Existence of non-negative solutions for semilinear elliptic systems via variational methods

Somayeh Khademloo[a], **Shapur Heidarkhani**[b]

[a]Department of Basic Sciences, Babol University of Technology
47148-71167, Babol, Iran
s.khademloo@nit.ac.ir

[b]Department of Mathematics, Faculty of Sciences, Razi University
Kermanshah 67149, Iran
s.heidarkhani@razi.ac.ir

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Abstract. In this paper we consider a semilinear elliptic system with nonlinearities, indefinite weight functions and critical growth terms in bounded domains. The existence result of nontrivial nonnegative solutions is obtained by variational methods.

Keywords: variational methods, Robin boundary condition, first eigenvalue.

1 Introduction

In this paper we consider the existence results of the following two coupled semilinear equation

$$\begin{aligned} -\Delta u &= \lambda a u - \big(pav^2u|u|^{p-2} + 2au|v|^p\big), \quad x \in \Omega,\\ -\Delta v &= \lambda a v - \big(pau^2v|v|^{p-2} + 2av|u|^p\big), \quad x \in \Omega,\\ (1-\alpha)\frac{\partial u}{\partial n} + \alpha u &= (1-\alpha)\frac{\partial v}{\partial n} + \alpha v = 0, \quad x \in \partial\Omega, \end{aligned} \tag{1}$$

where α and λ are real parameters, $p < 2^* - 2$, for $2^* = \frac{2N}{N-2}$, Ω is an open bounded domain in $\mathbb{R}^N$, $N \geq 3$ with a smooth boundary $\partial\Omega$, and $a : \Omega \to \mathbb{R}$ is a sign changing weight function.

This work is motivated by the results in the literature for the single equation case, namely the equation of the form

$$-\Delta u = \lambda g(x)\big(1 + |u|^p\big)u, \quad x \in \Omega, \tag{2}$$

$$(1-\alpha)\frac{\partial u}{\partial n} + \alpha u = 0, \qquad x \in \partial\Omega, \tag{3}$$

with the sign changing weight function g. See [1–4] and references therein for the case where $\alpha \neq 0$, and [5] for the case where $\alpha = 1$. Recently in [6] some existence results were established for the case when a single equation is replace by a system of equations. Also we refer to [7] where Δ is replaced by Δ_p. In this work we extend this studies to classes of Robin boundary conditions. We prove our existence results via variational methods.

The system (1) is posed in the framework of the Sobolev space $H = H^1(\Omega)\times H^1(\Omega)$ with the norm

$$\|(u,v)\|_H = \biggl(\int\limits_\Omega \bigl(|\nabla u|^2 + u^2\bigr) + \int\limits_\Omega \bigl(|\nabla v|^2 + v^2\bigr)\biggr)^{\frac{1}{2}}.$$

Moreover a pair of functions $(u,v)\in H$ is said to be a weak solution of the system (1) if

$$\begin{aligned}&\int\limits_\Omega \nabla u\nabla\phi_1 + \int\limits_\Omega \nabla v\nabla\phi_2 - \lambda\int\limits_\Omega a(u\phi_1 + v\phi_2) + \frac{\alpha}{1-\alpha}\int\limits_{\partial\Omega}(u\phi_1 + v\phi_2)\\ &\quad + p\int\limits_\Omega \bigl(av^2|u|^{p-2}u\phi_1 + au^2|v|^{p-2}v\phi_2\bigr) + 2\int\limits_\Omega \bigl(au|v|^p\phi_1 + av|u|^p\phi_2\bigr) = 0\end{aligned}$$

for all $(\phi_1,\phi_2)\in H$. Thus the corresponding energy functional to the system (1) is defined by

$$\begin{aligned}J_\lambda(u,v) &= \frac{1}{2}\biggl(\int\limits_\Omega \bigl(|\nabla u|^2 - \lambda a u^2\bigr) + \frac{\alpha}{1-\alpha}\int\limits_{\partial\Omega} u^2\biggr)\\ &\quad + \frac{1}{2}\biggl(\int\limits_\Omega \bigl(|\nabla v|^2 - \lambda a v^2\bigr) + \frac{\alpha}{1-\alpha}\int\limits_{\partial\Omega} v^2\biggr) + \int\limits_\Omega a\bigl(v^2|u|^p + u^2|v|^p\bigr)\\ &= \frac{1}{2}\bigl(L(u) + L(v)\bigr) + G_1(u,v) + G_2(u,v),\end{aligned}$$

where

$$L(t) = \int\limits_\Omega \bigl(|\nabla t|^2 - \lambda a t^2\bigr) + \frac{\alpha}{1-\alpha}\int\limits_{\partial\Omega} t^2$$

for $t = u$ or v, and

$$G_1(u,v) = \int\limits_\Omega a v^2|u|^p$$

and

$$G_2(u,v) = \int\limits_\Omega a u^2|v|^p.$$

It is well known that the weak solutions of the system (1) are the critical points of the Euler functional J_λ.

Let I be the Euler functional associated with an elliptic problem on a Banach space X. If I is bounded below and has a minimizer on X, thus this minimizer is a critical point of I, and so is a solution of the corresponding elliptic problem. However, the Euler functional J_λ is not bounded below on the whole space H, but is bounded on an appropriate subset, and a minimizer on this set (if it exists) gives a solution to the system (1).

Then we introduce the following notation:

For any functional $f : H \to \mathbb{R}$ we denote by $f'(u,v)(h_1,h_2)$ the Gâteaux derivative of f at (u,v) in the direction of $(h_1,h_2) \in H$, and

$$f^{(1)}(u,v)h_1 = f'(u+\varepsilon h_1, v)|_{\varepsilon=0},$$
$$f^{(2)}(u,v)h_2 = f'(u, v+\delta h_2)|_{\delta=0}.$$

In fact we have

$$f'(u,v)(h_1,h_2) = f^{(1)}(u,v)h_1 + f^{(2)}(u,v)h_2.$$

2 Notations and preliminaries

First we consider the eigenvalue problem

$$\mu(\alpha,\lambda) = \inf\left\{ \int\limits_\Omega \left(|\nabla u|^2 - \lambda a u^2\right) + \frac{\alpha}{1-\alpha}\int\limits_{\partial\Omega} u^2;\ u \in H^1(\Omega), \int\limits_\Omega u^2 = 1 \right\}.$$

Note that $\mu(\alpha,0) > 0$ on $\alpha \in [\alpha_0, 1]$ for some small negative α_0, and $\lambda \mapsto \mu(\alpha,\lambda)$ has exactly two zeroes λ_α^- and λ_α^+ and those are principal eigenvalues of the following linear problem

$$\begin{aligned} -\Delta u &= \lambda a u, && x \in \Omega,\\ (1-\alpha)\frac{\partial u}{\partial n} + \alpha u &= 0, && x \in \partial\Omega \end{aligned} \tag{4}$$

(for more details, see [1]). Define

$$\left\|(u,v)\right\|_\lambda = \left(L(u) + L(v)\right)^{\frac{1}{2}}.$$

We prove that for $\lambda \in (\lambda_\alpha^-, \lambda_\alpha^+)$, $\|.\|_\lambda$ defines a norm in H which is equivalent to the usual norm for H. Our proof is motivated by that of [3].

