On the Dynamics of Controlled Magnetohydrodynamic Systems*

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Abstract. In this paper we study the long time behavior of solutions for an optimal control problem associated with the viscous incompressible electrically conducting fluid modeled by the magnetohydrodynamic (MHD) equations in a bounded two dimensional domain through the adjustment of distributed controls. We first construct a quasi-optimal solution for the MHD systems which possesses exponential decay in time. We then derive some preliminary estimates for the long-time behavior of all admissible solutions of the MHD systems. Next we prove the existence of a solution for the optimal control problem for both finite and infinite time intervals. Finally, we establish the long-time decay properties of the solutions for the optimal control problem.

Keywords: dynamics, optimal control, MHD systems.

1 Introduction

Magnetohydrodynamics (MHD) is the branch of continuum mechanics that studies the macroscopic interaction of electrically conducting fluids and electromagnetic fields. The subject is of great interest for its numerous practical applications which includes motion of liquid metals, fusion technology, design of novel submarine propulsion devices and plasma physics. The motion of Newtonian fluids is governed by the Navier-Stokes equations and electromagnetic effects are governed by Maxwell's equations. Under a number of physical assumptions valid for the problems of interest these two general systems can be reduced to the MHD systems, see for e.g. [1–6].

The main goal of this paper is to study the dynamics of solutions to an optimal control problem in magneto-hydrodynamics. Optimal control of fluids to alter flows to achieve a desired effect remains an active research area due to its importance for the design and performance of fluid dynamical systems. The past decade has seen significant developments in theoretical and computational analysis in this area, see for e.g. [7–10].

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The study of long-time behavior of solutions of optimal control problems associated with MHD systems is of great importance in many fluid dynamic applications such as stabilization and drag minimization. There has been an extensive study in the literature of the asymptotic behaviors and dynamics of solutions for dissipative systems including Navier-Stokes systems and coupled Navier-Stokes systems such as MHD systems. The asymptotic behavior of solutions for the controlled Navier-Stokes system was studied in [11, 12]. In this article we study the long time behavior of solutions for optimal control problems associated with the magneto-hydrodynamic equations. This work was motivated by the desire to match a candidate flow field and magnetic field with a desired one by appropriately controlling the applied current and distributed force.

We formulate the optimal control problem as follows. Let us assume that a viscous incompressible and electrically conducting fluid fills a two dimensional bounded simply connected region Ω of class C^2 or convex with boundary $\partial\Omega$ whose unit normal will be denoted by $\mathbf{n}$. We further assume that the magnetic field lies in the plane where the fluid motion occurs and the electric current density is a vector field normal to this plane. The macroscopic state of the fluid can be described by the fluid velocity $\mathbf{u} = (u_1(\mathbf{x},t), u_2(\mathbf{x},t), 0)$, pressure $p = p(\mathbf{x},t)$, magnetic field $\mathbf{B} = (B_1(\mathbf{x},t), B_2(\mathbf{x},t), 0)$ and electric current density $\mathbf{j} = (0, 0, \mathrm{j}(\mathbf{x},t))$. The non-dimensional form of the viscous incompressible MHD equations is (see for e.g. [1])

$$\begin{aligned}
&\frac{\partial \mathbf{u}}{\partial t} - \frac{1}{Re}\nabla^2\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - S(\operatorname{curl}\mathbf{B})\times\mathbf{B} = \mathbf{f} && \text{in } \Omega\times(0,\infty),\\
&\frac{\partial \mathbf{B}}{\partial t} + \frac{1}{Re_m}\operatorname{curl}(\operatorname{curl}\mathbf{B}) - \operatorname{curl}(\mathbf{u}\times\mathbf{B}) = \operatorname{curl}\mathbf{j} && \text{in } \Omega\times(0,\infty),\\
&\nabla\cdot\mathbf{u} = 0 && \text{in } \Omega\times(0,\infty),\\
&\nabla\cdot\mathbf{B} = 0 && \text{in } \Omega\times(0,\infty),
\end{aligned} \tag{1}$$

where $\mathbf{u}, \mathbf{B}, \mathbf{j}$ and p are non-dimensional quantities corresponding to the normalization by reference units denoted by ℓ^*, T^*, $U^* := \ell^*/T^*$, j^*, B^*, ρ^* for lengths, times, velocities, currents, magnetic fields and densities. Moreover, $\mathbf{f}$ and $\mathbf{j}$ are the applied distributed force and current, respectively. The three non-dimensional numbers that appear in (1) are Re – Reynolds number, $Re := \frac{U^*\ell^*}{\mu}$, where μ is the kinematic viscosity; Re_m – magnetic Reynolds number, $Re_m := \mu_0\sigma U^*\ell^*$, where μ_0 is the magnetic permeability and σ the electrical conductivity; and S – Alfan number or coupling number, $S := \frac{B^{*2}}{\mu_0\rho^* U^{*2}}$.

The system of equations (1) has to be supplemented with the initial conditions

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) \quad \text{and} \quad \mathbf{B}(\mathbf{x},0) = \mathbf{B}_0(\mathbf{x}) \quad \text{in } \Omega, \tag{1a}$$

along with proper boundary conditions. For the velocity we specify

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega\times(0,\infty), \tag{1b}$$

that is, there is no flow through the boundaries and a no-slip boundary condition is satisfied at the boundaries. For the magnetic field we specify

$$\mathbf{B}\cdot\mathbf{n} = 0 \quad \text{and} \quad (\operatorname{curl}\mathbf{B})\times\mathbf{n} = 0 \quad \text{on } \partial\Omega\times(0,\infty), \tag{1c}$$

that is, we assume the boundary is perfectly conducting (no tangential electric field and no normal magnetic field), see [5].

Our objective of matching the candidate flow field and magnetic field with the desired ones in ideal setting means matching the desired flow at each time instance. This warrants minimizing a cost functional defined in terms of a pointwise norm in t. However, such an ideal cost functional is too costly to realize physically and to compute numerically. Therefore it is natural to consider the time-averaged functional $\mathscr{J}$ defined by

$$\mathscr{J}(\mathbf{u},\mathbf{B},\mathbf{f},\operatorname{curl}\mathbf{j}) := \frac{1}{2}\int_0^\infty\int_\Omega \alpha_1\left|\mathbf{u}-\mathbf{u}^d\right|^2 + S\alpha_2\left|\mathbf{B}-\mathbf{B}^d\right|^2 + \beta_1\left|\mathbf{f}-\mathbf{f}^d\right|^2 + \beta_2\left|\operatorname{curl}\mathbf{j}-\operatorname{curl}\mathbf{j}^d\right|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t,$$

where $\mathbf{u}^d$ is some desired velocity field, $\mathbf{B}^d$ is some desired magnetic field, $\mathbf{f}^d$ is some desired distributed force and $\mathbf{j}^d$ is some desired current density. Also, $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ are given constants, the functions $\operatorname{curl}\mathbf{j}$ and $\mathbf{f}$ are the distributed controls and $|\cdot|$ denotes the usual Euclidean norm. We wish to find the controls $\operatorname{curl}\mathbf{j}$, $\mathbf{f}$ and the associated pair $(\mathbf{u},\mathbf{B})$ such that the cost functional $\mathscr{J}$ is minimized subject to MHD equations (1). The first term in the cost functional $\mathscr{J}$ measures the deviation between the candidate states and the desired states. Therefore, the physical objective of this minimization problem is to match a desired flow and magnetic field by adjusting the controls $\operatorname{curl}\mathbf{j}$ and $\mathbf{f}$. The cost functional reflects a trade-off between achieving the physical objective and minimizing the work involved in the control effort.

We will show in this paper that for large t, the time averaged optimizer will indeed give pointwise matching in t. We like to note here that exact controllability [13] is somewhat related to the problem of controlling the pointwise in t behavior for the solutions. But the controllability approach does not give information on the matching of the flow and magnetic fields over a time period, nor does it give any information beyond $t = T$.

The rest of the paper is structured as follows. In the rest of the section, we present some preliminary material. In Section 2, we construct a quasi-optimal control solution and obtain some preliminary estimates for optimal solutions. In Section 3, we prove the existence of an optimal solution on both finite and infinite time intervals and derive the optimality-system. Finally, in Section 4, we prove the decay of the controlled dynamics to the desired dynamics.

1.1 Preliminaries

1.1.1 Notations, function spaces and inequalities

We denote by $\mathrm{L}^q(\Omega)$, $1 < q < \infty$, $(\mathrm{L}^\infty(\Omega))$ the space of real functions defined on Ω with q-th power absolutely integrable (or essentially bounded functions) and that are equipped with the norm $\|u\|_{\mathrm{L}^q} := [\int_\Omega |u|^q \,\mathrm{d}\mathbf{x}]^{\frac{1}{q}}$ or $\|u\|_{\mathrm{L}^\infty} := ess.\sup_\Omega |u(\mathbf{x})|$. For $q = 2$, $\mathrm{L}^2(\Omega)$ is a Hilbert space with inner product $\int_\Omega uv \,\mathrm{d}\mathbf{x}$ and $\|u\| := [\int_\Omega |u|^2 \,\mathrm{d}\mathbf{x}]^{\frac{1}{2}}$. The standard notation $\mathrm{H}^m(\Omega), m = 0, 1, 2, \ldots,$ is used for the Sobolev space of functions in $\mathrm{L}^2(\Omega)$ with square integrable derivatives of order $\leq m$ $(\mathrm{H}^0(\Omega) = \mathrm{L}^2(\Omega))$. In particular for $m = 1$, $\mathrm{H}_0^1(\Omega)$ denotes the space of functions in $\mathrm{H}^1(\Omega)$ which vanish on the boundary $\partial\Omega$, whereas in $\mathrm{H}_n^1(\Omega)$ only the normal component of the function is assumed to vanish along the boundary. These spaces have the associated norm $\|u\|_1 := [\|u\|^2 + \sum_{i=1}^2 \|\frac{\partial u}{\partial x_i}\|^2]^{\frac{1}{2}}$. We shall be concerned with two dimensional vector functions with components in one of these spaces. We shall use the notation $\mathbf{L}^q(\Omega) := \mathrm{L}^q(\Omega) \times \mathrm{L}^q(\Omega)$, $\mathbf{H}^m(\Omega) := \mathrm{H}^m(\Omega) \times \mathrm{H}^m(\Omega)$, $\mathbf{H}_0^1(\Omega) := \mathrm{H}_0^1(\Omega) \times \mathrm{H}_0^1(\Omega)$. For integer $m \geq 0$, $\mathbf{H}^m(\Omega)$ is equipped with the norm $\|\mathbf{u}\|_m := [\sum_{i=1}^2 \|u_i\|_m^2]^{\frac{1}{2}}$. Also, the inner-product for functions belonging to $\mathbf{L}^2(\Omega) = \mathbf{H}^0(\Omega) := \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Omega)$ is given by $\int_\Omega \mathbf{u}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}$. For details, concerning these spaces, see for e.g. [14] and [15]. For the purpose of dealing with the linear constraints in the MHD equations, we introduce the following spaces of divergence free vector fields

$$\mathbf{H}_u = \mathbf{H}_B := \{\mathbf{w} \in \mathbf{L}^2(\Omega) \colon \nabla\cdot\mathbf{w} = 0 \text{ and } \mathbf{w}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\},$$
$$\mathbf{V}_u := \mathbf{H}_u \cap \mathbf{H}_0^1(\Omega) \quad \text{and} \quad \mathbf{V}_B := \mathbf{H}_B \cap \mathbf{H}_n^1(\Omega).$$

We then define the Hilbert spaces $\mathbf{H} := \mathbf{H}_u \times \mathbf{H}_B$ and $\mathbf{V} := \mathbf{V}_u \times \mathbf{V}_B$ endowed with the following inner-products and corresponding norms

$$\big((\mathbf{u}_1,\mathbf{B}_1),(\mathbf{u}_2,\mathbf{B}_2)\big)_{\mathbf{H}} := \int_\Omega \mathbf{u}_1\cdot\mathbf{u}_2\,\mathrm{d}\mathbf{x} + S\int_\Omega \mathbf{B}_1\cdot\mathbf{B}_2\,\mathrm{d}\mathbf{x} \qquad \forall(\mathbf{u}_1,\mathbf{B}_1),(\mathbf{u}_2,\mathbf{B}_2)\in\mathbf{H},$$
$$\big((\mathbf{u}_1,\mathbf{B}_1),(\mathbf{u}_2,\mathbf{B}_2)\big)_{\mathbf{V}} := \frac{1}{Re}\int_\Omega \nabla\mathbf{u}_1 : \nabla\mathbf{u}_2\,\mathrm{d}\mathbf{x} + \frac{S}{Re_m}\int_\Omega \operatorname{curl}\mathbf{B}_1\cdot\operatorname{curl}\mathbf{B}_2\,\mathrm{d}\mathbf{x} \qquad \forall(\mathbf{u}_1,\mathbf{B}_1),(\mathbf{u}_2,\mathbf{B}_2)\in\mathbf{V},$$

where $(\nabla\mathbf{u})_{ij} := \frac{\partial u_i}{\partial x_i}$ is the Jacobian matrix and $\nabla\mathbf{u} : \nabla\mathbf{w} := \sum_{i=1,j=1}^2 \frac{\partial u_i}{\partial x_j}\frac{\partial w_i}{\partial x_j}$. We have the inclusion $\mathbf{V} \subset \mathbf{H}$ which is compact and dense by Rellich theorem, see [16].

