \quad \Gamma_3(t) = \frac{t}{T} \Gamma_5,$$

$$\Gamma_4 = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.001 \end{pmatrix}, \quad \Gamma_5 = \begin{pmatrix} 0.05 & 0 & 0 \\ 0 & 0.001 & 0 \\ 0 & 0 & 0.00002 \end{pmatrix}.$$

According to Theorem 1 and the designed controller (6) and parameter updating laws given by (7)–(10), it is concluded that hybrid function projective synchronization can be achieved between Lorenz system and Chen system. This is verified by the simulation results shown in Fig. 3. Furthermore, Fig. 4 depicts the time evolution of the controller, and Fig. 5 and Fig. 7 show the boundedness of the estimates of unknown time-varying parameters $\Phi(t)$ and $\Psi(t)$, respectively, and Fig. 6 and Fig. 8 display the boundedness of the estimates of unknown constant parameters Θ and Ξ , respectively. Moreover, the Fig. 9 and Fig. 10 draw the errors of estimates and the real values for constant parameters Θ and Ξ , respectively. These show that the parameter identification characteristic can be obtained for the unknown constant parameters, however, the similar property of the unknown time-varying parameters cannot be guaranteed yet. The problem will be studied in the near future.

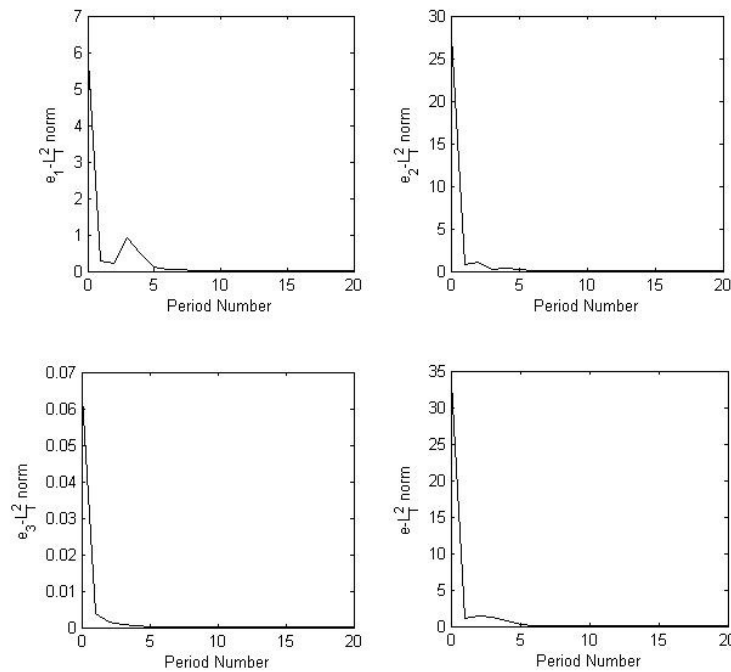


Fig. 3. The evolution of the tracking errors in L_T^2 norm.