Since $\|.\|_\lambda$ corresponds to the bilinear form

$$\begin{aligned} \left\langle (u_1,v_1),(u_2,v_2)\right\rangle = {} & \int\limits_\Omega (\nabla u_1 \nabla u_2 - \lambda a u_1 u_2) + \frac{\alpha}{1-\alpha}\int\limits_{\partial\Omega} u_1u_2 \\ & + \int\limits_\Omega (\nabla v_1 \nabla v_2 - \lambda a v_1 v_2) + \frac{\alpha}{1-\alpha}\int\limits_{\partial\Omega} v_1v_2, \end{aligned}$$

in order to prove that $\|.\|_\lambda$ is a norm, it is sufficient to prove that $\langle (u,v),(u,v)\rangle \geq 0$ for all $(u,v) \in H - \{(0,0)\}$. It follows from the variational characterization of $\mu(\alpha,\lambda)$ that

$$\left\langle (u,v),(u,v)\right\rangle = L(u) + L(v) \geq \mu(\alpha,\lambda)\biggl(\int\limits_\Omega \left(u^2+v^2\right)\biggr). \tag{5}$$

Hence if $\lambda \in (\lambda_\alpha^-, \lambda_\alpha^+)$, then $\mu(\alpha, \lambda) > 0$ and so $\|(u,v)\|_\lambda > 0$ whenever $u, v \neq 0$, thus $\|.\|_\lambda$ is a norm.

Moreover for $(u,v) \in H$, there exists a constant $k > 0$ such that

$$\|(u,v)\|_\lambda \le k\|(u,v)\|_H.$$

In fact by using the embedding of $H_1(\Omega)$ into $L^2(\partial\Omega)$, we have

$$\begin{aligned}
&\|(u,v)\|_\lambda^2 \\
&\quad \le \left| \int_\Omega \left(|\nabla u|^2 - \lambda a u^2\right) + \frac{\alpha}{1-\alpha} \int_{\partial\Omega} u^2 \right| + \left| \int_\Omega \left(|\nabla v|^2 - \lambda a v^2\right) + \frac{\alpha}{1-\alpha} \int_{\partial\Omega} v^2 \right| \\
&\quad \le \int_\Omega |\nabla u|^2 + |\lambda|\left(a^+\right) \int_\Omega u^2 + \frac{\alpha}{1-\alpha} c_1 \int_\Omega |\nabla u|^2 \\
&\qquad + \int_\Omega |\nabla v|^2 + |\lambda|\left(a^+\right) \int_\Omega v^2 + \frac{\alpha}{1-\alpha} c_1 \int_\Omega |\nabla v|^2 \\
&\quad \le c_2 \|(u,v)\|_H^2,
\end{aligned}$$

where c_1 is the best Sobolev constant of the embedding of $H_1(\Omega)$ into $L^2(\partial\Omega)$,

$$c_2 = \max\left\{1 + \frac{\alpha c_1}{1-\alpha}, \lambda_\alpha^+ a^+\right\}$$

and $a^+ = \sup_{x\in\Omega} |a(x)| > 0$. Now suppose that there exists a sequence $\{(u_n, v_n)\} \subseteq H$ such that $\|(u_n, v_n)\|_\lambda \to 0$ and $\|(u_n, v_n)\|_H = 1$. Since $\{(u_n, v_n)\}$ is bounded in H, there exists $(u,v) \in H$ such that $(u_n, v_n) \rightharpoonup (u,v)$ in H. Applying compactly embedding of H in $L^2(\Omega) \times L^2(\Omega)$ and $L^2(\partial\Omega) \times L^2(\partial\Omega)$, we have $(u_n, v_n) \to (u,v)$ in $L^2(\Omega) \times L^2(\Omega)$ and $L^2(\partial\Omega) \times L^2(\partial\Omega)$, respectively. Taking $\|(u_n, v_n)\|_\lambda \to 0$ into account, from (5), we have $(u_n, v_n) \to (0,0)$ in $L^2(\Omega) \times L^2(\Omega)$, and so $u = v = 0$. This implies $(u_n, v_n) \to (0,0)$ in $L^2(\partial\Omega) \times L^2(\partial\Omega)$, which concludes that

$$\lim_{n\to\infty} \left(\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right) \mathrm{d}x \right) = 0.$$

This contradicts with the fact $\|(u_n, v_n)\|_H = 1$ for all n. Hence, $\|.\|_\lambda$ and $\|.\|_H$ are equivalent norms.

Now we consider the Nehari minimizing problem

$$\mathcal{N}(\lambda) = \inf\left\{J_\lambda(u,v);\ (u,v) \in M_\lambda\right\},$$

where

$$M_\lambda = \left\{(u,v) \in H - \{(0,0)\};\ \left\langle J'_\lambda(u,v), (u,v)\right\rangle = J_\lambda^{(1)}(u,v)u + J_\lambda^{(2)}(u,v)v = 0\right\}.$$

It is clear that all critical points of J_λ must lie on M_λ which is well-known as the Nehari manifold (see [2,8]). We will see below that local minimizers of J_λ on M_λ contains every non-zero solution of the system (1). First we claim that for $\lambda \in (\lambda_\alpha^-, \lambda_\alpha^+) - \{0\}$ we have $M_\lambda \neq \emptyset$. In fact, since the function a changes sign, we can choose non-zero function $(u_0, v_0) \in H$ such that $G_1(u_0, v_0) = \int a v_0^2 |u_0|^p > 0$ and $G_2(u_0, v_0) = \int a u_0^2 |v_0|^p > 0$.

Let

$$t^p = \frac{-L(u_0) - L(v_0)}{(2+p)(G_1(u_0, v_0) + G_2(u_0, v_0))},$$

then

$$\begin{aligned} &\big\langle J_\lambda'(tu_0, tv_0), (tu_0, tv_0)\big\rangle \\ &\quad = t^2\big[L(u_0) + L(v_0) + t^p(p+2)\big(G_1(u_0, v_0) + G_2(u_0, v_0)\big)\big] = 0, \end{aligned}$$

and so $(u, v) = t(u_0, v_0) \in M_\lambda$.

Define $\mathcal{F}_\lambda(u, v) = \langle J_\lambda'(u, v), (u, v)\rangle$. Then for $(u, v) \in M_\lambda$,

$$\begin{aligned} \big\langle \mathcal{F}_\lambda'(u, v), (u, v)\big\rangle &= 2\big(L(u) + L(v)\big) + (p+2)^2(G_1 + G_2)(u, v) \\ &= p(p+2)(G_1 + G_2)(u, v) = -p\big(L(u) + L(v)\big). \end{aligned}$$

Set

$$\mathcal{K}_\lambda^+ = \inf\big\{L(u) + L(v);\ (u, v) \in H,\ (G_1 + G_2)(u, v) = 1\big\}$$

and

$$\mathcal{K}_\lambda^- = \inf\big\{L(u) + L(v);\ (u, v) \in H,\ (G_1 + G_2)(u, v) = -1\big\}.$$

By a similar way we can define $\mathcal{K}_0^+$ and $\mathcal{K}_0^-$.

For $\alpha \in [0, 1]$, we have that $\lambda \to \mathcal{K}_\lambda^+$ is a concave continuous curve on the interval $[\lambda_\alpha^-, \lambda_\alpha^+]$. By using a similar arguments we have these facts for $\mathcal{K}_\lambda^-$.