We next introduce the temporal spatial function space $\mathrm{L}^q(0,T;\mathbf{Z})$ defined on $Q_T := \Omega\times(0,T)$ equipped with the norm $\|\mathbf{u}\|_{\mathrm{L}^q(0,T;\mathbf{Z})} := \int_0^T \|\mathbf{u}(t)\|_{\mathbf{Z}}^q\,\mathrm{d}t]^{1/q}$, where $q \in [1,\infty)$ and $\mathbf{Z} := \mathbf{H}^m(\Omega)$ or $\mathbf{V}$. The solenoidal temporal-spatial function space

$$\mathbf{W}^{(1)}(Q_T) := \left\{(\mathbf{u},\mathbf{B}) \in \mathrm{L}^2(0,T;\mathbf{V}) \colon \left(\frac{\partial\mathbf{u}}{\partial t},\frac{\partial\mathbf{B}}{\partial t}\right) \in \mathrm{L}^2(0,T;\mathbf{V}^*)\right\}$$

with the norm $\|\mathbf{v}\|_{\mathbf{W}^{(1)}(Q_T)} := [\|\mathbf{v}\|_{\mathrm{L}^2(0,T;\mathbf{V})}^2 + \|\frac{\partial\mathbf{v}}{\partial t}\|_{\mathrm{L}^2(0,T;\mathbf{V}^*)}^2]^{\frac{1}{2}}$, where $\mathbf{V}^*$ is the dual of $\mathbf{V}$. For functions $\mathbf{u}$ and $\mathbf{B}$ in the temporal-spatial space, we often use the notation $\mathbf{u}(t) := \mathbf{u}(\cdot,t)$ and $\mathbf{B}(t) := \mathbf{B}(\cdot,t)$ to stand for the restriction of $\mathbf{u}$ and $\mathbf{B}$ at time t as a function defined over the spatial domain Ω.

For $(\mathbf{u},\mathbf{B}) \in \mathbf{V}$, we deduce from Poincare inequalities $\|\mathbf{u}\| \leq \lambda_u^{-\frac{1}{2}}\|\nabla\mathbf{u}\|$ and $\|\mathbf{B}\| \leq \lambda_B^{-\frac{1}{2}}\|\operatorname{curl}\mathbf{B}\|$ that

$$\kappa\|(\mathbf{u},\mathbf{B})\|_{\mathbf{H}}^2 \leq \|(\mathbf{u},\mathbf{B})\|_{\mathbf{V}}^2 \;\; \forall(\mathbf{u},\mathbf{B})\in\mathbf{V}, \quad \text{where} \quad \kappa := \min\left\{\frac{\lambda_u}{Re},\frac{\lambda_B}{Re_m}\right\}. \tag{2}$$

We will also make use of the Young's inequality

$$ab \le \frac{\epsilon}{q}a^q + \frac{\epsilon^{-\frac{r}{q}}}{r}b^r, \quad 1 < q, r < \infty, \qquad \frac{1}{q} + \frac{1}{r} = 1, \quad a, b \ge 0.$$

In addition to the well known Gronwall's inequality [17], we will also use the uniform Gronwall's inequality (see [18]):

Lemma 1 (Uniform Gronwall's Inequality)**.** *Assume that positively locally integrable functions $y(t)$, $g(t)$ and $h(t)$ satisfy*

$$\frac{\mathrm{d}y}{\mathrm{d}t} \le gy + h, \quad t \ge 0$$

and moreover

$$\int_t^{t+\epsilon} g(s)\,\mathrm{d}s \le a_1, \quad \int_t^{t+\epsilon} h(s)\,\mathrm{d}s \le a_2, \quad \int_t^{t+\epsilon} y(s)\,\mathrm{d}s \le a_3,$$

where ϵ, a_1, a_2 and a_3 are positive constants. Then

$$y(t+\epsilon) \le (a_3/\epsilon + a_2)e^{a_1}, \quad t \ge 0.$$

1.1.2 Weak Formulation of the MHD equations

The weak form of equations (1) is obtained using standard arguments. Under the assumptions of smoothness of solutions, multiplying the first and second equations in (1) by divergence free test functions $\mathbf{w}$ and $\boldsymbol{\Upsilon}$, respectively, integrating by parts and adding the results after multiplying the second equation by S lead to

$$\left(\left(\frac{\partial \mathbf{u}}{\partial t}, \frac{\partial \mathbf{B}}{\partial t}\right), (\mathbf{w}, \boldsymbol{\Upsilon})\right)_{\mathbf{H}} + a\big((\mathbf{u},\mathbf{B}),(\mathbf{w},\boldsymbol{\Upsilon})\big) + b\big((\mathbf{u},\mathbf{B}),(\mathbf{u},\mathbf{B}),(\mathbf{w},\boldsymbol{\Upsilon})\big)$$
$$= ((\mathbf{f}, \operatorname{curl}\mathbf{j}), (\mathbf{w}, \boldsymbol{\Upsilon}))_{\mathbf{H}},$$

which by continuity holds for all $\mathbf{w} \in \mathbf{V}_u$ and $\boldsymbol{\Upsilon} \in \mathbf{V}_B$. Here

$$a\big((\mathbf{u}_1,\mathbf{B}_1),(\mathbf{u}_2,\mathbf{B}_2)\big) = \frac{1}{Re}\int_\Omega \nabla\mathbf{u}_1 : \nabla\mathbf{u}_2\,\mathrm{d}\mathbf{x} + \frac{S}{Re_m}\int_\Omega \operatorname{curl}\mathbf{B}_1 \cdot \operatorname{curl}\mathbf{B}_2\,\mathrm{d}\mathbf{x}$$

and

$$b\big((\mathbf{u}_1,\mathbf{B}_1),(\mathbf{u}_2,\mathbf{B}_2),(\mathbf{u}_3,\mathbf{B}_3)\big) = \int_\Omega (\mathbf{u}_1\cdot\nabla)\mathbf{u}_2\cdot\mathbf{u}_3\,\mathrm{d}\mathbf{x} - S\int_\Omega \operatorname{curl}\mathbf{B}_2 \times \mathbf{B}_1 \cdot \mathbf{u}_3\,\mathrm{d}\mathbf{x}$$
$$- S\int_\Omega \mathbf{u}_2 \times \mathbf{B}_1 \cdot \operatorname{curl}\mathbf{B}_3\,\mathrm{d}\mathbf{x}$$

for all $(\mathbf{u}_i, \mathbf{B}_i) \in \mathbf{V}$ $(i = 1, 2, 3)$. This suggest the following weak form for the MHD equations (1):

Definition 1. *Given* $T \in (0,\infty)$, $(\mathbf{u}_0,\mathbf{B}_0) \in \mathbf{H}$ *and* $(\mathbf{f}, \mathrm{curl}\,\mathbf{j}) \in \mathrm{L}^2(0,T;\mathbf{V}^*)$, $(\mathbf{u},\mathbf{B})$ *is said to be a solution of the MHD equations if and only if* $(\mathbf{u},\mathbf{B}) \in \mathbf{W}^{(1)}(Q_T)$ *and* $(\mathbf{u},\mathbf{B})$ *satisfies*

$$\begin{aligned} \left(\frac{\partial}{\partial t}(\mathbf{u},\mathbf{B}),(\mathbf{w},\boldsymbol{\Upsilon})\right)_{\mathbf{H}} &+ a\big((\mathbf{u},\mathbf{B}),(\mathbf{w},\boldsymbol{\Upsilon})\big) + b\big((\mathbf{u},\mathbf{B}),(\mathbf{u},\mathbf{B}),(\mathbf{w},\boldsymbol{\Upsilon})\big) \\ &= \big((\mathbf{f},\mathrm{curl}\,\mathbf{j}),(\mathbf{w},\boldsymbol{\Upsilon})\big)_{\mathbf{H}} \ \forall(\mathbf{w},\boldsymbol{\Upsilon}) \in \mathbf{V} \ \textit{(almost everywhere)} \quad t \in (0,T) \end{aligned} \tag{3}$$

and

$$\big(\mathbf{u}(0),\mathbf{B}(0)\big) = (\mathbf{u}_0,\mathbf{B}_0) \quad \textit{in } \mathbf{H}. \tag{4}$$

Note that $(\mathbf{u},\mathbf{B}) \in \mathbf{W}^{(1)}(Q_T)$ *implies* $(\mathbf{u},\mathbf{B}) \in \mathrm{C}([0,T];\mathbf{H})$. *Therefore* (4) *makes sense. The bilinear form* $a(\cdot,\cdot)$ *is continuous and coercive on* $\mathbf{V}$*:*

(i) $\big|a\big((\mathbf{u},\mathbf{B}),(\mathbf{w},\boldsymbol{\Upsilon})\big)\big| \le 2\|(\mathbf{u},\mathbf{B})\|_{\mathbf{V}}\|(\mathbf{w},\boldsymbol{\Upsilon})\|_{\mathbf{V}} \quad \forall(\mathbf{u},\mathbf{B}),(\mathbf{w},\boldsymbol{\Upsilon}) \in \mathbf{V}$,

(ii) $\big|a\big((\mathbf{u},\mathbf{B}),(\mathbf{u},\mathbf{B})\big)\big| = \|(\mathbf{u},\mathbf{B})\|^2_{\mathbf{V}} \quad \forall(\mathbf{u},\mathbf{B}) \in \mathbf{V}$.

Therefore $a(\cdot,\cdot)$ *defines a continuous positive operator* $\mathscr{A} : \mathbf{V} \to \mathbf{V}^*$ *as*

$$\big(\mathscr{A}(\mathbf{u}_1,\mathbf{B}_1),(\mathbf{u}_2,\mathbf{B}_2)\big)_{\mathbf{H}} := a\big((\mathbf{u}_1,\mathbf{B}_1),(\mathbf{u}_2,\mathbf{B}_2)\big) \quad \forall(\mathbf{u}_1,\mathbf{B}_1),(\mathbf{u}_2,\mathbf{B}_2) \in \mathbf{V}.$$

It can be shown that $\mathscr{A}$ extends to an unbounded self-adjoint operator in $\mathbf{H}$ with a domain $D(\mathscr{A}) := \{(\mathbf{u},\mathbf{B}) \in \mathbf{V} : \mathscr{A}(\mathbf{u},\mathbf{B}) \in \mathbf{H}\}$ dense in $\mathbf{V}$.

It is well-known that if $\partial\Omega$ is Lipschitz continuous, $T \in (0,\infty)$, $(\mathbf{f},\mathrm{curl}\,\mathbf{j}) \in \mathrm{L}^2(0,T;\mathbf{V}^*)$ and $(\mathbf{u}_0,\mathbf{B}_0) \in \mathbf{H}$, then there exists a unique solution to (3), (4) and it satisfies $(\mathbf{u},\mathbf{B}) \in \mathrm{C}([0,T];\mathbf{H}) \cap \mathrm{L}^2(0,T;\mathbf{V})$ and $(\mathbf{u}_t,\mathbf{B}_t) \in \mathrm{L}^2(0,T;\mathbf{V}^*)$. If $\partial\Omega$ is C^2, $(\mathbf{u}_0,\mathbf{B}_0) \in \mathbf{V}$ and $(\mathbf{f},\mathrm{curl}\,\mathbf{j}) \in \mathrm{L}^2(0,T;\mathbf{H})$, then $(\mathbf{u},\mathbf{B}) \in \mathrm{C}([0,T];\mathbf{V}) \cap \mathrm{L}^2(0,T;D(\mathscr{A}))$, see for e.g. [19].

For $T = \infty$, we define a solution of the MHD equations as follows.

Definition 2. *Given* $(\mathbf{u}_0,\mathbf{B}_0) \in \mathbf{H}$ *and* $(\mathrm{curl}\,\mathbf{j},\mathbf{f}) \in \mathrm{L}^2_{\mathrm{loc}}(0,\infty;\mathbf{V}^*)$, $(\mathbf{u},\mathbf{B})$ *is said to be solution of the MHD equations on* $(0,\infty)$ *if and only if* $(\mathbf{u},\mathbf{B}) \in \mathrm{L}^2_{\mathrm{loc}}(0,\infty;\mathbf{V}) \cap \mathrm{L}^\infty(0,\infty;\mathbf{H})$, $\frac{\partial}{\partial t}(\mathbf{u},\mathbf{B}) \in \mathrm{L}^2_{\mathrm{loc}}(0,\infty;\mathbf{V}^*)$ *and* $(\mathbf{u},\mathbf{B})$ *satisfies* (3),(4) *with* $T = \infty$.