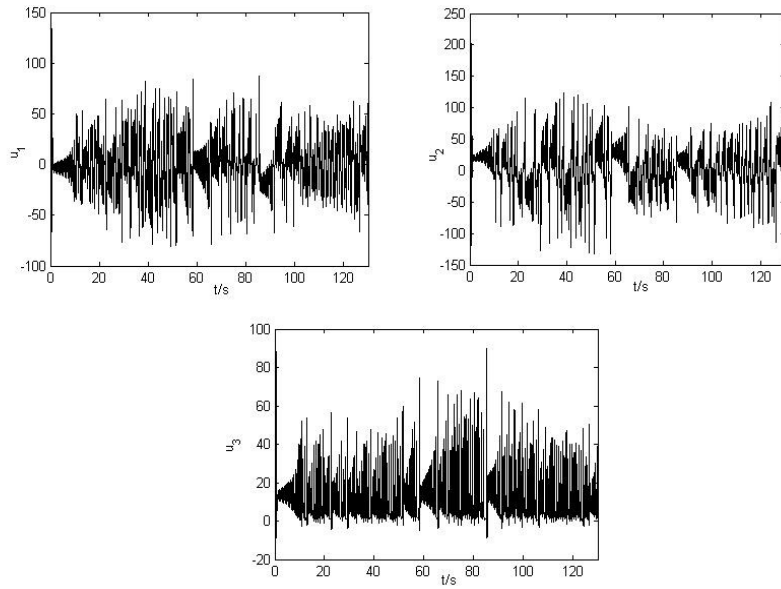
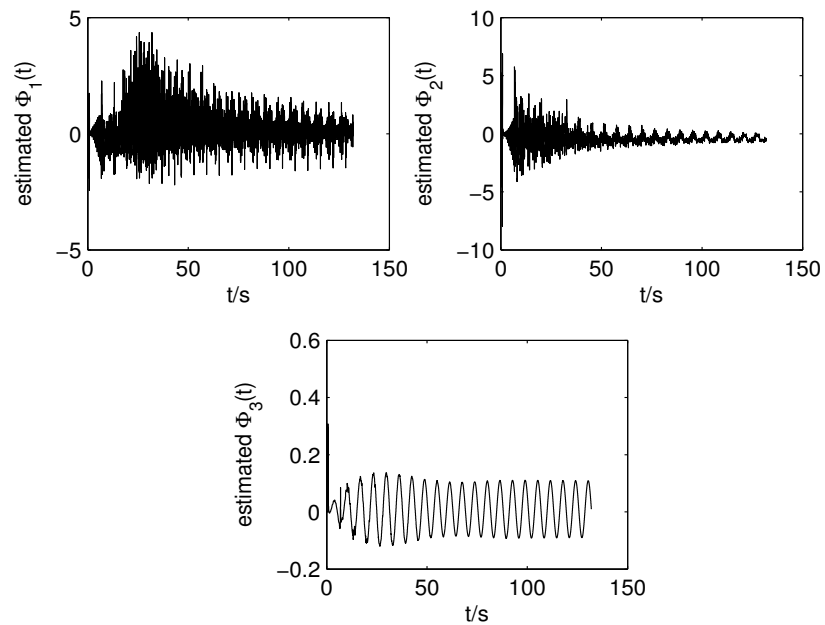


Fig. 4. Time evolution of the controller.

Fig. 5. Time evolution of the estimated time-varying parameter $\Phi(t)$.

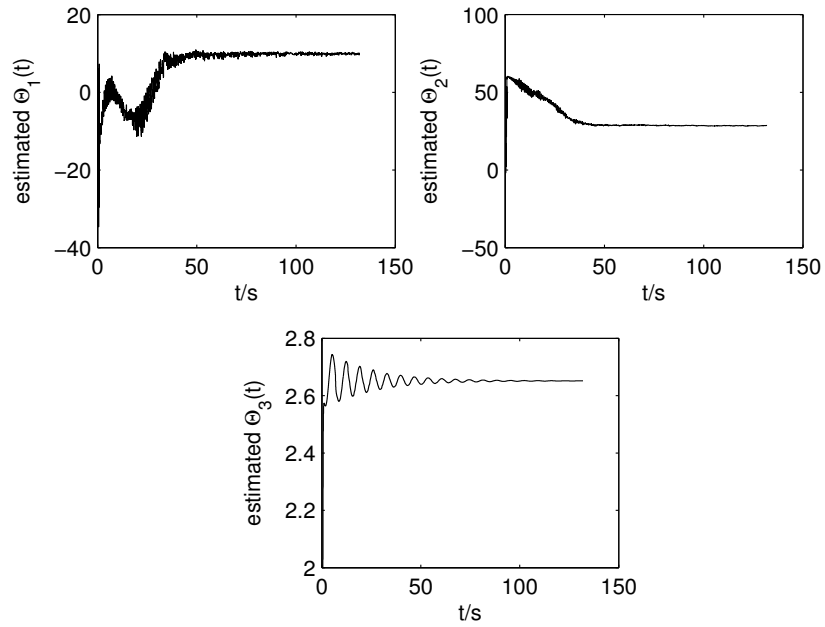


Fig. 6. Time evolution of the estimated constant parameter θ .