To state our main result, we now present some important properties of $\mathcal{K}_\lambda^+$ and $\mathcal{K}_\lambda^-$.

Lemma 1. $\mathcal{K}_0^+ > 0$ *and* $\mathcal{K}_0^- > 0$ *for* $\alpha \in (0, 1]$.

Proof. Suppose otherwise, that is there exists sequence $(u_n, v_n) \in H$ such that

$$\lim_{n\to\infty} \big(L(u_n) + L(v_n)\big) = 0, \quad (G_1 + G_2)(u_n, v_n) = 1.$$

By the Sobolev embedding theorem and Hölder inequality, there exists $\varepsilon_0 < \frac{2}{N-2}$ such that

$$1 = \left| \int_\Omega \big(a v_n^2 |u_n|^p + a u_n^2 |v_n|^p\big) \right| \le a^+ \int_\Omega \big(v_n^2 |u_n|^p + u_n^2 |v_n|^p\big)$$

$$\le a^+\Bigg(\bigg[\int\limits_\Omega (v_n^2)^{\frac{2^*}{2}-\varepsilon_0}\bigg]^{\frac{2}{2^*-2\varepsilon_0}}\bigg[\int\limits_\Omega \big(|u_n|^p\big)^{\frac{2^*-2\varepsilon_0}{2^*-2(\varepsilon_0+1)}}\bigg]^{\frac{2^*-2(\varepsilon_0+1)}{2^*-2\varepsilon_0}}$$

$$+\bigg[\int\limits_\Omega (u_n^2)^{\frac{2^*}{2}-\varepsilon_0}\bigg]^{\frac{2}{2^*-2\varepsilon_0}}\bigg[\int\limits_\Omega \big(|v_n|^p\big)^{\frac{2^*-2\varepsilon_0}{2^*-2(\varepsilon_0+1)}}\bigg]^{\frac{2^*-2(\varepsilon_0+1)}{2^*-2\varepsilon_0}}\Bigg)$$

$$\le a^+\big(2\big\|(u_n,v_n)\big\|_H^{2^*}\big) \le c'a^+\big\|(u_n,v_n)\big\|_\lambda^{2^*} \to 0$$

as $n\to\infty$, which is a contradiction.

By the same argument we have $\mathcal{K}_0^- > 0$. □

Lemma 2. $\mathcal{K}^+_{\lambda_\alpha^+} = \mathcal{K}^-_{\lambda_\alpha^-} = 0$.

Proof. Suppose that φ^+ is a positive eigenfunction of the linear problem (4) corresponding to the principal eigenvalue λ_α^+. Then $L(\varphi^+) = 0$. On the other hand, if $\lambda \ne 0$ be the principal eigenvalue of (4) with corresponding positive principal eigenfunction φ, then $\lambda\int_\Omega a\varphi^{p+1} > 0$ (see [3]). So in this case, since $\lambda_\alpha^+ > 0$, we have $\int_\Omega a(\varphi^+)^{p+1} > 0$. We now let

$$u = \frac{\varphi^+}{(2\int_\Omega a(\varphi^+)^{p+2})^{\frac{1}{p+2}}}.$$

Then

$$(G_1+G_2)(u,u) = 2\int\limits_\Omega a|u|^{p+2} = 1,$$

and

$$L(u)+L(u) = \frac{2L(\varphi^+)}{2^{\frac{2}{p+2}}(\int_\Omega a(\varphi^+)^{p+2})^{\frac{2}{p+2}}} = 0,$$

i.e., $\mathcal{K}^+_{\lambda_\alpha^+} = 0$. By using a similar way we can prove that $\mathcal{K}^-_{\lambda_\alpha^-} = 0$. Since $\lambda \to \mathcal{K}_\lambda^+$ is a concave continuous curve, we also have $0 < \mathcal{K}_\lambda^+ < \mathcal{K}_0^+$ for $\lambda \in (0,\lambda_\alpha^+)$, and $0 < \mathcal{K}_\lambda^- < \mathcal{K}_0^-$ for all $\lambda \in (\lambda_\alpha^-, 0)$. □

Lemma 3. *For $\lambda \in (\lambda_\alpha^-, \lambda_\alpha^+)$, put*

$$Y = \bigg\{(u,v)\in H;\ (G_1+G_2)(u,v) = -\frac{1}{p+2}\bigg\}.$$

Then $\|(u,v)\|_\lambda^{\frac{2}{p}}(u,v) \in M_\lambda$ if $(u,v)\in Y$, and $\|(u,v)\|_\lambda^{-\frac{2}{p+2}}(u,v)\in Y$ if $(u,v)\in M_\lambda$.

Proof. First suppose that $(u,v)\in Y$, then

$$\begin{aligned}
&\mathcal{F}_\lambda\big(\big\|(u,v)\big\|_\lambda^{\frac{2}{p}}(u,v)\big)\\
&\quad= \big\langle J_\lambda'\big(\big\|(u,v)\big\|_\lambda^{\frac{2}{p}}u, \big\|(u,v)\big\|_\lambda^{\frac{2}{p}}v\big), \big(\big\|(u,v)\big\|_\lambda^{\frac{2}{p}}u, \big\|(u,v)\big\|_\lambda^{\frac{2}{p}}v\big)\big\rangle\\
&\quad= \big(\big\|(u,v)\big\|_\lambda^{\frac{4}{p}}\big)\big(L(u)+L(v)\big) + \big\|(u,v)\big\|_\lambda^{\frac{2(2+p)}{p}}(p+2)(G_1+G_2)(u,v)\\
&\quad= \big\|(u,v)\big\|_\lambda^{\frac{4}{p}}\big\|(u,v)\big\|_\lambda^2 - \big\|(u,v)\big\|_\lambda^{\frac{2(2+p)}{p}} = 0,
\end{aligned}$$

i.e., $\|(u,v)\|_\lambda^{\frac{2}{p}}(u,v) \in M_\lambda$.

Now if $(u,v) \in M_\lambda$, we get

$$L(u) + L(v) + (p+2)(G_1 + G_2)(u,v) = \langle J'_\lambda(u,v)(u,v)\rangle = 0,$$

thus

$$\begin{aligned}
&(G_1 + G_2)\big(\|(u,v)\|_\lambda^{-\frac{2}{p+2}} u, \|(u,v)\|_\lambda^{-\frac{2}{p+2}} v\big) \\
&\quad = \|(u,v)\|_\lambda^{-2}(G_1+G_2)(u,v) = \|(u,v)\|_\lambda^{-2}\frac{-L(u)-L(v)}{p+2} \\
&\quad = -\frac{\|(u,v)\|_\lambda^{-2}\|(u,v)\|_\lambda^{2}}{p+2} = -\frac{1}{p+2},
\end{aligned}$$

i.e., $\|(u,v)\|_\lambda^{-\frac{2}{p+2}}(u,v) \in Y$. □

For $\lambda \in (\lambda_\alpha^-, \lambda_\alpha^+)$, we define $Q_\lambda : Y \to \mathbb{R}$ by $Q_\lambda(u,v) = \|(u,v)\|_\lambda^2$. Then for $(u,v) \in Y$ we have

$$Q_\lambda(u,v) = \left(\frac{2(p+2)}{p} J_\lambda\left(\|(u,v)\|_\lambda^{\frac{2}{p}}(u,v)\right)\right)^{\frac{p}{p+2}}.$$

Indeed,

$$\begin{aligned}
&J_\lambda\big(\|(u,v)\|_\lambda^{\frac{2}{p}}(u,v)\big) \\
&\quad = \frac{1}{2}\|(u,v)\|_\lambda^{\frac{4}{p}}\big(L(u)+L(v)\big) + \|(u,v)\|_\lambda^{\frac{2(p+2)}{p}}(G_1+G_2)(u,v).
\end{aligned}$$

Since $(G_1+G_2)(u,v) = -\frac{1}{p+2}$ for $(u,v) \in Y$, we conclude

$$J_\lambda\big(\|(u,v)\|_\lambda^{\frac{2}{p}}(u,v)\big) = \left(\frac{1}{2} - \frac{1}{p+2}\right)\|(u,v)\|_\lambda^{\frac{2(p+2)}{p}},$$

which means that the claim is true.