The following properties of the trilinear form will be important for the forthcoming analysis. The skew symmetric property of $b(\cdot,\cdot,\cdot)$, i.e.,

$$\begin{aligned} b\big((\mathbf{u}_1,\mathbf{B}_1),(\mathbf{u}_2,\mathbf{B}_2),(\mathbf{u}_3,\mathbf{B}_3)\big) &= -\, b\big((\mathbf{u}_1,\mathbf{B}_1),(\mathbf{u}_3,\mathbf{B}_3),(\mathbf{u}_2,\mathbf{B}_2)\big) \\ &\forall(\mathbf{u}_1,\mathbf{B}_1) \in \mathbf{H} \quad \text{and} \quad (\mathbf{u}_2,\mathbf{B}_2),(\mathbf{u}_3,\mathbf{B}_3) \in \mathbf{V}, \end{aligned}$$

follows easily from the skew-symmetric property of the trilinear form $b_u(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3) := \int_\Omega(\mathbf{u}_1\cdot\nabla)\mathbf{u}_2\cdot\mathbf{u}_3\,\mathrm{d}\mathbf{x}$ for all $(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3) \in \mathbf{H}_u \times \mathbf{V}_u \times \mathbf{V}_u$ and the algebraic cancellation of the last two terms in the definition of $b(\cdot,\cdot,\cdot)$. Moreover, the trilinear form $b(\cdot,\cdot,\cdot)$

posses the following continuity properties:

(i) $|b(\mathbf{\Phi}_1, \mathbf{\Phi}_2, \mathbf{\Phi}_3)| \le c_a \|\mathbf{\Phi}_1\|_{\mathbf{H}}^{\frac{1}{2}} \|\mathbf{\Phi}_1\|_{\mathbf{V}}^{\frac{1}{2}} \|\mathbf{\Phi}_2\|_{\mathbf{V}}^{\frac{1}{2}} \|\mathscr{A}\mathbf{\Phi}_2\|_{\mathbf{H}}^{\frac{1}{2}} \|\mathbf{\Phi}_3\|_{\mathbf{H}}$
$\forall \mathbf{\Phi}_1 \in \mathbf{V}, \mathbf{\Phi}_2 \in D(\mathscr{A}), \mathbf{\Phi}_3 \in \mathbf{H}$,

(ii) $|b(\mathbf{\Phi}_1, \mathbf{\Phi}_2, \mathbf{\Phi}_3)| \le c_b \|\mathbf{\Phi}_1\|_{\mathbf{H}}^{\frac{1}{2}} \|\mathbf{\Phi}_1\|_{\mathbf{V}}^{\frac{1}{2}} \|\mathbf{\Phi}_2\|_{\mathbf{V}} \|\mathbf{\Phi}_3\|_{\mathbf{H}}^{\frac{1}{2}} \|\mathbf{\Phi}_3\|_{\mathbf{V}}^{\frac{1}{2}}$
$\forall \mathbf{\Phi}_1, \mathbf{\Phi}_2, \mathbf{\Phi}_3 \in \mathbf{V}$

for some constants $c_a, c_b > 0$.

1.1.3 Statement of the optimal control problem

In order that the optimal control solution of the MHD equations is close to the desired $(\mathbf{u}^d, \mathbf{B}^d)$, we must place some restrictions on the desired current density $\mathbf{j}^d$ and desired distributed force $\mathbf{f}^d$ present in the cost functional $\mathscr{J}$. We therefore choose

$$\mathbf{f}^d := \frac{\partial \mathbf{u}^d}{\partial t} - \frac{1}{Re}\nabla^2 \mathbf{u}^d + \big(\mathbf{u}^d \cdot \nabla\big)\mathbf{u}^d - S\big(\operatorname{curl} \mathbf{B}^d\big) \times \mathbf{B}^d,$$
$$\operatorname{curl} \mathbf{j}^d := \frac{\partial \mathbf{B}^d}{\partial t} + \frac{1}{Re_m} \operatorname{curl}\big(\operatorname{curl} \mathbf{B}^d\big) - \operatorname{curl}\big(\mathbf{u}^d \times \mathbf{B}^d\big).$$

Furthermore, we will assume throughout this paper that

$$(\mathbf{u}^d, \mathbf{B}^d) \in \mathrm{L}^\infty(0,\infty;\mathbf{H}) \quad \text{and} \quad (\mathbf{f}^d, \operatorname{curl} \mathbf{j}^d) \in \mathrm{L}^\infty(0,\infty;\mathbf{L}^2(\Omega))^2.$$

For each $T \in (0,\infty]$, we define the cost functional $\mathscr{J}_T$ by

$$\mathscr{J}_T(\mathbf{u}, \mathbf{B}, \mathbf{f}, \operatorname{curl} \mathbf{j}) := \frac{1}{2}\int_0^T\!\!\int_\Omega \alpha_1 \big|\mathbf{u} - \mathbf{u}^d\big|^2 + S\alpha_2 \big|\mathbf{B} - \mathbf{B}^d\big|^2 + \beta_1 \big|\mathbf{f} - \mathbf{f}^d\big|^2 + \beta_2 \big|\operatorname{curl} \mathbf{j} - \operatorname{curl} \mathbf{j}^d\big|^2 \, \mathrm{d}\mathbf{x}\, \mathrm{d}t, \tag{5}$$

for all $(\mathbf{u}, \mathbf{B}) \in (\mathbf{u}^d, \mathbf{B}^d) + \mathrm{L}^2(Q_T)$, $\mathbf{f} \in \mathbf{f}^d + \mathrm{L}^2(Q_T)$ and $\operatorname{curl} \mathbf{j} \in \operatorname{curl} \mathbf{j}^d + \mathrm{L}^2(Q_T)$. We will denote $\mathscr{J}_\infty$ simply by $\mathscr{J}$.

We define the admissible elements as follows with $\mathbf{X}_T$ and $\mathbf{Y}_T$ denoting, respectively, the function spaces

$$\mathbf{X}_T := \begin{cases} \mathbf{W}^{(1)}(Q_T) & \text{if } T \in (0,\infty), \\ \Big\{ (\mathbf{u}, \mathbf{B}) \in \mathrm{L}^2_{\mathrm{loc}}(0,\infty;\mathbf{V}) \cap \mathrm{L}^\infty(0,\infty;\mathbf{H}) : \\ \quad \frac{\partial}{\partial t}(\mathbf{u}, \mathbf{B}) \in \mathrm{L}^2_{\mathrm{loc}}(0,\infty;\mathbf{V}^*) \Big\} & \text{if } T = \infty \end{cases}$$

and

$$\mathbf{Y}_T := \begin{cases} \mathrm{L}^2(0,T;\mathbf{V}^*) & \text{if } T \in (0,\infty), \\ \mathrm{L}^2_{\mathrm{loc}}(0,\infty;\mathbf{V}^*) & \text{if } T = \infty. \end{cases}$$

Definition 3. *For a given* $T \in (0,\infty]$*, a quadruple* $(\mathbf{u},\mathbf{B},\mathbf{f},\operatorname{curl}\mathbf{j}) \in \mathbf{X}_T \times \mathbf{Y}_T$ *is called an admissible element if* $\mathscr{J}_T(\mathbf{u},\mathbf{B},\mathbf{f},\operatorname{curl}\mathbf{j}) < \infty$ *and* $(\mathbf{u},\mathbf{B},\mathbf{f},\operatorname{curl}\mathbf{j})$ *satisfies* (3), (4)*. The set of all admissible elements is denoted by* $\mathbf{U}_{\rm ad}(T)$*.*

Now for each $T \in (0,\infty]$, we state the optimal control problem on $(0,T)$ as follows: (OP-CON) find $(\mathbf{u},\mathbf{B},\mathbf{f},\operatorname{curl}\mathbf{j}) \in \mathbf{U}_{\rm ad}(T)$ such that the cost functional $\mathscr{J}_T$ in (5) is minimized.

We note here that our optimal control problem has nontrivial solutions since in general the initial state $(\mathbf{u}_0,\mathbf{B}_0)$ is certain distance away from the desired flow and thus the cost functional generally has a positive minimum.

With the change of variables $(\mathbf{v},\boldsymbol{\Xi}) := (\mathbf{u},\mathbf{B}) - (\mathbf{u}^d,\mathbf{B}^d)$, $\mathbf{h} := \mathbf{f} - \mathbf{f}^d$ and $\mathbf{g} := \operatorname{curl}\mathbf{j} - \operatorname{curl}\mathbf{j}^d$ system (3), (4) is equivalent to $(\mathbf{v},\boldsymbol{\Xi}) \in \mathbf{X}_T$ and $(\mathbf{h},\mathbf{g}) \in \mathbf{Y}_T$ satisfying

$$\begin{aligned}&\left(\frac{\partial}{\partial t}(\mathbf{v},\boldsymbol{\Xi}),(\mathbf{w},\boldsymbol{\Upsilon})\right)_{\mathbf{H}} + a\big((\mathbf{v},\boldsymbol{\Xi}),(\mathbf{w},\boldsymbol{\Upsilon})\big) + b\big((\mathbf{v},\boldsymbol{\Xi}),(\mathbf{v},\boldsymbol{\Xi}),(\mathbf{w},\boldsymbol{\Upsilon})\big)\\ &+ b\big((\mathbf{u}^d,\mathbf{B}^d),(\mathbf{v},\boldsymbol{\Xi}),(\mathbf{w},\boldsymbol{\Upsilon})\big) + b\big((\mathbf{v},\boldsymbol{\Xi}),(\mathbf{u}^d,\mathbf{B}^d),(\mathbf{w},\boldsymbol{\Upsilon})\big)\\ &\quad = \big((\mathbf{h},\mathbf{g}),(\mathbf{w},\boldsymbol{\Upsilon})\big)_{\mathbf{H}} \quad \forall(\mathbf{w},\boldsymbol{\Upsilon}) \in \mathbf{V} \ \text{(almost everywhere)}\ t \in (0,T)\end{aligned} \tag{6}$$

and

$$\big(\mathbf{v}(0),\boldsymbol{\Xi}(0)\big) = (\mathbf{u}_0,\mathbf{B}_0) - \big(\mathbf{u}_0^d,\mathbf{B}_0^d\big) \quad \text{in } \mathbf{H}. \tag{7}$$

This allows us to give another physical interpretation to the optimal control problem; i.e., one seeks a candidate flow field and magnetic field $(\mathbf{v},\boldsymbol{\Xi})$ and a candidate controls $\mathbf{h}$ and $\mathbf{g}$ such that the time averaged total energy of the electrically conducting fluid and the total work done by the control is minimized. Here we define the total energy of the electrically conducting fluid as $K(\mathbf{v},\boldsymbol{\Xi}) := \frac{1}{2}\int_\Omega |\mathbf{v}|^2 + S|\boldsymbol{\Xi}|^2\,\mathrm{d}\mathbf{x}$, consisting physically of two parts corresponding to the kinetic energy and the magnetic energy proportional to the coupling parameter S.

2 Preliminary estimates for the dynamics

2.1 A quasi-optimizer

In this Section, we will derive a sharp bound for the value of

$$\inf_{(\mathbf{u},\mathbf{B},\mathbf{f},\operatorname{curl}\mathbf{j}) \in \mathbf{U}_{\rm ad}(T)} \mathscr{J}_T(\mathbf{u},\mathbf{B},\mathbf{f},\operatorname{curl}\mathbf{j})$$

which is uniform in T. This estimate will then be used to estimate the dynamics of the optimal control solution. We next construct a quasi-optimizer $(\widetilde{\mathbf{u}},\widetilde{\mathbf{B}},\widetilde{\mathbf{f}},\operatorname{curl}\widetilde{\mathbf{j}}) \in \mathbf{U}_{\rm ad}(\infty)$ for $\mathscr{J}_\infty(\cdot,\cdot,\cdot,\cdot)$ and derive some preliminary estimates for the optimal solutions. By a quasi-optimizer we mean an element $(\widetilde{\mathbf{u}},\widetilde{\mathbf{B}},\widetilde{\mathbf{f}},\operatorname{curl}\widetilde{\mathbf{j}}) \in \mathcal{U}_{\rm ad}(\infty)$ satisfying $\|(\widetilde{\mathbf{u}},\widetilde{\mathbf{B}}) - (\mathbf{u}^d(t),\mathbf{B}^d(t))\|_{\mathbf{H}} \to 0$ as $t \to \infty$. The existence of such an element is shown in the following theorem.

Theorem 1. *There exists a quadruple* $(\widetilde{\mathbf{u}}, \widetilde{\mathbf{B}}, \widetilde{\mathbf{f}}, \operatorname{curl}\widetilde{\mathbf{j}}) \in \mathbf{U}_{\mathrm{ad}}(\infty)$ *such that*

$$\big\|\big(\widetilde{\mathbf{u}} - \mathbf{u}^d(t), \widetilde{\mathbf{B}} - \mathbf{B}^d(t)\big)\big\|_{\mathbf{H}}^2 \leq \big\|\big(\mathbf{u}_0 - \mathbf{u}_0^d, \mathbf{B}_0 - \mathbf{B}_0^d\big)\big\|_{\mathbf{H}}^2 e^{-(\gamma-\omega)t} \tag{8}$$

for some $\gamma > \omega$*, where*

$$\omega := c_b^2 \big\|\big(\mathbf{u}^d, \mathbf{B}^d\big)\big\|_{L^\infty(0,T;\mathbf{V})}^2 - \kappa$$

and

$$\mathscr{J}_T(\widetilde{\mathbf{u}}, \widetilde{\mathbf{B}}, \widetilde{\mathbf{f}}, \operatorname{curl}\widetilde{\mathbf{j}}) \leq c_0 \big\|\big(\mathbf{u}_0 - \mathbf{u}_0^d, \mathbf{B}_0 - \mathbf{B}_0^d\big)\big\|_{\mathbf{H}}^2 \big(1 - e^{-(\gamma-\omega)T}\big) \quad \forall T \in (0, \infty], \tag{9}$$

where

$$c_0 := \frac{\big(4\alpha + \gamma^2 \max\big\{\beta_1, \frac{\beta_2}{S}\big\}\big)}{8(\gamma - \omega)}$$

and $\alpha := \max\{\alpha_1, \alpha_2\}$.