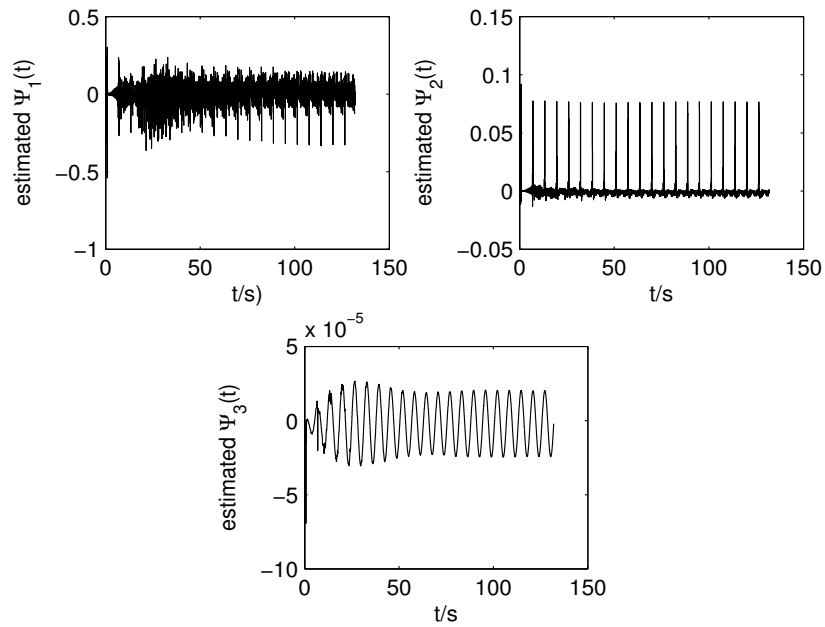


Fig. 7. Time evolution of the estimated time-varying parameter $\Psi(t)$.

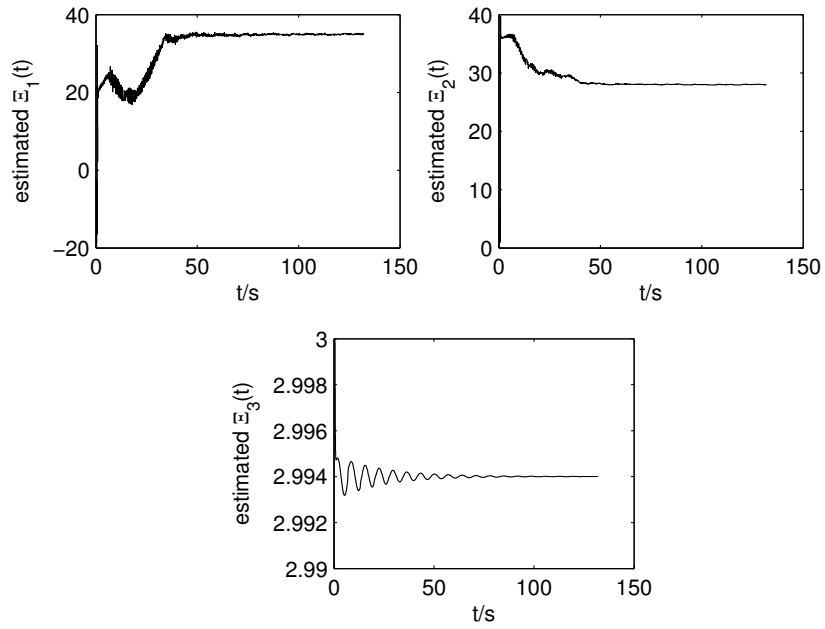


Fig. 8. Time evolution of the estimated constant parameter Ξ .