Using a similar argument, if $(u,v) \in M_\lambda$, then

$$J_\lambda(u,v) = \frac{p}{2(p+2)} Q_\lambda\big(\|(u,v)\|_\lambda^{-\frac{2}{p+2}}(u,v)\big)^{\frac{p+2}{p}}.$$

Indeed,

$$\begin{aligned}
&\frac{p}{2(p+2)} Q_\lambda\big(\|(u,v)\|_\lambda^{-\frac{2}{p+2}}(u,v)\big)^{\frac{p+2}{p}} \\
&\quad = \frac{p}{2(p+2)}\big(\|(u,v)\|_\lambda^{-\frac{4}{p+2}}\|(u,v)\|_\lambda^2\big)^{\frac{p+2}{p}} = \frac{p}{2(p+2)}\big(\|(u,v)\|_\lambda^{\frac{2p}{p+2}}\big)^{\frac{p+2}{p}} \\
&\quad = \frac{p}{2(p+2)}\|(u,v)\|_\lambda^2 = J_\lambda(u,v).
\end{aligned}$$

The latest equality follows from that

$$L(u) + L(v) + (p+2)(G_1 + G_2)(u,v) = 0$$

for $(u,v) \in M_\lambda$, i.e.,

$$(G_1 + G_2)(u,v) = -\frac{1}{p+2}\big(L(u) + L(v)\big) = -\frac{1}{p+2}\big\|(u,v)\big\|_\lambda^2,$$

and so

$$\begin{aligned} J_\lambda(u,v) &= \frac{1}{2}\big(L(u) + L(v)\big) + (G_1 + G_2)(u,v) \\ &= \frac{1}{2}\big\|(u,v)\big\|_\lambda^2 - \frac{1}{p+2}\big\|(u,v)\big\|_\lambda^2 = \frac{p}{2(p+2)}\big\|(u,v)\big\|_\lambda^2. \end{aligned}$$

In fact we proved the following result:

Lemma 4. *Define* $q_\lambda = \inf_{(u,v)\in Y} Q_\lambda(u,v)$ *and* $j_\lambda = \inf_{(u,v)\in M_\lambda} J_\lambda(u,v)$. *Then,* $q_\lambda = (\frac{2(p+2)}{p} j_\lambda)^{\frac{p}{p+2}}$.

3 Main results

Lemma 5. $(u,v) = (0,0)$ *is not a limit point of* M_λ *if* $\lambda \in (\lambda_\alpha^-, \lambda_\alpha^+)$, $\alpha \neq 0$.

Proof. Let $\{(u_n, v_n)\}$ in M_λ, so that $\|(u_n, v_n)\|_\lambda \to 0$ as $n \to \infty$.

Now let $(u_n', v_n') = \frac{(u_n, v_n)}{\|(u_n, v_n)\|_\lambda}$, then $\|(u_n', v_n')\|_\lambda = 1$, i.e., $\{(u_n', v_n')\}$ is a bounded sequence in $L^{2^*}(\Omega) \times L^{2^*}(\Omega)$ equipped with the norm

$$\big\|(u,v)\big\|_{L^{2^*}(\Omega)\times L^{2^*}(\Omega)} = \bigg(\int\limits_\Omega \big(|u|^{2^*} + |v|^{2^*}\big)\bigg)^{\frac{1}{2^*}},$$

therefore

$$\begin{aligned} 0 &= \frac{\langle J_\lambda'(u_n, v_n), (u_n, v_n)\rangle}{\|(u_n, v_n)\|_\lambda^2} \\ &= \frac{L(u_n) + L(v_n)) + (p+2)(G_1 + G_2)(u_n, v_n)}{\|(u_n, v_n)\|_\lambda^2} \\ &= 1 + (p+2)\frac{(G_1 + G_2)(u_n, v_n)}{\|(u_n, v_n)\|_\lambda^2} \\ &= 1 + (p+2)\big\|(u_n, v_n)\big\|_\lambda^p (G_1 + G_2)(u_n', v_n'). \end{aligned}$$

Moreover by the boundedness of $\{(u_n', v_n')\}$ in H, and applying the inequality mentioned in Lemma 1, we derive that the sequence $\{(G_1+G_2)(u_n', v_n')\}$ is bounded in H, and so the right hand side of the last equality tends to 1. This contradiction proves the lemma. □

Theorem 1. *Let $\alpha \in (0,1]$ or $\int_\Omega a \neq 0$ if $\alpha = 0$. Then there exist two positive constants δ_1, δ_2 such that if $\{(u_n, v_n)\}$ is a minimizing sequence of J_λ on M_λ, then*

$$\liminf_{n\to\infty} \left| \int\limits_\Omega a\bigl(u_n^2 + v_n^2\bigr) \right| > 0,$$

where $\lambda_\alpha^- < \lambda < \lambda_\alpha^- + \delta_1$ or $\lambda_\alpha^+ - \delta_2 < \lambda < \lambda_\alpha^+$.

Proof. Let φ^- and φ^+ be the corresponding eigenfunctions to the principal eigenvalues λ_α^- and λ_α^+, respectively.

We can assume that

$$\int\limits_\Omega a|\varphi^+|^{p+2} = 1, \qquad \int\limits_\Omega a|\varphi^-|^{p+2} = -1,$$
$$\int\limits_\Omega a(\varphi^+)^2 > 0 \quad \text{and} \quad \int\limits_\Omega a(\varphi^-)^2 < 0.$$

Now let

$$\delta_2 = \lambda_\alpha^+ - \frac{\int_\Omega |\nabla\varphi^+|^2 + \frac{\alpha}{1-\alpha}\int_{\partial\Omega}(\varphi^+)^2 - \mathcal{K}_0^+}{\int_\Omega a(\varphi^+)^2} = \frac{\mathcal{K}_0^+}{\int_\Omega a(\varphi^+)^2}.$$