Proof. Let $t \in (0, \infty)$ be arbitrary. Let $(\widetilde{\mathbf{u}}, \widetilde{\mathbf{B}}) \in \mathbf{X}_\infty$ be the solution to (3), (4) with linear feedback:

$$\widetilde{\mathbf{f}} := \mathbf{f}^d - \frac{\gamma}{2}\big(\widetilde{\mathbf{u}} - \mathbf{u}^d\big), \quad \operatorname{curl}\widetilde{\mathbf{j}} = \operatorname{curl}\mathbf{j}^d - \frac{\gamma}{2}\big(\widetilde{\mathbf{B}} - \mathbf{B}^d\big)$$

for some fixed constant $\gamma > 0$. Existence of such a solution can be shown using the techniques for the MHD equations (see [19]). By using the change of variables $(\widetilde{\mathbf{v}}, \widetilde{\boldsymbol{\Xi}}) := (\widetilde{\mathbf{u}}, \widetilde{\mathbf{B}}) - (\mathbf{u}^d, \mathbf{B}^d)$, we see that the feedback controlled system is

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}\big(\widetilde{\mathbf{v}}, \widetilde{\boldsymbol{\Xi}}\big), (\mathbf{w}, \boldsymbol{\Upsilon})\right)_{\mathbf{H}} + a\big(\big(\widetilde{\mathbf{v}}, \widetilde{\boldsymbol{\Xi}}\big), (\mathbf{w}, \boldsymbol{\Upsilon})\big) + b\big(\big(\widetilde{\mathbf{v}}, \widetilde{\boldsymbol{\Xi}}\big), \big(\widetilde{\mathbf{v}}, \widetilde{\boldsymbol{\Xi}}\big), (\mathbf{w}, \boldsymbol{\Upsilon})\big) \\
&\quad + b\big(\big(\mathbf{u}^d, \mathbf{B}^d\big), \big(\widetilde{\mathbf{v}}, \widetilde{\boldsymbol{\Xi}}\big), (\mathbf{w}, \boldsymbol{\Upsilon})\big) + b\big(\big(\widetilde{\mathbf{v}}, \widetilde{\boldsymbol{\Xi}}\big), \big(\mathbf{u}^d, \mathbf{B}^d\big), (\mathbf{w}, \boldsymbol{\Upsilon})\big) \\
&\qquad = -\frac{\gamma}{2}\big(\big(\widetilde{\mathbf{v}}, \widetilde{\boldsymbol{\Xi}}\big), (\mathbf{w}, \boldsymbol{\Upsilon})\big)_{\mathbf{H}} \quad \forall (\mathbf{w}, \boldsymbol{\Upsilon}) \in \mathbf{V} \ \text{(almost everywhere)} \ t \in (0, \infty)
\end{aligned} \tag{10}$$

and

$$\big(\mathbf{v}(0), \boldsymbol{\Xi}(0)\big) = (\mathbf{u}_0, \mathbf{B}_0) - \big(\mathbf{u}_0^d, \mathbf{B}_0^d\big) \quad \text{in } \mathbf{H}. \tag{11}$$

Setting $\mathbf{w} = \widetilde{\mathbf{v}}$ and $\boldsymbol{\Upsilon} = \widetilde{\boldsymbol{\Xi}}$ in (10) and using the skew-symmetry property of $b(\cdot,\cdot,\cdot)$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|\big(\widetilde{\mathbf{v}}, \widetilde{\boldsymbol{\Xi}}\big)\big\|_{\mathbf{H}}^2 + \big\|\big(\widetilde{\mathbf{v}}, \widetilde{\boldsymbol{\Xi}}\big)\big\|_{\mathbf{V}}^2 + \frac{\gamma}{2}\big\|\big(\widetilde{\mathbf{v}}, \widetilde{\boldsymbol{\Xi}}\big)\big\|_{\mathbf{H}}^2 = -b\big(\big(\widetilde{\mathbf{v}}, \widetilde{\boldsymbol{\Xi}}\big), \big(\mathbf{u}^d, \mathbf{B}^d\big), \big(\widetilde{\mathbf{v}}, \widetilde{\boldsymbol{\Xi}}\big)\big). \tag{12}$$

By the continuity property (iii) of the trilinear form, we have

$$\begin{aligned}
\big|b\big(\big(\widetilde{\mathbf{v}}, \widetilde{\boldsymbol{\Xi}}\big), \big(\mathbf{u}^d, \mathbf{B}^d\big), \big(\widetilde{\mathbf{v}}, \widetilde{\boldsymbol{\Xi}}\big)\big)\big| &\leq c_b \big\|\big(\widetilde{\mathbf{v}}, \widetilde{\boldsymbol{\Xi}}\big)\big\|_{\mathbf{H}} \big\|\big(\widetilde{\mathbf{v}}, \widetilde{\boldsymbol{\Xi}}\big)\big\|_{\mathbf{V}} \big\|\big(\mathbf{u}^d, \mathbf{B}^d\big)\big\|_{L^\infty(0,T;\mathbf{V})} \\
&\leq \frac{1}{2}\big\|\big(\widetilde{\mathbf{v}}, \widetilde{\boldsymbol{\Xi}}\big)\big\|_{\mathbf{V}}^2 + \frac{c_b^2}{2}\big\|\big(\mathbf{u}^d, \mathbf{B}^d\big)\big\|_{L^\infty(0,T;\mathbf{V})}^2 \big\|\big(\widetilde{\mathbf{v}}, \widetilde{\boldsymbol{\Xi}}\big)\big\|_{\mathbf{H}}^2
\end{aligned}$$

so that from (12) and the inequality (2), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|\big(\widetilde{\mathbf{v}}(t),\widetilde{\boldsymbol{\Xi}}(t)\big)\big\|_{\mathbf{H}}^2+\Big(\gamma+\kappa-c_b^2\big\|\big(\mathbf{u}^d,\mathbf{B}^d\big)\big\|_{L^\infty(0,T;\mathbf{V})}^2\Big)\big\|\big(\widetilde{\mathbf{v}}(t),\widetilde{\boldsymbol{\Xi}}(t)\big)\big\|_{\mathbf{H}}^2\le 0.$$

Therefore if γ satisfies $\gamma>\omega$ we can apply Gronwall's inequality to obtain

$$\big\|\big(\widetilde{\mathbf{v}},\widetilde{\boldsymbol{\Xi}}\big)\big\|_{\mathbf{H}}^2\le\big\|\big(\mathbf{u}_0-\mathbf{u}_0^d,\mathbf{B}_0-\mathbf{B}_0^d\big)\big\|_{\mathbf{H}}^2\,e^{-(\gamma-\omega)t}$$

which proves inequality (8). Moreover, we see that for each $T\in(0,\infty]$, by (8),

$$\begin{aligned}&\mathscr{J}_T\big(\widetilde{\mathbf{v}},\widetilde{\boldsymbol{\Xi}},\widetilde{\mathbf{h}},\widetilde{\mathbf{g}}\big)\\&\quad\le\frac{\alpha}{2}\int_0^T\big\|\big(\widetilde{\mathbf{v}},\widetilde{\boldsymbol{\Xi}}\big)\big\|_{\mathbf{H}}^2\,\mathrm{d}t+\frac{\beta_1\gamma^2}{8}\int_0^T\|\widetilde{\mathbf{v}}\|^2\,\mathrm{d}t+\frac{\beta_2\gamma^2}{8}\int_0^T\big\|\widetilde{\boldsymbol{\Xi}}\big\|^2\,\mathrm{d}t\\&\quad\le\frac{\big(4\alpha+\gamma^2\max\big\{\beta_1,\frac{\beta_2}{S}\big\}\big)}{8(\gamma-\omega)}\big\|\big(\mathbf{u}_0-{\mathbf{u}^d}_0,\mathbf{B}_0-\mathbf{B}_0^d\big)\big\|_{\mathbf{H}}^2\big(1-e^{-(\gamma-\omega)T}\big)\end{aligned}$$

which proves the inequality (9). □

We note here that from the results of Theorem 1, the quasi-optimizer constructed there satisfies $\|(\widetilde{\mathbf{u}},\widetilde{\mathbf{B}})(t)-(\mathbf{u}^d,\mathbf{B}^d)\|_{\mathbf{H}}\to 0$ as $t\to\infty$ exponentially and $\mathscr{J}_\infty(\widetilde{\mathbf{u}},\widetilde{\mathbf{B}},\widetilde{\mathbf{f}},\operatorname{curl}\widetilde{\mathbf{j}})$ is bounded. Also, it is quite straight forward to compute this feedback solution via an initial value solve. However, since the value of γ may be large, the work required to implement this control may be large. We will later show that the true optimizer only satisfies $\|(\widetilde{\mathbf{u}},\widetilde{\mathbf{B}})(t)-(\mathbf{u}^d,\mathbf{B}^d)(t)\|_{\mathbf{H}}\to 0$ as $t\to\infty$ but it also minimizes the work involved to realize it.

2.2 Estimates for the dynamics of admissible elements

In this section, we will derive some preliminary estimates for the dynamics of all solutions of (3), (4). These estimates will allow us to derive preliminary estimates for the dynamics of the optimal solutions. We begin with the $\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))$ estimates in terms of the initial data and the functional values.

Theorem 2. *Let $T\in(0,\infty]$. Assume that $(\mathbf{u},\mathbf{B},\mathbf{f},\operatorname{curl}\mathbf{j})\in\mathbf{U}_{\mathrm{ad}}(T)$. Then $\forall t\in[0,T]$,*

$$\begin{aligned}\big\|\big(\mathbf{u}-\mathbf{u}^d,\mathbf{B}-\mathbf{B}^d\big)\big\|_{\mathbf{H}}^2&\le\big\|\big(\mathbf{u}_0-\mathbf{u}_0^d,\mathbf{B}_0-\mathbf{B}_0^d\big)\big\|_{\mathbf{H}}^2\\&\quad+2\max\left\{\frac{\big(c_b^2\big\|\big(\mathbf{u}^d,\mathbf{B}^d\big)\big\|_{\mathbf{V}}^2+1\big)}{\alpha_1},\frac{\big(c_b^2\big\|\big(\mathbf{u}^d,\mathbf{B}^d\big)\big\|_{\mathbf{V}}^2+1\big)}{\alpha_2},\frac{1}{\beta_1},\frac{S}{\beta_2}\right\}\\&\quad\times\mathscr{J}_T(\mathbf{u},\mathbf{B},\mathbf{f},\operatorname{curl}\mathbf{j}).\end{aligned}\tag{13}$$

If in addition, $\mathscr{J}_T(\mathbf{u},\mathbf{B},\mathbf{f},\operatorname{curl}\mathbf{j})\le\mathscr{J}_T(\widetilde{\mathbf{u}},\widetilde{\mathbf{B}},\widetilde{\mathbf{f}},\operatorname{curl}\widetilde{\mathbf{j}})$, then

$$\big\|\big(\mathbf{u}-\mathbf{u}^d,\mathbf{B}-\mathbf{B}^d\big)\big\|_{\mathbf{H}}^2\le c_1\big\|\big(\mathbf{u}_0-\mathbf{u}_0^d,\mathbf{B}_0-\mathbf{B}_0^d\big)\big\|_{\mathbf{H}}^2,\tag{14}$$

where $c_1 := (1 + 2\max\{\frac{(c_b^2\|(\mathbf{u}^d,\mathbf{B}^d)\|_{\mathbf{V}}^2+1)}{\alpha_1}, \frac{(c_b^2\|(\mathbf{u}^d,\mathbf{B}^d)\|_{\mathbf{V}}^2+1)}{\alpha_2}, \frac{1}{\beta_1}, \frac{S}{\beta_2}\}c_0)$ *with* c_0 *defined in Theorem* 1.

Proof. Setting $\mathbf{w} = \mathbf{v}$ and $\boldsymbol{\Upsilon} = \boldsymbol{\Xi}$ in (6) and using the skew-symmetry and continuity properties of $b(\cdot,\cdot,\cdot)$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\|(\mathbf{v},\boldsymbol{\Xi})\|_{\mathbf{H}}^2 + \|(\mathbf{v},\boldsymbol{\Xi})\|_{\mathbf{V}}^2\\ &= -b\big((\mathbf{v},\boldsymbol{\Xi}),\big(\mathbf{u}^d,\mathbf{B}^d\big),(\mathbf{v},\boldsymbol{\Xi})\big) + \big((\mathbf{h},\mathbf{g}),(\mathbf{v},\boldsymbol{\Xi})\big)\\ &\le c_b\|(\mathbf{v},\boldsymbol{\Xi})\|_{\mathbf{H}}\|(\mathbf{v},\boldsymbol{\Xi})\|_{\mathbf{V}}\big\|\big(\mathbf{u}^d,\mathbf{B}^d\big)\big\|_{\mathbf{V}} + \|\mathbf{h}\|\|\mathbf{v}\| + S\|\mathbf{g}\|\|\boldsymbol{\Xi}\|\\ &\le \frac{c_b^2}{2}\|(\mathbf{v},\boldsymbol{\Xi})\|_{\mathbf{H}}^2\big\|\big(\mathbf{u}^d,\mathbf{B}^d\big)\big\|_{\mathbf{V}}^2 + \frac{1}{2}\|(\mathbf{v},\boldsymbol{\Xi})\|_{\mathbf{V}}^2 + \frac{1}{2}\|(\mathbf{h},\mathbf{g})\|_{\mathbf{H}}^2 + \frac{1}{2}\|(\mathbf{v},\boldsymbol{\Xi})\|_{\mathbf{H}}^2\end{aligned}$$

so that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&\|(\mathbf{v},\boldsymbol{\Xi})\|_{\mathbf{H}}^2 + \|(\mathbf{v},\boldsymbol{\Xi})\|_{\mathbf{V}}^2\\ &\le \max\left\{\frac{\big(c_b^2\big\|\big(\mathbf{u}^d,\mathbf{B}^d\big)\big\|_{\mathbf{V}}^2+1\big)}{\alpha_1}, \frac{\big(c_b^2\big\|\big(\mathbf{u}^d,\mathbf{B}^d\big)\big\|_{\mathbf{V}}^2+1\big)}{\alpha_2}, \frac{1}{\beta_1}, \frac{S}{\beta_2}\right\}\\ &\quad\times\big(\alpha_1\|\mathbf{v}\|^2 + S\alpha_2\|\boldsymbol{\Xi}\|^2 + \beta_1\|\mathbf{h}\|^2 + \beta_2\|\mathbf{g}\|^2\big).\end{aligned} \tag{15}$$