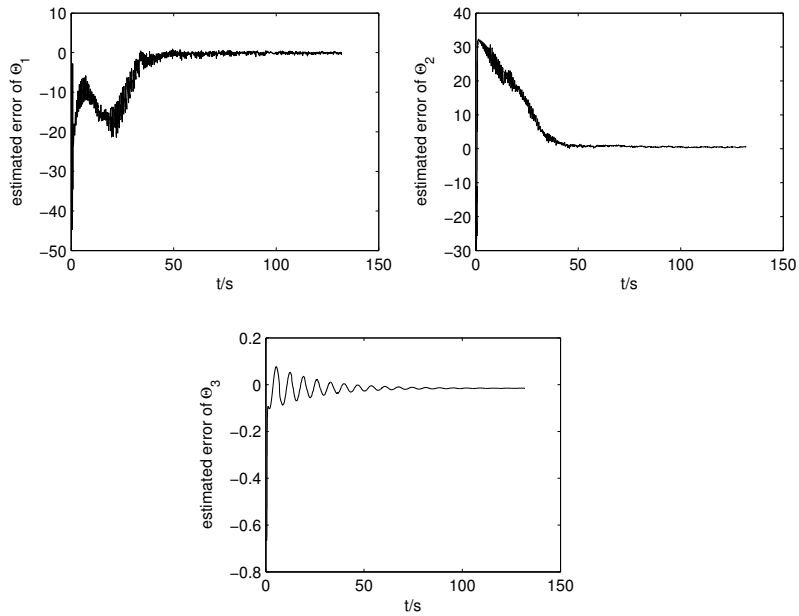


Fig. 9. Errors of estimates and the real values of parameters Θ .

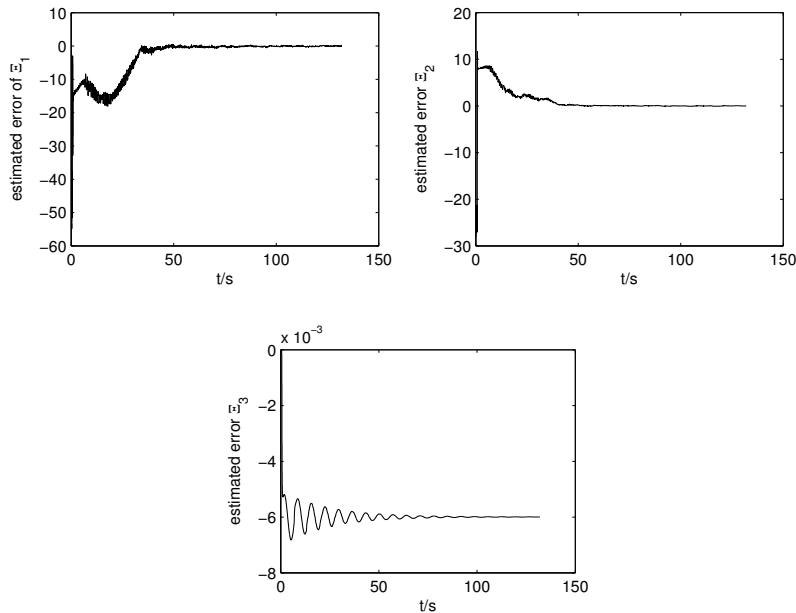


Fig. 10. Errors of estimates and the real values of constant parameters Ξ .

6 Conclusions

In this paper, we have proposed an adaptive learning control method for HFPS of two different chaotic systems both with unknown periodic time-varying parameters. Based on the Lyapunov–Krasovskii stability theory, the adaptive learning controller and the differential-difference mixed parameter learning laws are constructed for global stability of the closed loop system. The asymptotic synchronization of the error dynamics between the driving and the response systems and the boundedness of the all parameter estimates are also obtained. The proposed approach has been successfully applied to HFPS between the Lorenz system and Chen system. The feasibility and effectiveness of the proposed approach are confirmed through theoretical analysis and numerical simulations. The consistency of estimated parameters will be further studied in the future.

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