Then for $\lambda \in (\lambda_\alpha^+ - \delta_2, \lambda_\alpha^+)$, if the sequence $\{(u_n, v_n)\}$ be a minimizing sequence of J_λ on M_λ, we derive that

$$\begin{aligned}
J_\lambda(u_n, v_n) &= \frac{p}{2(p+2)}\bigl[Q_\lambda\bigl(\bigl\|(u_n, v_n)\bigr\|_\lambda^{-\frac{2}{p+2}}(u_n, v_n)\bigr)\bigr]^{\frac{p+2}{p}} \\
&= \frac{p}{2(p+2)}\bigl\|\bigl\|(u_n, v_n)\bigr\|_\lambda^{-\frac{2}{p+2}}(u_n, v_n)\bigr\|_\lambda^{\frac{2(p+2)}{p}} \\
&= \frac{p}{2(p+2)}\bigl\|(u_n, v_n)\bigr\|_\lambda^{-\frac{4}{p}}\bigl\|(u_n, v_n)\bigr\|_\lambda^{\frac{2(p+2)}{p}} \\
&= \frac{p}{2(p+2)}\bigl\|(u_n, v_n)\bigr\|_\lambda^2.
\end{aligned}$$

This implies $\{(u_n, v_n)\}$ is bounded in H and so there exist a subsequence, which for convenience we again denote by $\{(u_n, v_n)\}$, and $\{(u, v)\} \in H$ such that $(u_n, v_n) \rightharpoonup (u, v)$ in H. Since H may be compactly embedded in $L^2(\Omega) \times L^2(\Omega)$, we have $(u_n, v_n) \to (u, v)$ in $L^2(\Omega) \times L^2(\Omega)$. For $\lambda > 0$, by using Lemma 4, if $\{(u_n, v_n)\}$ be a minimizing sequence of J_λ on M_λ, then $\|(u_n, v_n)\|_\lambda^{-\frac{2}{p+2}}\{(u_n, v_n)\}$ is a minimizing sequence of Q_λ on Y, and so we get

$$\begin{aligned}
\inf_Y Q_\lambda &= \lim_{n\to\infty} Q_\lambda\bigl(\bigl\|(u_n, v_n)\bigr\|_\lambda^{-\frac{2}{p+2}}(u_n, v_n)\bigr) \\
&= \lim_{n\to\infty} \bigl\|(u_n, v_n)\bigr\|_\lambda^{-\frac{4}{p+2}}\bigl(L(u_n) + L(v_n)\bigr)
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \bigl(L\bigl(\|(u_n,v_n)\|_\lambda^{-\frac{2}{p+2}} u_n\bigr) + L\bigl(\|(u_n,v_n)\|_\lambda^{-\frac{2}{p+2}} v_n\bigr)\bigr) \\
&< q < \mathcal{K}_0^+
\end{aligned}
$$

for some $q > 0$. Also we have

$$
(G_1+G_2)\bigl(\|(u_n,v_n)\|_\lambda^{-\frac{2}{p+2}} u_n, \|(u_n,v_n)\|_\lambda^{-\frac{2}{p+2}} v_n\bigr) = -\frac{1}{p+2}.
$$

Now by Lemma 5, we obtain $\|(u_n,v_n)\|_\lambda \not\to 0$, i.e., $L(u_n)+L(v_n) \not\to 0$, and if $\int_\Omega a(u_n^2+v_n^2) \to 0$, then we get

$$
\|(u_n,v_n)\|_\lambda^{-\frac{4}{p+2}} \biggl[\int_\Omega \bigl(|\nabla u_n|^2 + |\nabla v_n|^2\bigr) + \int_{\partial\Omega} \bigl(u_n^2+v_n^2\bigr)\biggr] \not\to 0.
$$

So by taking $\lambda = 0$, we have

$$
\lim_{n\to\infty} \|(u_n,v_n)\|_\lambda^{-\frac{4}{p+2}} \bigl(L(u_n)+L(v_n)\bigr) = \mathcal{K}_0^+ \le q < \mathcal{K}_0^+,
$$

which is a contradiction. Therefore,

$$
\lim_{n\to\infty} \biggl|\int_\Omega a\bigl(u_n^2+v_n^2\bigr)\biggr| > 0.
$$

Now let

$$
\delta_1 = \frac{\int_\Omega |\nabla\varphi^-|^2 + \frac{\alpha}{1-\alpha}\int_{\partial\Omega}(\varphi^-)^2 - \mathcal{K}_0^-}{\int_\Omega a(\varphi^-)^2} - \lambda_\alpha^- = -\frac{\mathcal{K}_0^-}{\int_\Omega a(\varphi^-)^2}.
$$

By a similar argument we get a same result for $\lambda \in (\lambda_\alpha^-, \lambda_\alpha^- + \delta_1)$. □

Lemma 6. *The production of two Hilbert spaces, is a Hilbert space.*

Proof. For Hilbert spaces H_1 and H_2, define

$$
\bigl\langle (u_1,u_2),(v_1,v_2)\bigr\rangle_{H_1\times H_2} = \langle u_1,v_1\rangle_{H_1} + \langle u_2,v_2\rangle_{H_2}.
$$

It is easy to see that, the mentioned bilinear form defines an inner product on $H_1\times H_2$. □

To state our significant proposition form [9], first let B_ϵ be a ball in the Hilbert space H, centered at 0 and of radial ϵ.

Proposition 1. *Let Φ be a C^1-functional on a Hilbert space $X = X_1 \times X_2$, where X_1 and X_2 are Hilbert spaces, and let Γ be a closed subset in X such that for any $(u,v) \in \Gamma$ with $\Phi'(u,v) \neq 0$ and $\epsilon > 0$ small arbitrary, there exists Frechet differentiable function*

$s_{(u,v)} : B_\epsilon \to \mathbb{R}$ such that by setting $t_{(u,v)}(\delta) = s_{(u,v)}(\delta \frac{\Phi'(u,v)}{\|\Phi'(u,v)\|_X})$ for $0 \le \delta \le \epsilon$, we have $t_{(u,v)}(0) = 1$, and

$$t_{(u,v)}(\delta)\left(u - \delta\frac{\Phi^{(1)}(u,v)}{\|\Phi^{(1)}(u,v)\|_{X_1}}, v - \delta\frac{\Phi^{(2)}(u,v)}{\|\Phi^{(2)}(u,v)\|_{X_2}}\right) \in \Gamma.$$

If Φ is bounded below on Γ, then for any minimizing sequence $\{(u_n, v_n)\}$ for Φ in Γ, there exists another minimizing sequence $\{(u_n^*, v_n^*)\}$ for Φ in Γ such that

$$\Phi(u_n^*, v_n^*) \le \Phi(u_n, v_n), \quad \lim_{n\to\infty} \big\|(u_n^*, v_n^*) - (u_n, v_n)\big\| = 0$$

and

$$\big\|\Phi'(u_n^*, v_n^*)\big\|_X \\ \le \frac{1}{n}\big(\sqrt{2} + \big\|(u_n^*, v_n^*)\big\|\big|t'_{(u_n^*,v_n^*)}(0)\big|\big) + \big|t'_{(u_n^*,v_n^*)}(0)\big|\big|\big\langle \Phi'(u_n^*, v_n^*), (u_n^*, v_n^*)\big\rangle\big|.$$

Proof. Let $\eta = \inf_{\gamma\in\Gamma} \Phi(\gamma)$. Using Ekland variational principle, we get a minimizing sequence $\{(u_n^*, v_n^*)\}$ in Γ, which

(i) $\Phi(u_n^*, v_n^*) \le \Phi(u_n, v_n) < \eta + \frac{1}{n}$,

(ii) $\lim_{n\to\infty} \|(u_n^*, v_n^*) - (u_n, v_n)\| = 0$,

(iii) $\Phi(x, y) \ge \Phi(u_n^*, v_n^*) - \frac{1}{n}\big\|(x, y) - (u_n^*, v_n^*)\|$ for all $(x, y) \in \Gamma$.