Multiplying this inequality by $e^{\kappa t}$ and integrating over $(0,t)$, we obtain

$$\begin{aligned}\|(\mathbf{v},\boldsymbol{\Xi})\|_{\mathbf{H}}^2 &\le \|(\mathbf{v}_0,\boldsymbol{\Xi}_0)\|_{\mathbf{H}}^2 e^{-\kappa t}\\ &\quad + 2\max\left\{\frac{\big(c_b^2\big\|\big(\mathbf{u}^d,\mathbf{B}^d\big)\big\|_{\mathbf{V}}^2+1\big)}{\alpha_1}, \frac{\big(c_b^2\big\|\big(\mathbf{u}^d,\mathbf{B}^d\big)\big\|_{\mathbf{V}}^2+1\big)}{\alpha_2}, \frac{1}{\beta_1}, \frac{S}{\beta_2}\right\}\\ &\quad\times\int_0^t\left(\frac{\alpha_1}{2}\|\mathbf{v}\|^2 + S\frac{\alpha_2}{2}\|\boldsymbol{\Xi}\|^2 + \frac{\beta_1}{2}\|\mathbf{h}\|^2 + \frac{\beta_2}{2}\|\mathbf{g}\|^2\right)e^{-\kappa(t-s)}\,\mathrm{d}s\end{aligned}$$

which yields (13). Finally (14) follows from (13) and (9). ☐

Theorem 3. *Let* $T \in (0,\infty]$ *and* $(\mathbf{u},\mathbf{B},\mathbf{f},\mathrm{curl}\,\mathbf{j}) \in \mathbf{U}_{\mathrm{ad}}(T)$. *Assume that* $\mathscr{J}_T(\mathbf{u},\mathbf{B},\mathbf{f},\mathrm{curl}\,\mathbf{j}) \le \mathscr{J}_T(\tilde{\mathbf{u}},\widetilde{\mathbf{B}},\widetilde{\mathbf{f}},\mathrm{curl}\,\widetilde{\mathbf{j}})$. *Then for each* $\epsilon > 0$,

$$\big(\mathbf{u}-\mathbf{u}^d,\mathbf{B}-\mathbf{B}^d\big) \in \mathrm{L}^2(0,T;\mathbf{V}) \cap \mathrm{L}^\infty(\epsilon,T;\mathbf{V}), \tag{16}$$

$$\int_0^T \big\|\big(\mathbf{u}(s)-\mathbf{u}^d(s),\mathbf{B}(s)-\mathbf{B}^d(s)\big)\big\|_{\mathbf{V}}^2\,\mathrm{d}s \le c_1\big\|\big(\mathbf{u}_0-\mathbf{u}_0^d,\mathbf{B}_0-\mathbf{B}_0^d\big)\big\|_{\mathbf{H}}^2 \tag{17}$$

and

$$\big\|(\mathbf{u},\mathbf{B}) - \big(\mathbf{u}^d,\mathbf{B}^d\big)\big\|_{\mathbf{V}}^2 \le \widetilde{c}_1(\epsilon)\big\|\big(\mathbf{u}_0-\mathbf{u}_0^d,\mathbf{B}_0-\mathbf{B}_0^d\big)\big\|_{\mathbf{H}}^2 \quad \forall t \ge \epsilon \tag{18}$$

where c_1 *is given in Theorem* 2 *and*

$$\begin{aligned}\widetilde{c}_1(\epsilon) :=& \exp\left\{\frac{6^3 c_a^4 c_1}{2}\left\|\left(\mathbf{u}_0-\mathbf{u}_0^d,\mathbf{B}_0-\mathbf{B}_0^d\right)\right\|_{\mathbf{H}}^4\right\}\\ &\times\left(\frac{c_1}{\epsilon}+\left[c_1 4\frac{c_a^2}{\sqrt{\kappa}}\left\|\left(\mathbf{u}^d,\mathbf{B}^d\right)\right\|_{\mathrm{L}^\infty(0,T;\mathbf{V})}\left\|\mathscr{A}\left(\mathbf{u}^d,\mathbf{B}^d\right)\right\|_{\mathrm{L}^\infty(0,T;\mathbf{H})}\right.\right.\\ &\qquad+8c_0\max\left\{\frac{1}{\beta_1},\frac{S}{\beta_2}\right\}\\ &\qquad\left.\left.+\frac{6^3}{2}c_a^4 c_1\left\|\left(\mathbf{u}^d,\mathbf{B}^d\right)\right\|_{\mathrm{L}^\infty(0,T;\mathbf{H})}^2\left\|\left(\mathbf{u}^d,\mathbf{B}^d\right)\right\|_{\mathrm{L}^\infty(0,T;\mathbf{V})}^2\right]\right).\end{aligned}$$

Proof. Let $T\in(0,\infty]$. We easily see that inequality (17) follows by integrating (15) over the time interval $(0,T)$. Let us now prove (18). Inserting $(\mathbf{w},\mathbf{\Upsilon})=\mathscr{A}(\mathbf{v},\mathbf{\Xi})$ in (6), yields

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\mathbf{v},\mathbf{\Xi})\|_{\mathbf{V}}^2+\|\mathscr{A}(\mathbf{v},\mathbf{\Xi})\|_{\mathbf{H}}^2\le& \left|b\big((\mathbf{v},\mathbf{\Xi}),(\mathbf{v},\mathbf{\Xi}),\mathscr{A}(\mathbf{v},\mathbf{\Xi})\big)\right|\\ &+\left|b\big((\mathbf{v},\mathbf{\Xi}),\left(\mathbf{u}^d,\mathbf{B}^d\right),\mathscr{A}(\mathbf{v},\mathbf{\Xi})\big)\right|\\ &+\left|b\big(\left(\mathbf{u}^d,\mathbf{B}^d\right),(\mathbf{v},\mathbf{\Xi}),\mathscr{A}(\mathbf{v},\mathbf{\Xi})\big)\right|\\ &+\left|\big((\mathbf{h},\mathbf{g}),\mathscr{A}(\mathbf{v},\mathbf{\Xi})\big)_{\mathbf{H}}\right|.\end{aligned}$$

Using the continuity and skew symmetry properties of the trilinear form $b(\cdot,\cdot,\cdot)$ (see § 1.1.1) yields that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\mathbf{v},\mathbf{\Xi})\|_{\mathbf{V}}^2+\|\mathscr{A}(\mathbf{v},\mathbf{\Xi})\|_{\mathbf{H}}^2\\ &\quad\le c_a\|(\mathbf{v},\mathbf{\Xi})\|_{\mathbf{V}}\|\mathscr{A}(\mathbf{v},\mathbf{\Xi})\|_{\mathbf{H}}^{\frac{3}{2}}\|(\mathbf{v},\mathbf{\Xi})\|_{\mathbf{H}}^{\frac{1}{2}}\\ &\qquad+c_a\|(\mathbf{v},\mathbf{\Xi})\|_{\mathbf{V}}^{\frac{1}{2}}\|(\mathbf{v},\mathbf{\Xi})\|_{\mathbf{H}}^{\frac{1}{2}}\left\|\left(\mathbf{u}^d,\mathbf{B}^d\right)\right\|_{\mathbf{V}}^{\frac{1}{2}}\left\|\mathscr{A}\left(\mathbf{u}^d,\mathbf{B}^d\right)\right\|_{\mathbf{H}}^{\frac{1}{2}}\|\mathscr{A}(\mathbf{v},\mathbf{\Xi})\|_{\mathbf{H}}\\ &\qquad+c_a\left\|\left(\mathbf{u}^d,\mathbf{B}^d\right)\right\|_{\mathbf{H}}^{\frac{1}{2}}\left\|\left(\mathbf{u}^d,\mathbf{B}^d\right)\right\|_{\mathbf{V}}^{\frac{1}{2}}\|(\mathbf{v},\mathbf{\Xi})\|_{\mathbf{V}}^{\frac{1}{2}}\|\mathscr{A}(\mathbf{v},\mathbf{\Xi})\|_{\mathbf{H}}^{\frac{3}{2}}\\ &\qquad+\left|\big((\mathbf{h},\mathbf{g}),\mathscr{A}(\mathbf{v},\mathbf{\Xi})\big)\right|.\end{aligned}$$

Using Young's inequality yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|(\mathbf{v},\mathbf{\Xi})\|_{\mathbf{V}}^2+\|\mathscr{A}(\mathbf{v},\mathbf{\Xi})\|_{\mathbf{H}}^2\le 2\Bigg[&2\|(\mathbf{h},\mathbf{g})\|_{\mathbf{H}}^2+\frac{6^3c_a^4}{4}\|(\mathbf{v},\mathbf{\Xi})\|_{\mathbf{H}}^2\|(\mathbf{v},\mathbf{\Xi})\|_{\mathbf{V}}^4+\\ &+\frac{2c_a^2}{\sqrt{\kappa}}\|(\mathbf{v},\mathbf{\Xi})\|_{\mathbf{V}}^2\left\|\left(\mathbf{u}^d,\mathbf{B}^d\right)\right\|_{\mathbf{V}}\left\|\mathscr{A}\left(\mathbf{u}^d,\mathbf{B}^d\right)\right\|_{\mathbf{H}}\\ &+\frac{6^3}{4}c_a^4\left\|\left(\mathbf{u}^d,\mathbf{B}^d\right)\right\|_{\mathbf{H}}^2\left\|\left(\mathbf{u}^d,\mathbf{B}^d\right)\right\|_{\mathbf{V}}^2\|(\mathbf{v},\mathbf{\Xi})\|_{\mathbf{V}}^2\Bigg].\end{aligned}$$

Thus using (14), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|(\mathbf{v},\boldsymbol{\Xi})\|_{\mathbf{V}}^2+\|\mathscr{A}(\mathbf{v},\boldsymbol{\Xi})\|_{\mathbf{H}}^2 \\
&\qquad\le 4\|(\mathbf{h},\mathbf{g})\|_{\mathbf{H}}^2+\left[\frac{6^3c_a^4c_1}{2}\|(\mathbf{v}_0,\boldsymbol{\Xi}_0)\|_{\mathbf{H}}^2\|(\mathbf{v},\boldsymbol{\Xi})\|_{\mathbf{V}}^2\right]\|(\mathbf{v},\boldsymbol{\Xi})\|_{\mathbf{V}}^2 \\
&\qquad\quad+\left[\frac{4c_a^2}{\sqrt{\kappa}}\big\|\big(\mathbf{u}^d,\mathbf{B}^d\big)\big\|_{\mathbf{V}}\big\|\mathscr{A}\big(\mathbf{u}^d,\mathbf{B}^d\big)\big\|_{\mathbf{H}}\right. \\
&\qquad\qquad\left.+\frac{6^3}{2}c_a^4\big\|\big(\mathbf{u}^d,\mathbf{B}^d\big)\big\|_{\mathbf{H}}^2\big\|\big(\mathbf{u}^d,\mathbf{B}^d\big)\big\|_{\mathbf{V}}^2\right]\|(\mathbf{v},\boldsymbol{\Xi})\|_{\mathbf{V}}^2.
\end{aligned}$$

We introduce

$$\begin{aligned}
y(t)&:=\|(\mathbf{v},\boldsymbol{\Xi})\|_{\mathbf{V}}^2,\\
g(t)&:=\left[\frac{6^3c_a^4c_1}{2}\|(\mathbf{v}_0,\boldsymbol{\Xi}_0)\|_{\mathbf{H}}^2\|(\mathbf{v},\boldsymbol{\Xi})\|_{\mathbf{V}}^2\right]
\end{aligned}$$

and

$$\begin{aligned}
h(t):=&\left[\frac{4c_a^2}{\sqrt{\kappa}}\big\|\big(\mathbf{u}^d,\mathbf{B}^d\big)\big\|_{\mathbf{V}}\big\|\mathscr{A}\big(\mathbf{u}^d,\mathbf{B}^d\big)\big\|_{\mathbf{H}}+\frac{6^3c_a^4}{2}\big\|\big(\mathbf{u}^d,\mathbf{B}^d\big)\big\|_{\mathbf{H}}^2\big\|\big(\mathbf{u}^d,\mathbf{B}^d\big)\big\|_{\mathbf{V}}^2\right]\\
&\times\|(\mathbf{v},\boldsymbol{\Xi})\|_{\mathbf{V}}^2+4\|(\mathbf{h},\mathbf{g})\|_{\mathbf{H}}^2.
\end{aligned}$$