Let us assume $\|\Phi'(u_n^*, v_n^*)\|_X > 0$ for large n. Apply the hypothesis on the set Γ with $(u, v) = (u_n^*, v_n^*)$ to find the function

$$t_n(\delta) = t_{(u_n^*,v_n^*)}(\delta) = s_{(u_n^*,v_n^*)}\left(\delta\frac{\Phi'(u_n^*, v_n^*)}{\|\Phi'(u_n^*, v_n^*)\|_X}\right).$$

Then,

$$(x_\delta, y_\delta) = t_n(\delta)\left(u_n^* - \delta\frac{\Phi^{(1)}(u_n^*, v_n^*)}{\|\Phi^{(1)}(u_n^*, v_n^*)\|_{X_1}}, v_n^* - \delta\frac{\Phi^{(2)}(u_n^*, v_n^*)}{\|\Phi^{(2)}(u_n^*, v_n^*)\|_{X_2}}\right) \in \Gamma$$

for all small enough $\delta \ge 0$. By the mean value theorem we have

$$\begin{aligned}
&\frac{1}{n}\big\|(x_\delta, y_\delta) - (u_n^*, v_n^*)\big\| \\
&\quad\ge \Phi(u_n^*, v_n^*) - \Phi(x_\delta, y_\delta) = \big\langle \Phi'(x_\delta, y_\delta), (u_n^*, v_n^*) - (x_\delta, y_\delta)\big\rangle + o(\delta) \\
&\quad= \big\langle \Phi'(x_\delta, y_\delta), (u_n^*, v_n^*)\big\rangle - \big\langle \Phi'(x_\delta, y_\delta), (x_\delta, y_\delta)\big\rangle + o(\delta) \\
&\quad= \langle \Phi'(x_\delta, y_\delta), (u_n^*, v_n^*)\rangle - t_n(\delta)\big\langle \Phi'(x_\delta, y_\delta), (u_n^*, v_n^*)\big\rangle \\
&\qquad+ \delta t_n(\delta)\left\langle \Phi'(x_\delta, y_\delta), \left(\frac{\Phi^{(1)}(u_n^*, v_n^*)}{\|\Phi^{(1)}(u_n^*, v_n^*)\|_{X_1}}, \frac{\Phi^{(2)}(u_n^*, v_n^*)}{\|\Phi^{(2)}(u_n^*, v_n^*)\|_{X_2}}\right)\right\rangle + o(\delta) \\
&\quad= \big(1 - t_n(\delta)\big)\big\langle \Phi'(x_\delta, y_\delta), (u_n^*, v_n^*)\big\rangle \\
&\qquad+ \delta t_n(\delta)\left\langle \Phi'(x_\delta, y_\delta), \left(\frac{\Phi^{(1)}(u_n^*, v_n^*)}{\|\Phi^{(1)}(u_n^*, v_n^*)\|_{X_1}}, \frac{\Phi^{(2)}(u_n^*, v_n^*)}{\|\Phi^{(2)}(u_n^*, v_n^*)\|_{X_2}}\right)\right\rangle + o(\delta),
\end{aligned}$$

where $\frac{o(\delta)}{\delta} \to 0$ as $\delta \to 0$. So

$$\begin{aligned}
&\frac{1}{n\delta}\left\|\left(t_n(\delta)u_n^* - \delta t_n(\delta)\frac{\Phi^{(1)}(u_n^*,v_n^*)}{\|\Phi^{(1)}(u_n^*,v_n^*)\|_{X_1}} - u_n^*,\right.\right.\\
&\qquad\left.\left. t_n(\delta)v_n^* - \delta t_n(\delta)\frac{\Phi^{(2)}(u_n^*,v_n^*)}{\|\Phi^{(2)}(u_n^*,v_n^*)\|_{X_2}} - v_n^*\right)\right\|\\
&\quad\ge \frac{1-t_n(\delta)}{\delta}\langle\Phi'(x_\delta,y_\delta),(u_n^*,v_n^*)\rangle\\
&\qquad + t_n(\delta)\left\langle\Phi'(x_\delta,y_\delta),\left(\frac{\Phi^{(1)}(u_n^*,v_n^*)}{\|\Phi^{(1)}(u_n^*,v_n^*)\|_{X_1}},\frac{\Phi^{(2)}(u_n^*,v_n^*)}{\|\Phi^{(2)}(u_n^*,v_n^*)\|_{X_2}}\right)\right\rangle + \frac{o(\delta)}{\delta}.
\end{aligned}$$

Now we pass to the limit as $\delta \to 0$, and we derive

$$\begin{aligned}
&\lim_{\delta\to 0}\frac{1}{n}\left\|\left(\left[\frac{t_n(\delta)-1}{\delta}\right]u_n^*,\left[\frac{t_n(\delta)-1}{\delta}\right]v_n^*\right)\right\|\\
&+\lim_{\delta\to 0}\frac{1}{n}\left|t_n(\delta)\right|\left\|\left(\frac{\Phi^{(1)}(u_n^*,v_n^*)}{\|\Phi^{(1)}(u_n^*,v_n^*)\|_{X_1}},\frac{\Phi^{(2)}(u_n^*,v_n^*)}{\|\Phi^{(2)}(u_n^*,v_n^*)\|_{X_2}}\right)\right\|\\
&\quad=\frac{1}{n}\left|t_n'(0)\right|\left\|(u_n^*,v_n^*)\right\| + \frac{1}{n}\left(\left\|\frac{\Phi^{(1)}(u_n^*,v_n^*)}{\|\Phi^{(1)}(u_n^*,v_n^*)\|_{X_1}}\right\|^2 + \left\|\frac{\Phi^{(2)}(u_n^*,v_n^*)}{\|\Phi^{(2)}(u_n^*,v_n^*)\|_{X_2}}\right\|^2\right)^{\frac{1}{2}}\\
&\quad=\frac{1}{n}\left(\sqrt{2}+\left|t_n'(0)\right|\left\|(u_n^*,v_n^*)\right\|\right)\\
&\quad\ge -t_n'(0)\left\langle\Phi'(u_n^*,v_n^*),(u_n^*,v_n^*)\right\rangle\\
&\qquad+\left\langle\Phi'(u_n^*,v_n^*),\left(\frac{\Phi^{(1)}(u_n^*,v_n^*)}{\|\Phi^{(1)}(u_n^*,v_n^*)\|_{X_1}},\frac{\Phi^{(2)}(u_n^*,v_n^*)}{\|\Phi^{(2)}(u_n^*,v_n^*)\|_{X_2}}\right)\right\rangle\\
&\quad= -t_n'(0)\langle\Phi'(u_n^*,v_n^*),(u_n^*,v_n^*)\rangle\\
&\qquad+\left\langle\Phi^{(1)}(u_n^*,v_n^*),\frac{\Phi^{(1)}(u_n^*,v_n^*)}{\|\Phi^{(1)}(u_n^*,v_n^*)\|_{X_1}}\right\rangle + \left\langle\Phi^{(2)}(u_n^*,v_n^*),\frac{\Phi^{(2)}(u_n^*,v_n^*)}{\|\Phi^{(2)}(u_n^*,v_n^*)\|_{X_2}}\right\rangle\\
&= -t_n'(0)\langle\Phi'(u_n^*,v_n^*),(u_n^*,v_n^*)\rangle + \left\|\Phi^{(1)}(u_n^*,v_n^*)\right\|_{X_1} + \left\|\Phi^{(2)}(u_n^*,v_n^*)\right\|_{X_2}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\left\|\Phi'(u_n^*,v_n^*)\right\|_X\\
&\quad=\left\|\Phi^{(1)}(u_n^*,v_n^*)+\Phi^{(2)}(u_n^*,v_n^*)\right\|_X \le \left\|\Phi^{(1)}(u_n^*,v_n^*)\right\|_{X_1} + \left\|\Phi^{(2)}(u_n^*,v_n^*)\right\|_{X_2}\\
&\quad\le\frac{1}{n}\left(\sqrt{2}+\left|t_n'(0)\right|\left\|(u_n^*,v_n^*)\right\|\right) + t_n'(0)\langle\Phi'(u_n^*,v_n^*),(u_n^*,v_n^*)\rangle.
\end{aligned}$$

This completes the proof. □

Let $\alpha \in (0,1]$ or that $\int_\Omega a \ne 0$ for $\alpha = 0$. Then for $\lambda \in (\lambda_\alpha^-, \lambda_\alpha^+)$ we have the following results:

Lemma 7. *J_λ is bounded below on M_λ.*

Proof. Firstly we note that for $(u,v) \in M_\lambda$,

$$(G_1 + G_2)(u,v) = -\frac{1}{p+2}\big(L(u) + L(v)\big). \tag{6}$$

Now for $\lambda \in (\lambda_\alpha^-, \lambda_\alpha^+)$, let $w = \frac{u}{(\int_\Omega u^2)^{\frac{1}{2}}}$. Then

$$\mu(\alpha,\lambda) \le L(w) = \frac{L(u)}{\int_\Omega u^2} = \frac{\int_\Omega(|\nabla u|^2 - \lambda a u^2) + \frac{\alpha}{1-\alpha}\int_{\partial\Omega} u^2}{\int_\Omega u^2},$$

and so $L(u) > 0$ for $\lambda \in (\lambda_\alpha^-, \lambda_\alpha^+)$. A similar argument shows that $L(v) > 0$ for $\lambda \in (\lambda_\alpha^-, \lambda_\alpha^+)$, and (6) implies $(G_1 + G_2)(u,v) < 0$ for $(u,v) \in M_\lambda$. Hence, $J_\lambda(u,v) = -\frac{p}{2}(G_1 + G_2)(u,v) > 0$, and J_λ is bounded below on M_λ. □

Theorem 2. *There exists a minimizing sequence $\{(u_n^*, v_n^*)\}$ of J_λ on M_λ such that*

$$\lim_{n\to\infty} \big\|J'_\lambda(u_n^*, v_n^*)\big\|_\lambda = 0.$$

Proof. For $(u,v) \in M_\lambda$, let $\Psi : R \times H \to R$ such that

$$\Psi\big(s,(w_1,w_2)\big) = \mathcal{F}_\lambda(su - w_1, sv - w_2).$$

Since $(u,v) \in M_\lambda$, we have $\Psi(1,(0,0)) = 0$. Also,

$$\begin{aligned}\frac{\partial}{\partial s}\Psi\big(1,(0,0)\big) &= 2\big(L(u) + L(v)\big) + (p+2)^2(G_1 + G_2)(u,v)\\ &= 2\big(-(p+2)(G_1 + G_2)(u,v)\big) + (p+2)^2(G_1 + G_2)(u,v)\\ &= p(p+2)(G_1 + G_2)(u,v).\end{aligned}$$

Now we want to apply the Implicit function theorem at $(1,(0,0))$ to get for any $\delta > 0$ small enough, a differentiable function $s_{(u,v)} : B_\delta \to R$ such that

$$s_{(u,v)}(0,0) = 1, \quad s_{(u,v)}(w_1,w_2)\big((u,v) - (w_1,w_2)\big) \in M_\lambda,$$
$$\Big\langle s'_{(u,v)}(0,0),(w_1,w_2)\Big\rangle = \frac{\langle \mathcal{F}'_\lambda(u,v),(w_1,w_2)\rangle}{\langle \mathcal{F}'_\lambda(u,v),(u,v)\rangle}$$

for all $(w_1,w_2) \in B_\delta$, where B_δ is defined before.

Now for every $(u,v) \in H$, let

$$(x,y)_{(u,v)} = \frac{J'_\lambda(u,v)}{\|J'_\lambda(u,v)\|_\lambda},$$
$$t_{(u,v)}(\rho) = s_{(u,v)}\big(\rho(x,y)_{(u,v)}\big) \quad \text{for } 0 \le \rho \le \delta.$$

Then we have the following results:

(i) $t_{(u,v)}(0) = 1$,

(ii) $t'_{(u,v)}(0) = \langle s'_{(u,v)}(0,0),(x,y)_{(u,v)}\rangle$,

(iii) $\mathcal{F}_\lambda(s_{(u,v)}(\rho(x,y)_{(u,v)})((u,v)-\rho(x,y)_{(u,v)}))=0$,

(iv) $t_{(u,v)}(\rho)((u,v)-\rho(x,y)_{(u,v)})=s_{(u,v)}(\rho(x,y)_{(u,v)})((u,v)-\rho(x,y)_{(u,v)})\in M_\lambda$.

From Proposition 1, there exists a minimizing sequence $\{(u_n^*,v_n^*)\}$ for J_λ such that

$$J_\lambda(u_n^*,v_n^*)\le J_\lambda(u_n,v_n)<\inf_{M_\lambda}J_\lambda+\frac{1}{n},$$
$$\lim_{n\to\infty}\|(u_n^*,v_n^*)-(u_n,v_n)\|=0$$

and

$$\|J'_\lambda(u_n^*,v_n^*)\|_\lambda\le\frac{1}{n}\bigl(\sqrt{2}+\|(u_n^*,v_n^*)\|_\lambda|t'_{(u_n^*,v_n^*)}(0)|\bigr)+|t'_{(u_n^*,v_n^*)}(0)\mathcal{F}_\lambda(u_n^*,v_n^*)|.$$

Since

$$J_\lambda(u_n^*,v_n^*)=\frac{p}{2(p+2)}\|(u_n^*,v_n^*)\|_\lambda^2<\inf_{M_\lambda}J_\lambda+\frac{1}{n},$$

so that $\{(u_n^*,v_n^*)\}$ is a bounded sequence in H, i.e., there exits $c_1>0$ such that $\|(u_n^*,v_n^*)\|_\lambda<c_1$ for all n. Then,

$$\|J'_\lambda(u_n^*,v_n^*)\|_\lambda\le\frac{1}{n}\bigl(\sqrt{2}+|t'_{(u_n^*,v_n^*)}(0)|c_1\bigr).$$

Moreover,

$$\begin{aligned}|t'_{(u_n^*,v_n^*)}(0)|&=|\langle s'_{(u_n^*,v_n^*)}(0,0),(x,y)_{(u_n^*,v_n^*)}\rangle|=\frac{|\langle\mathcal{F}'_\lambda(u_n^*,v_n^*),(x,y)_{(u_n^*,v_n^*)}\rangle|}{|\langle\mathcal{F}'_\lambda(u_n^*,v_n^*),(u_n^*,v_n^*)\rangle|}\\&=\frac{|\langle\mathcal{F}'_\lambda(u_n^*,v_n^*),(x,y)_{(u_n^*,v_n^*)}\rangle|}{|p(p+2)(G_1+G_2)(u_n^*,v_n^*)|}=\frac{|\langle\mathcal{F}'_\lambda(u_n^*,v_n^*),(x,y)_{(u_n^*,v_n^*)}\rangle|}{p\|(u_n^*,v_n^*)\|_\lambda^2}\end{aligned}$$

and

$$\liminf_{n\to\infty}\|(u_n^*,v_n^*)\|_\lambda>0,$$

since $(0,0)$ is not a limit point of M_λ.