For each $\epsilon>0$, by Theorem 2 and (17), we have

$$\begin{aligned}
&\int_t^{t+\epsilon} y(s)\,\mathrm{d}s\le c_1\big\|(\mathbf{u}_0,\mathbf{B}_0)-\big(\mathbf{u}_0^d,\mathbf{B}_0^d\big)\big\|_{\mathbf{H}}^2,\\
&\int_t^{t+\epsilon} g(s)\,\mathrm{d}s\le \frac{6^3c_a^4c_1}{2}\big\|(\mathbf{u}_0,\mathbf{B}_0)-\big(\mathbf{u}_0^d,\mathbf{B}_0^d\big)\big\|_{\mathbf{H}}^4
\end{aligned}$$

and

$$\begin{aligned}
\int_t^{t+\epsilon} h(s)\,\mathrm{d}s\le{}& 8\max\left\{\frac{1}{\beta_1},\frac{S}{\beta_2}\right\}\mathscr{J}_T\\
&+c_1\left[4\frac{c_a^2}{\sqrt{\kappa}}\big\|\big(\mathbf{u}^d,\mathbf{B}^d\big)\big\|_{\mathrm{L}^\infty(0,T;\mathbf{V})}\big\|\mathscr{A}\big(\mathbf{u}^d,\mathbf{B}^d\big)\big\|_{\mathrm{L}^\infty(0,T;\mathbf{H})}\right.\\
&\qquad\left.\frac{6^3}{2}c_a^4\big\|\big(\mathbf{u}^d,\mathbf{B}^d\big)\big\|_{\mathrm{L}^\infty(0,T;\mathbf{H})}^2\big\|\big(\mathbf{u}^d,\mathbf{B}^d\big)\big\|_{\mathrm{L}^\infty(0,T;\mathbf{V})}^2\right]\\
&\times\big\|(\mathbf{u}_0,\mathbf{B}_0)-\big(\mathbf{u}_0^d,\mathbf{B}_0^d\big)\big\|_{\mathbf{H}}^2,
\end{aligned}$$

so that by (9)

$$\int_t^{t+\epsilon} h(s)\,\mathrm{d}s$$
$$\leq \Bigg[c_1 4 \frac{c_a^2}{\sqrt{\kappa}} \big\|\big(\mathbf{u}^d, \mathbf{B}^d\big)\big\|_{\mathrm{L}^\infty(0,T;\mathbf{V})} \big\|\mathscr{A}\big(\mathbf{u}^d, \mathbf{B}^d\big)\big\|_{\mathrm{L}^\infty(0,T;\mathbf{H})} + 8c_0 \max\left\{\frac{1}{\beta_1}, \frac{S}{\beta_2}\right\}$$
$$+ \frac{6^3}{2} c_a^4 c_1 \big\|\big(\mathbf{u}^d, \mathbf{B}^d\big)\big\|^2_{\mathrm{L}^\infty(0,T;\mathbf{H})} \big\|\big(\mathbf{u}^d, \mathbf{B}^d\big)\big\|^2_{\mathrm{L}^\infty(0,T;\mathbf{V})} \Bigg]$$
$$\times \big\|(\mathbf{u}_0, \mathbf{B}_0) - \big(\mathbf{u}_0^d, \mathbf{B}_0^d\big)\big\|^2_{\mathbf{H}}.$$

Now (18) follows from the uniform Gronwall's inequality in Lemma 1 and the last three estimates for y, g and h. ☐

The following theorem giving preliminary estimates for the optimal solutions is a consequence of Theorem 2 and Theorem 3.

Theorem 4. *Let* $T \in (0,\infty]$ *and* $(\mathbf{u}^*, \mathbf{B}^*, \mathbf{f}^*, \mathrm{curl}\,\mathbf{j}^*) \in \mathbf{U}_{\mathrm{ad}}(T)$ *be an optimal solution for (OP-CON). Then*

$$\big\|(\mathbf{u}^*, \mathbf{B}^*) - \big(\mathbf{u}^d, \mathbf{B}^d\big)\big\|^2_{\mathbf{H}} \leq c_0 \big\|\big(\mathbf{u}_0 - \mathbf{u}_0^d, \mathbf{B}_0 - \mathbf{B}_0^d\big)\big\|^2_{\mathbf{H}},$$
$$\int_0^T \big\|(\mathbf{u}^*, \mathbf{B}^*) - \big(\mathbf{u}^d, \mathbf{B}^d\big)\big\|^2_{\mathbf{V}}\,\mathrm{d}s \leq c_1 \big\|\big(\mathbf{u}_0 - \mathbf{u}_0^d, \mathbf{B}_0 - \mathbf{B}_0^d\big)\big\|^2_{\mathbf{H}},$$
$$\big\|(\mathbf{u}^*, \mathbf{B}^*) - \big(\mathbf{u}^d, \mathbf{B}^d\big)\big\|^2_{\mathbf{V}} \leq \widetilde{c}_1(\epsilon) \big\|\big(\mathbf{u}_0 - \mathbf{u}_0^d, \mathbf{B}_0 - \mathbf{B}_0^d\big)\big\|^2_{\mathbf{H}}, \quad \forall t \geq \epsilon,$$

where all the constants are as defined in Theorems 2 *and* 3.

3 Existence of optimal control

3.1 Finite time interval

In this section, we prove the existence of optimal solutions and derive some estimates for the adjoint states.

Theorem 5. *Let* $T \in (0,\infty)$ *and* $(\mathbf{u}^d, \mathbf{B}^d, \mathbf{f}^d, \mathrm{curl}\,\mathbf{j}^d) \in \mathbf{U}_{\mathrm{ad}}(T)$. *Then there exists an optimal solution* $(\widehat{\mathbf{u}}, \widehat{\mathbf{B}}, \widehat{\mathbf{f}}, \mathrm{curl}\,\widehat{\mathbf{j}}) \in \mathbf{U}_{\mathrm{ad}}(T)$ *for optimal control problem (OP-CON). That is, there exists* $(\widehat{\mathbf{f}}, \mathrm{curl}\,\widehat{\mathbf{j}}) \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))^2$ *and* $(\widehat{\mathbf{u}}, \widehat{\mathbf{B}}) \in \mathrm{C}([0,T];\mathbf{H}) \cap \mathrm{L}^2(0,T;\mathbf{V})$ *of the optimal control problem.*

Proof. First note that the set $\mathbf{U}_{\mathrm{ad}}(T)$ is non empty, for e.g., $(\mathbf{u}, \mathbf{B}, \mathbf{0}, \mathbf{0}) \in \mathbf{U}_{\mathrm{ad}}$. Let $\{(\mathbf{f}_n, \mathrm{curl}\,\mathbf{j}_n)\}$ be a minimizing sequence for the optimal control problem and denote $(\mathbf{u}_n, \mathbf{B}_n) = (\mathbf{u}(\mathbf{f}_n, \mathrm{curl}\,\mathbf{j}_n), \mathbf{B}(\mathbf{f}_n, \mathrm{curl}\,\mathbf{j}_n))$. The sequence $\{(\mathbf{f}_n, \mathrm{curl}\,\mathbf{j}_n)\}$ is bounded in $\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))$ and the corresponding solution $\{(\mathbf{u}_n, \mathbf{B}_n)\}$ is bounded in $\mathrm{C}([0,T];\mathbf{H}) \cap$

$L^2(0,T;\mathbf{V})$, see [19]. Therefore we can find subsequences, again denoted by $\{(\mathbf{u}_n,\mathbf{B}_n)\}$ and $\{(\mathbf{f}_n,\operatorname{curl}\mathbf{j}_n)\}$ such that

$$\begin{aligned}
&\mathbf{f}_n \to \widehat{\mathbf{f}} && \text{weakly in } L^2\big(0,T;\mathbf{L}^2(\Omega)\big),\\
&\operatorname{curl}\mathbf{j}_n \to \operatorname{curl}\widehat{\mathbf{j}} && \text{weakly in } L^2\big(0,T;\mathbf{L}^2(\Omega)\big),\\
&(\mathbf{u}_n,\mathbf{B}_n) \to \big(\widehat{\mathbf{u}},\widehat{\mathbf{B}}\big) && \text{weakly in } L^2(0,T;\mathbf{V}),\\
&(\mathbf{u}_n,\mathbf{B}_n) \to \big(\widehat{\mathbf{u}},\widehat{\mathbf{B}}\big) && \text{weak* in } L^\infty(0,T;\mathbf{H}).
\end{aligned}$$

Using lower semi-continuity yields that

$$\int_0^T \left\|\widehat{\mathbf{u}}-\mathbf{u}^d\right\|^2 \mathrm{d}t \le \lim_{n\to\infty}\inf \int_0^T \left\|\mathbf{u}_n-\mathbf{u}^d\right\|^2 \mathrm{d}t,$$

$$\int_0^T \left\|\widehat{\mathbf{B}}-\mathbf{B}^d\right\|^2 \mathrm{d}t \le \lim_{n\to\infty}\inf \int_0^T \left\|\mathbf{B}_n-\mathbf{B}^d\right\|^2 \mathrm{d}t,$$

$$\int_0^T \left\|\widehat{\mathbf{f}}-\mathbf{f}^d\right\|^2 \mathrm{d}t \le \lim_{n\to\infty}\inf \int_0^T \left\|\mathbf{f}_n-\mathbf{f}^d\right\|^2 \mathrm{d}t,$$

and

$$\int_0^T \left\|\operatorname{curl}\widehat{\mathbf{j}}-\operatorname{curl}\mathbf{j}^d\right\|^2 \mathrm{d}t \le \lim_{n\to\infty}\inf \int_0^T \left\|\operatorname{curl}\mathbf{j}_n-\operatorname{curl}\mathbf{j}^d\right\|^2 \mathrm{d}t,$$

which implies that

$$\mathscr{J}_T\big(\widehat{\mathbf{u}},\widehat{\mathbf{B}},\widehat{\mathbf{f}},\operatorname{curl}\widehat{\mathbf{j}}\big) \le \lim_{n\to\infty}\inf \mathscr{J}_T(\mathbf{u}_n,\mathbf{B}_n,\mathbf{f}_n,\operatorname{curl}\mathbf{j}_n).$$

Now using the standard arguments as in the Navier-Stokes theory [20], we can show, by using a fractional time-order Sobolev space a priori estimates, that $\{(\mathbf{u}_n,\mathbf{B}_n)\}$ converges strongly in $L^2(0,T;\mathbf{H})$ and that $(\widehat{\mathbf{u}},\widehat{\mathbf{B}})$ satisfies the weak form of the MHD equations. □

Theorem 6. *Let $T\in(0,\infty)$ and $(\mathbf{f},\operatorname{curl}\mathbf{j})$ be in $L^2(0,T;L^2(\Omega))^2$. Then the mapping $(\mathbf{f},\operatorname{curl}\mathbf{j}) \mapsto (\mathbf{u},\mathbf{B})(\mathbf{f},\operatorname{curl}\mathbf{j})$ is Gateaux differentiable as a function from $L^2(0,T;L^2(\Omega))^2$ to $L^2(0,T;\mathbf{V})$. Furthermore its Gateaux derivative $((\bar{\mathbf{u}},\bar{\mathbf{B}}))(\boldsymbol{\phi},\operatorname{curl}\boldsymbol{\psi}) := \frac{\mathscr{D}(\mathbf{u},\mathbf{B})}{\mathscr{D}(\mathbf{f},\operatorname{curl}\mathbf{j})}\cdot(\boldsymbol{\phi},\operatorname{curl}\boldsymbol{\psi})$, for every $(\boldsymbol{\phi},\operatorname{curl}\boldsymbol{\psi})\in L^2(0,T;L^2(\Omega))^2$, is the solution of the linear problem*

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}\big(\bar{\mathbf{u}},\bar{\mathbf{B}}\big),(\mathbf{w},\boldsymbol{\Upsilon})\right)_{\mathbf{H}} + a\big((\bar{\mathbf{u}},\bar{\mathbf{B}}),(\mathbf{w},\boldsymbol{\Upsilon})\big) + b\big((\bar{\mathbf{u}},\bar{\mathbf{B}}),(\mathbf{u},\mathbf{B}),(\mathbf{w},\boldsymbol{\Upsilon})\big)\\
&\qquad + b\big((\bar{\mathbf{u}},\bar{\mathbf{B}}),(\mathbf{u},\mathbf{B}),(\mathbf{w},\boldsymbol{\Upsilon})\big) = \big((\boldsymbol{\phi},\operatorname{curl}\boldsymbol{\psi}),(\mathbf{w},\boldsymbol{\Upsilon})\big)_{\mathbf{H}} \quad \forall(\mathbf{w},\boldsymbol{\Upsilon})\in\mathbf{V}, \qquad (19)\\
&\big(\bar{\mathbf{u}}(0),\bar{\mathbf{B}}(0)\big) = (\mathbf{0},\mathbf{0}) \quad \textit{in } \mathbf{H},
\end{aligned}$$

where $(\bar{\mathbf{u}}, \bar{\mathbf{B}}) \in L^\infty(0,T;\mathbf{H}) \cap \mathrm{L}^2(0,T;\mathbf{V})$.

Proof. Let $(\mathbf{f}, \operatorname{curl}\mathbf{j})$ and $(\boldsymbol{\phi}, \operatorname{curl}\boldsymbol{\psi})$ be given in $\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))^2$. We need to prove the following result

$$\begin{aligned}\lim_{\epsilon\to 0}\Bigg\| &\frac{(\mathbf{u},\mathbf{B})\big((\mathbf{f},\operatorname{curl}\mathbf{j}) + \epsilon(\boldsymbol{\phi},\operatorname{curl}\boldsymbol{\psi})\big) - (\mathbf{u},\mathbf{B})(\mathbf{f},\operatorname{curl}\mathbf{j})}{\epsilon} \\ &- \frac{\epsilon\big(\bar{\mathbf{u}},\bar{\mathbf{B}}\big)(\boldsymbol{\phi},\operatorname{curl}\boldsymbol{\psi})}{\epsilon}\Bigg\|_{\mathrm{L}^2(0,T;\mathbf{V})} = 0.\end{aligned} \tag{20}$$

First note that $(\bar{\mathbf{u}}, \bar{\mathbf{B}}))((\boldsymbol{\phi}, \operatorname{curl}\boldsymbol{\psi}))$ clearly satisfies equation (19) by direct differentiation. The fact that $(\mathbf{u},\mathbf{B})((\mathbf{f}\operatorname{curl}\mathbf{j}) + \epsilon(\boldsymbol{\phi},\operatorname{curl}\boldsymbol{\psi}))$ and $(\mathbf{u},\mathbf{B})((\mathbf{f},\operatorname{curl}\mathbf{j}))$ are two weak solutions imply that $(\widehat{\mathbf{u}},\widehat{\mathbf{B}}) := (\mathbf{u},\mathbf{B})((\mathbf{f},\operatorname{curl}\mathbf{j}) + \epsilon(\boldsymbol{\phi},\operatorname{curl}\boldsymbol{\psi})) - (\mathbf{u},\mathbf{B})((\mathbf{f},\operatorname{curl}\mathbf{j}))$ satisfies

$$\begin{aligned}&\left(\frac{\partial}{\partial t}(\widehat{\mathbf{u}},\widehat{\mathbf{B}}),(\mathbf{w},\boldsymbol{\Upsilon})\right)_{\mathbf{H}} + a\big((\widehat{\mathbf{u}},\widehat{\mathbf{B}}),(\mathbf{w},\boldsymbol{\Upsilon})\big) + b\big((\widehat{\mathbf{u}},\widehat{\mathbf{B}}),(\mathbf{u},\mathbf{B}),(\mathbf{w},\boldsymbol{\Upsilon})\big) \\ &\qquad + b\big((\mathbf{u},\mathbf{B}),\big(\widehat{\mathbf{u}},\widehat{\mathbf{B}}\big),(\mathbf{w},\boldsymbol{\Upsilon})\big) = \epsilon\big((\boldsymbol{\phi},\operatorname{curl}\boldsymbol{\psi}),(\mathbf{w},\boldsymbol{\Upsilon})\big)_{\mathbf{H}} \quad \forall(\mathbf{w},\boldsymbol{\Upsilon}) \in \mathbf{V}, \\ &\big(\widehat{\mathbf{u}}(0),\widehat{\mathbf{B}}(0)\big) = (\mathbf{0},\mathbf{0})\end{aligned} \tag{21}$$

Setting $(\mathbf{w},\boldsymbol{\Upsilon}) = (\widehat{\mathbf{u}},\widehat{\mathbf{B}})$ in (21) and using the skew adjoint property of the trilinear form $b(\cdot,\cdot,\cdot)$ yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|(\widehat{\mathbf{u}},\widehat{\mathbf{B}})\big\|^2 + \big\|(\widehat{\mathbf{u}},\widehat{\mathbf{B}})\big\|_{\mathbf{V}}^2 + b\big((\widehat{\mathbf{u}},\widehat{\mathbf{B}}),(\mathbf{u},\mathbf{B}),\big(\widehat{\mathbf{u}},\widehat{\mathbf{B}}\big)\big) \\ &\qquad = \epsilon\big((\boldsymbol{\phi},\operatorname{curl}\boldsymbol{\psi}),\big(\widehat{\mathbf{u}},\widehat{\mathbf{B}}\big)\big)_{\mathbf{H}}.\end{aligned}$$

By using the continuity properties of the trilinear form and Young's inequality it follows that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|(\widehat{\mathbf{u}},\widehat{\mathbf{B}})\big\|_{\mathbf{H}}^2 + \frac{1}{2}\big\|(\widehat{\mathbf{u}},\widehat{\mathbf{B}})\big\|_{\mathbf{V}}^2 \\ &\qquad \le \frac{\epsilon^2}{2}\|(\boldsymbol{\phi},\operatorname{curl}\boldsymbol{\psi})\|_{\mathbf{H}}^2 + \left(\frac{1}{2} + \frac{c_b^2}{2}\|(\mathbf{u},\mathbf{B})\|_{\mathbf{V}}^2\right)\big\|(\widehat{\mathbf{u}},\widehat{\mathbf{B}})\big\|_{\mathbf{H}}^2.\end{aligned}$$

Therefore by Gronwall's inequality we get

$$\begin{aligned}&\big\|(\widehat{\mathbf{u}},\widehat{\mathbf{B}})\big\|_{L^\infty(0,T;\mathbf{H})}^2 \\ &\qquad \le \Big[\|(\boldsymbol{\phi},\operatorname{curl}\psi)\|_{L^\infty(0,T;\mathbf{H})}^2 \exp\big\{T + c_b^2\|(\mathbf{u},\mathbf{B})\|_{\mathrm{L}^2(0,T;\mathbf{V})}^2\big\}\Big]\epsilon^2 =: c_2\epsilon^2.\end{aligned}$$

Now integrating the preceding differential inequality yields

$$\begin{aligned}&\big\|(\widehat{\mathbf{u}},\widehat{\mathbf{B}})\big\|_{\mathrm{L}^2(0,T;\mathbf{V})}^2 \\ &\qquad \le \epsilon^2\Big[\|(\boldsymbol{\phi},\operatorname{curl}\psi)\|_{\mathrm{L}^2(0,T;\mathbf{H})}^2 + \big(1 + c_b^2\|(\mathbf{u},\mathbf{B})\|_{L^2(0,T;\mathbf{V})}^2\big)c_2\Big] =: \epsilon^2 c_3.\end{aligned} \tag{22}$$

Now note that $(\breve{\mathbf{u}}, \breve{\mathbf{B}}) := (\widehat{\mathbf{u}}, \widehat{\mathbf{B}}) - \epsilon(\bar{\mathbf{u}}, \bar{\mathbf{B}})$ satisfies the equation

$$\left(\frac{\partial}{\partial t}\big(\breve{\mathbf{u}}, \breve{\mathbf{B}}\big), (\mathbf{w}, \boldsymbol{\Upsilon})\right)_{\mathbf{H}} + a\big(\big(\breve{\mathbf{u}}, \breve{\mathbf{B}}\big), (\mathbf{w}, \boldsymbol{\Upsilon})\big) + b\big(\big(\breve{\mathbf{u}}, \breve{\mathbf{B}}\big), (\mathbf{u}, \mathbf{B}), (\mathbf{w}, \boldsymbol{\Upsilon})\big)$$
$$+ b\big((\mathbf{u}, \mathbf{B}), \big(\breve{\mathbf{u}}, \breve{\mathbf{B}}\big), (\mathbf{w}, \boldsymbol{\Upsilon})\big) = b\big(\big(\widehat{\mathbf{u}}, \widehat{\mathbf{B}}\big), \big(\widehat{\mathbf{u}}, \widehat{\mathbf{B}}\big), (\mathbf{w}, \boldsymbol{\Upsilon})\big) \quad \forall (\mathbf{w}, \boldsymbol{\Upsilon}) \in \mathbf{V}.$$

Setting $(\mathbf{w}, \boldsymbol{\Upsilon}) = (\breve{\mathbf{u}}, \breve{\mathbf{B}})$ in this equation and using the skew symmetric and continuity properties of the trilinear form we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|\big(\breve{\mathbf{u}}, \breve{\mathbf{B}}\big)\big\|_{\mathbf{H}}^2 + \big\|\big(\breve{\mathbf{u}}, \breve{\mathbf{B}}\big)\big\|_{\mathbf{V}}^2$$
$$\leq 2c_b^2\big\|\big(\breve{\mathbf{u}}, \breve{\mathbf{B}}\big)\big\|_{\mathbf{H}}^2\big\|\big(\mathbf{u}, \mathbf{B}\big)\big\|_{\mathbf{V}}^2 + 2c_b^2\big\|\big(\widehat{\mathbf{u}}, \widehat{\mathbf{B}}\big)\big\|_{\mathbf{H}}^2\big\|\big(\widehat{\mathbf{u}}, \widehat{\mathbf{B}}\big)\big\|_{\mathbf{V}}^2.$$

From this, by using Gronwall's inequality, we obtain

$$\big\|\big(\breve{\mathbf{u}}, \breve{\mathbf{B}}\big)\big\|_{L^\infty(0,T;\mathbf{H})}^2 \leq \Big[2c_b^2 c_2 c_3 \exp\big\{2c_b^2\|(\mathbf{u}, \mathbf{B})\|_{L^2(0,T;\mathbf{V})}^2\big\}\Big]\epsilon^4 =: c_4\epsilon^4.$$

Integrating the preceding differential inequality we also obtain

$$\big\|\big(\breve{\mathbf{u}}, \breve{\mathbf{B}}\big)\big\|_{\mathrm{L}^2(0,T;\mathbf{V})}^2 \leq \Big[2c_b^2 c_4\|(\mathbf{u}, \mathbf{B})\|_{\mathrm{L}^2(0,T;\mathbf{V})}^2 + 2c_b^2 c_2 c_3\Big]\epsilon^4 =: c_5\epsilon^4. \tag{23}$$

This easily implies that

$$\Bigg\|\frac{(\mathbf{u}, \mathbf{B})\big((\mathbf{f}, \operatorname{curl}\mathbf{j}) + \epsilon(\boldsymbol{\phi}, \operatorname{curl}\boldsymbol{\psi})\big) - (\mathbf{u}, \mathbf{B})(\mathbf{f}, \operatorname{curl}\mathbf{j})}{\epsilon}$$
$$- \frac{\epsilon\big(\bar{\mathbf{u}}, \bar{\mathbf{B}}\big)(\mathbf{f}, \operatorname{curl}\boldsymbol{\psi})}{\epsilon}\Bigg\|_{\mathrm{L}^2(0,T;\mathbf{V})} \leq \sqrt{c_5}\,\epsilon.$$

Therefore the desired limit in (20) exists and thus the mapping is Gateaux differentiable. □

Lemma 2. *Assume* $T \in (0, \infty)$. *Let* $(\boldsymbol{\phi}, \operatorname{curl}\boldsymbol{\psi}) \in \mathrm{L}^2(0, T; \mathrm{L}^2(\Omega))^2$ *and let* $(\bar{\mathbf{u}}, \bar{\mathbf{B}})$ *be defined as in Theorem* 6. *Then, for every* $(\mathbf{e}, \mathbf{k}) \in \mathrm{L}^2(0, T; \mathrm{L}^2(\Omega))^2$, *we have*

$$\int_0^T \big((\mathbf{e}, \mathbf{k}), \big(\bar{\mathbf{u}}, \bar{\mathbf{B}}\big)\big)_{\mathbf{H}}\,\mathrm{d}t = \int_0^T \big((\boldsymbol{\phi}, \operatorname{curl}\boldsymbol{\psi}), (\boldsymbol{\zeta}, \boldsymbol{\Pi})\big)_{\mathbf{H}}\,\mathrm{d}t, \tag{24}$$

where $(\boldsymbol{\zeta}, \boldsymbol{\Pi}) \in \mathrm{L}^2(0, T; \mathbf{H}) \cap \mathrm{L}^2(0, T; \mathbf{V})$ *satisfies*

$$\begin{aligned} &-\left(\frac{\partial}{\partial t}(\boldsymbol{\zeta}, \boldsymbol{\Pi}), (\mathbf{w}, \boldsymbol{\Upsilon})\right)_{\mathbf{H}} + a\big((\boldsymbol{\zeta}, \boldsymbol{\Pi}), (\mathbf{w}, \boldsymbol{\Upsilon})\big) + b\big((\mathbf{w}, \boldsymbol{\Upsilon}), (\mathbf{u}, \mathbf{B}), (\boldsymbol{\zeta}, \boldsymbol{\Pi})\big) \\ &\quad + b\big((\mathbf{u}, \mathbf{B}), (\mathbf{w}, \boldsymbol{\Upsilon}), (\boldsymbol{\zeta}, \boldsymbol{\Pi})\big) = \big((\mathbf{e}, \mathbf{k}), (\mathbf{w}, \boldsymbol{\Upsilon})\big)_{\mathbf{H}} \quad \forall (\mathbf{w}, \boldsymbol{\Upsilon}) \in \mathbf{V}, \\ &\big(\boldsymbol{\zeta}(T), \boldsymbol{\Pi}(T)\big) = (\mathbf{0}, \mathbf{0}). \end{aligned} \tag{25}$$