So, if we show that $|t'_{(u_n^*,v_n^*)}(0)|$ is uniformly bounded on n, we are done.

By using the Hölder inequality, Sobolev embedding theorem, boundedness of the sequence $\{(u_n^*,v_n^*)\}$ and $\|(x,y)_{(u_n^*,v_n^*)}\|=1$, we have

$$|\langle\mathcal{F}'_\lambda(u_n^*,v_n^*),(x,y)_{(u_n^*,v_n^*)}\rangle|\le c_2\|(u_n^*,v_n^*)\|_\lambda+c_3.$$

This proves the theorem. □

We can now prove the main result of the paper:

Theorem 3. *Let* $\alpha\in(0,1]$ *or that* $\int_\Omega a\ne0$ *for* $\alpha=0$. *For any* $\lambda\in(\lambda_\alpha^-,\lambda_\alpha^-+\delta_1)\cup(\lambda_\alpha^+-\delta_2,\lambda_\alpha^+)$, $\lambda\ne0$, *The system* (1) *has a nontrivial nonnegative solution.*

Proof. Let $c = \inf J_\lambda(M_\lambda)$ and $\{(u_n, v_n)\}$ be a sequence in M_λ such that

$$\lim_{n\to\infty} J_\lambda(u_n, v_n) = c.$$

By using Theorem 2, we have $\lim_{n\to\infty} \|J'_\lambda(u_n, v_n)\|_\lambda = 0$. Then $\{(u_n, v_n)\}$ is bounded and we can find a weak limit point of the sequence in H, i.e., $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ both in $H^1(\Omega)$, for some $u, v \in H^1(\Omega)$, and so $u_n \to u$ and $v_n \to v$ both in $L^q(\Omega)$ for $q < \frac{2N}{N-2}$. In particular for any $(h_1, h_2) \in H$,

$$\begin{aligned}
&\big\langle J'_\lambda(u_n, v_n), (h_1, h_2)\big\rangle \\
&\quad = \int_\Omega \nabla u_n \nabla h_1 + \int_\Omega \nabla v_n \nabla h_2 - \lambda \int_\Omega a(u_n h_1 + v_n h_2) + \frac{\alpha}{1-\alpha} \int_{\partial\Omega} (u_n h_1 + v_n h_2) \\
&\qquad + p \int_\Omega a\big(v_n^2 |u_n|^{p-2} u_n h_1 + u_n^2 |v_n|^{p-2} v_n h_2\big) + 2 \int_\Omega a\big(u_n |v_n|^p h_1 + v_n |u_n|^p h_2\big),
\end{aligned}$$

which converges to

$$\begin{aligned}
&\big\langle J'_\lambda(u, v), (h_1, h_2)\big\rangle \\
&\quad = \int_\Omega \nabla u \nabla h_1 + \int_\Omega \nabla v \nabla h_2 - \lambda \int_\Omega a(u h_1 + v h_2) + \frac{\alpha}{1-\alpha} \int_{\partial\Omega} (u h_1 + v h_2) \\
&\qquad + p \int_\Omega a\big(v^2 |u|^{p-2} u h_1 + u^2 |v|^{p-2} v h_2\big) + 2 \int_\Omega a\big(u |v|^p h_1 + v |u|^p h_2\big)
\end{aligned}$$

as $n \to \infty$. So, we derive

$$\begin{aligned}
&\big|\big\langle J'_\lambda(u, v), (h_1, h_2)\big\rangle\big| \\
&\quad = \lim_{n\to\infty} \big\langle J'_\lambda(u_n, v_n), (h_1, h_2)\big\rangle \le \lim_{n\to\infty} \big\|J'_\lambda(u_n, v_n)\big\|_\lambda \big\|(h_1, h_2)\big\| = 0,
\end{aligned}$$

that means $\langle J'_\lambda(u, v), (h_1, h_2)\rangle = 0$ for all $(h_1, h_2) \in Y$. Therefore, (u, v) is a weak solution for the system (1).

In particular, $\langle J'_\lambda(u, v), (u, v)\rangle = 0$. Since $\liminf |\int_\Omega (a u_n^2 + a v_n^2)| > 0$, we have $(u, v) \neq (0, 0)$. Hence, $(u, v) \in M_\lambda$.

On the other hand, J_λ is weakly lower semicontinuous, and so we have

$$c \le J_\lambda(u, v) \le \lim_{n\to\infty} J_\lambda(u_n, v_n) = c,$$

which follows that $J_\lambda(u, v) = c$ and that $\|(u_n, v_n)\|_\lambda \to \|(u, v)\|_\lambda$ which implies that $u_n \to u$ and $v_n \to v$ both in $H^1(\Omega)$.

Since J'_λ is continuous at (u, v), we get $J'_\lambda(u, v) = 0$. One can check that $J_\lambda(u, v) = J_\lambda(|u|, |v|)$, so (u, v) is a nontrivial nonnegative solution for the system (1). □

References

1. G.A. Afrouzi, K.J. Brown, On principal eigenvalues for boundary value problems with indefinite weight and Robin boundary conditions, *Proc. Am. Math. Soc.*, **127**, pp. 125–130, 1999.

2. G.A. Afrouzi, S. Khademloo, The Nehari manifold for a class of indefinite weight semilinear elliptic equations, *Bull. Iran. Math. Soc.*, **33**, pp. 49–59, 2007.

3. B. Ko, K.J. Brown, The existence of positive solutions for a class of indefinite weight semilinear elliptic boundary value problems, *Nonlinear Anal.*, **39**, pp. 587–597, 2000.

4. B. Ko, S. Kang, On the positive solutions for a class of indefinite weight semilinear elliptic boundary value problems with critical Sobolev exponent, *J. Korean Math. Soc.*, **44**, pp. 249–260, 2007.

5. K.J. Brown, Y. Zhang, The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function, *J. Differ. Equations*, **193**, pp. 481–499, 2003.

6. T.F. Wu, A semilinear elliptic problem involving nonlinear boundary condition and sign changing potential, *Electron. J. Differ. Equ.*, **2006**, pp. 1–15, 2006.

7. G.A. Afrouzi, S.H. Rasouli, A remark on the existence of multiple solutions to a multiparameter nonlinear elliptic system, *Nonlinear Anal.*, **71**, pp. 445–455, 2009.

8. Z. Nehari, On a class of nonlinear second-order diffrential equations, *Trans. Am. Math. Soc.*, **95**, pp. 101–123, 1960.

9. N. Ghoussoub, *Duality and perturbation method in critical point theory*, Cambridge Traces in Mathematics, Vol. 107, Cambridge Univ. Press, Cambridge, 1993.