Proof. We proceed as follows using integration by parts

$$
\begin{aligned}
\int_0^T & \big((\mathbf{e},\mathbf{k}),(\bar{\mathbf{u}},\bar{\mathbf{B}})\big)_{\mathbf{H}}\,\mathrm{d}t \\
&= \int_0^T \Bigg[-\left(\frac{\partial}{\partial t}(\boldsymbol{\zeta},\boldsymbol{\Pi}),(\bar{\mathbf{u}},\bar{\mathbf{B}})\right)_{\mathbf{H}} + a\big((\boldsymbol{\zeta},\boldsymbol{\Pi}),(\bar{\mathbf{u}},\bar{\mathbf{B}})\big) \\
&\qquad + b\big((\bar{\mathbf{u}},\bar{\mathbf{B}}),(\mathbf{u},\mathbf{B}),(\boldsymbol{\zeta},\boldsymbol{\Pi})\big) + b\big((\mathbf{u},\mathbf{B}),(\bar{\mathbf{u}},\bar{\mathbf{B}}),(\boldsymbol{\zeta},\boldsymbol{\Pi})\big)\Bigg]\,\mathrm{d}t \\
&= \int_0^T \Bigg[\Big(\frac{\partial}{\partial t}(\bar{\mathbf{u}},\bar{\mathbf{B}}),(\boldsymbol{\zeta},\boldsymbol{\Pi})\big)_{\mathbf{H}} + a\big((\boldsymbol{\zeta},\boldsymbol{\Pi}),(\bar{\mathbf{u}},\bar{\mathbf{B}})\big) \\
&\qquad + b\big((\bar{\mathbf{u}},\bar{\mathbf{B}}),(\mathbf{u},\mathbf{B}),(\boldsymbol{\zeta},\boldsymbol{\Pi})\big) + b\big((\mathbf{u},\mathbf{B}),(\bar{\mathbf{u}},\bar{\mathbf{B}}),(\boldsymbol{\zeta},\boldsymbol{\Pi})\big)\Bigg]\,\mathrm{d}t \\
&= \int_0^T \big((\boldsymbol{\phi},\operatorname{curl}\boldsymbol{\psi}),(\boldsymbol{\zeta},\boldsymbol{\Pi})\big)_{\mathbf{H}}\,\mathrm{d}t
\end{aligned}
$$

by (19). □

Theorem 7. *Let* $T \in (0,\infty)$ *and let* $(\widehat{\mathbf{u}},\widehat{\mathbf{B}},\widehat{\mathbf{f}},\operatorname{curl}\widehat{\mathbf{j}}) \in \mathbf{U}_{\mathrm{ad}}(T)$ *be a solution to the optimal control problem. Then the following equality holds:*

$$
\widehat{\boldsymbol{\zeta}} + \beta_1\big(\widehat{\mathbf{f}} - \mathbf{f}^d\big) = 0 \quad \textit{and} \quad \widehat{\boldsymbol{\Pi}} + \beta_2\big(\operatorname{curl}\widehat{\mathbf{j}} - \operatorname{curl}\mathbf{j}^d\big) = 0, \tag{26}
$$

where adjoint state variables $(\widehat{\boldsymbol{\zeta}},\widehat{\boldsymbol{\Pi}}) \in L^\infty(0,T;\mathbf{H}) \cap \mathrm{L}^2(0,T;\mathbf{V})$ *is the solution of the linear adjoint problem:*

$$
\begin{aligned}
&-\left(\frac{\partial}{\partial t}(\widehat{\boldsymbol{\zeta}},\widehat{\boldsymbol{\Pi}}),(\mathbf{w},\boldsymbol{\Upsilon})\right)_{\mathbf{H}} + a\big((\widehat{\boldsymbol{\zeta}},\widehat{\boldsymbol{\Pi}}),(\mathbf{w},\boldsymbol{\Upsilon})\big) + b\big((\mathbf{w},\boldsymbol{\Upsilon}),(\widehat{\mathbf{u}},\widehat{\mathbf{B}}),(\widehat{\boldsymbol{\zeta}},\widehat{\boldsymbol{\Pi}})\big) \\
&+ b\big((\widehat{\mathbf{u}},\widehat{\mathbf{B}}),(\mathbf{w},\boldsymbol{\Upsilon}),(\widehat{\boldsymbol{\zeta}},\widehat{\boldsymbol{\Pi}})\big) \\
&\qquad = \Big(\big(\alpha_1(\widehat{\mathbf{u}} - \mathbf{u}^d),\alpha_2(\widehat{\mathbf{B}} - \mathbf{B}^d)\big),(\mathbf{w},\boldsymbol{\Upsilon})\Big)_{\mathbf{H}} \quad \forall(\mathbf{w},\boldsymbol{\Upsilon}) \in \mathbf{V}, \\
&\big(\widehat{\boldsymbol{\zeta}}(T),\widehat{\boldsymbol{\Pi}}(T)\big) = (\mathbf{0},\mathbf{0}) \quad \textit{in } \mathbf{H}.
\end{aligned} \tag{27}
$$

Proof. Let $(\widehat{\mathbf{u}},\widehat{\mathbf{B}},\widehat{\mathbf{f}},\operatorname{curl}\widehat{\mathbf{j}})$ be an optimal solution. Then the Gateaux derivative

$$
\begin{aligned}
&\frac{\mathscr{D}\mathscr{J}\big(\widehat{\mathbf{u}},\widehat{\mathbf{B}},\widehat{\mathbf{f}},\operatorname{curl}\widehat{\mathbf{j}}\big)}{\mathscr{D}\big(\widehat{\mathbf{f}},\operatorname{curl}\widehat{\mathbf{j}}\big)} \cdot (\boldsymbol{\phi},\operatorname{curl}\boldsymbol{\psi}) \\
&\qquad = \int_0^T \int_\Omega \big[\alpha_1(\widehat{\mathbf{u}} - \mathbf{u}^d)\bar{\mathbf{u}} + S\alpha_2(\widehat{\mathbf{B}} - \mathbf{B}^d)\bar{B}\big] \\
&\qquad\qquad + \beta_2\big[(\operatorname{curl}\widehat{\mathbf{j}} - \operatorname{curl}\mathbf{j}^d)\operatorname{curl}\boldsymbol{\psi}\big] + \beta_1\big[(\widehat{\mathbf{f}} - \mathbf{f}^d)\boldsymbol{\phi}\big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t,
\end{aligned}
$$

where $(\bar{\mathbf{u}}, \bar{\mathbf{B}}, \operatorname{curl}\boldsymbol{\psi})$ is the solution of the system (19). Since $(\widehat{\mathbf{u}}, \widehat{\mathbf{B}}, \widehat{\mathbf{f}}, \operatorname{curl}\widehat{\mathbf{j}})$ is an optimal solution and the Gateaux derivative exists, this derivative must be zero in all directions $(\boldsymbol{\phi}, \operatorname{curl}\boldsymbol{\psi}) \in \mathrm{L}^2(0,T;\mathbf{H})$. Therefore by setting $(\mathbf{e},\mathbf{k}) = (\alpha_1(\widehat{\mathbf{u}} - \mathbf{u}^d), \alpha_2(\widehat{\mathbf{B}} - \mathbf{B}^d))$ in (24) yields

$$\int_0^T \int_\Omega \big[\widehat{\boldsymbol{\zeta}} + \beta_1\big(\widehat{\mathbf{f}} - \mathbf{f}^d\big)\big]\boldsymbol{\phi} + S\big[\widehat{\boldsymbol{\Pi}} + \beta_2\big(\operatorname{curl}\widehat{\mathbf{j}} - \operatorname{curl}\mathbf{j}^d\big)\big]\operatorname{curl}\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = 0.$$

This implies (26). □

For the adjoint state variables $(\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\Pi}})$, we obtain the following estimates on finite intervals.

Theorem 8. *For $T \in (0,\infty)$, let $(\widehat{\mathbf{u}}, \widehat{\mathbf{B}}, \widehat{\mathbf{f}}, \operatorname{curl}\widehat{\mathbf{j}}) \in \mathbf{U}_{\mathrm{ad}}(T)$ be a solution to (OP-CON) and that $(\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\Pi}}) \in L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V})$ be the solution to* (26), (27)*. Then, for each $\epsilon > 0$,*

$$\big\|(\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\Pi}})\big\|_{\mathbf{H}}^2 + \int_t^T \big\|(\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\Pi}})\big\|_{\mathbf{V}}^2 ds \le 2M\mathscr{J}_T \quad \forall t \in [\epsilon, T],$$

where

$$\rho_1(\epsilon) := \sqrt{\widetilde{c_1}(\epsilon)}\big\|(\mathbf{u}_0, \mathbf{B}_0) - \big(\mathbf{u}_0^d, \mathbf{B}_0^d\big)\big\|_{\mathbf{H}} + \big\|\big(\mathbf{u}^d, \mathbf{B}^d\big)\big\|_{L^\infty(0,\infty;\mathbf{V})} < \infty$$

and

$$M := \max\left\{\frac{\alpha_1}{\kappa}, \frac{\alpha_2}{\kappa}, C_b^2\rho_1^2(\epsilon)\beta_1, Sc_b^2\rho_1^2(\epsilon)\beta_2\right\}.$$

Proof. Setting $(\mathbf{w}, \boldsymbol{\Upsilon}) = (\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\Pi}})$ in (27), we have

$$\begin{aligned} -\frac{1}{2}\frac{d}{dt}&\big\|(\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\Pi}})\big\|_{\mathbf{H}}^2 + \big\|(\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\Pi}})\big\|_{\mathbf{V}}^2 \\ &= \big((\alpha_1\widehat{\mathbf{v}}, \alpha_2\boldsymbol{\Xi}), \big(\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\Pi}}\big)\big)_{\mathbf{H}} - b\big(\big(\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\Pi}}\big), \big(\widehat{\mathbf{u}}, \widehat{\mathbf{B}}\big), \big(\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\Pi}}\big)\big) \\ &\le \|(\alpha_1\widehat{\mathbf{v}}, \alpha_2\boldsymbol{\Xi})\|_{\mathbf{H}}\big\|(\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\Pi}})\big\|_{\mathbf{H}} + c_b\big\|(\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\Pi}})\big\|_{\mathbf{H}}\big\|(\widehat{\mathbf{u}}, \widehat{\mathbf{B}})\big\|_{\mathbf{V}}\big\|(\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\Pi}})\big\|_{\mathbf{V}}. \end{aligned}$$

Now using Young's inequality yields

$$-\frac{1}{2}\frac{d}{dt}\big\|(\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\Pi}})\big\|_{\mathbf{H}}^2 + \frac{1}{2}\big\|(\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\Pi}})\big\|_{\mathbf{V}}^2 \le \frac{1}{\kappa}\|(\alpha_1\widehat{\mathbf{v}}, \alpha_2\boldsymbol{\Xi})\|_{\mathbf{H}}^2 + c_b^2\big\|(\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\Pi}})\big\|_{\mathbf{H}}^2\big\|(\widehat{\mathbf{u}}, \widehat{\mathbf{B}})\big\|_{\mathbf{V}}^2.$$

From the estimate (18) and the triangle inequality $\|(\widehat{\mathbf{u}}, \widehat{\mathbf{B}})\|_{\mathbf{V}} \le \rho_1(\epsilon)$. Using this and (26) in the previous differential inequality yields

$$\begin{aligned} -\frac{1}{2}\frac{d}{dt}&\big\|(\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\Pi}})\big\|_{\mathbf{H}}^2 + \frac{1}{2}\big\|(\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\Pi}})\big\|_{\mathbf{V}}^2 \\ &\le \frac{1}{\kappa}\big[\alpha_1^2\|\widehat{\mathbf{v}}\|^2 + S\alpha_2^2\big\|\widehat{\boldsymbol{\Xi}}\big\|^2\big] + c_b^2\rho_1(\epsilon)^2\big[\beta_1^2\big\|\hat{\mathbf{h}}\big\|^2 + S\beta_2^2\|\hat{\mathbf{g}}\|^2\big] \\ &\le M\big[\alpha_1\|\widehat{\mathbf{v}}\|^2 + S\alpha_2\|\widehat{\boldsymbol{\Xi}}\|^2 + \beta_1\big\|\hat{\mathbf{h}}\big\|^2 + \beta_2\|\hat{\mathbf{g}}\|^2\big]. \end{aligned}$$

Integrating both sides over the interval (t, T) yields the desired estimate. □

3.2 The infinite time interval case

In this section, we will utilize the results of the previous section to prove the existence of optimal solutions to (OP-CON) in the infinite time interval.

Theorem 9. *There exists an optimal solution* $(\widehat{\mathbf{u}}, \widehat{\mathbf{B}}, \widehat{\mathbf{f}}, \operatorname{curl}\widehat{\mathbf{j}}) \in \mathbf{U}_{\mathrm{ad}}(\infty)$ *for (OP-CON) with* $T = \infty$.

Proof. For each $T \in (0, \infty)$, by using Theorem 4, we select a $(\mathbf{u}_T, \mathbf{B}_T, \mathbf{f}_T, \operatorname{curl}\mathbf{j}_T)$ which solves (OP-CON) and satisfies

$$\begin{aligned}
&\mathscr{J}_T(\mathbf{u}_T, \mathbf{B}_T, \mathbf{f}_T, \operatorname{curl}\mathbf{j}_T) \\
&\quad = \inf_{(\mathbf{w},\boldsymbol{\Upsilon},\boldsymbol{\phi},\operatorname{curl}\boldsymbol{\psi})\in\mathbf{U}_{\mathrm{ad}}(T)} \mathscr{J}(\mathbf{w}, \boldsymbol{\Upsilon}, \boldsymbol{\phi}, \operatorname{curl}\boldsymbol{\psi}), \ \ \forall(\mathbf{w}, \boldsymbol{\Upsilon}, \boldsymbol{\phi}, \operatorname{curl}\boldsymbol{\psi}) \in \mathbf{U}_{\mathrm{ad}}(T),\\
&\left(\frac{\partial}{\partial t}(\mathbf{u}_T, \mathbf{B}_T), (\mathbf{w}, \boldsymbol{\Upsilon})\right)_{\mathbf{H}} + a\big((\mathbf{u}_T, \mathbf{B}_T), (\mathbf{w}, \boldsymbol{\Upsilon})\big) + b\big((\mathbf{u}_T, \mathbf{B}_T), (\mathbf{u}_T, \mathbf{B}_T), (\mathbf{w}, \boldsymbol{\Upsilon})\big)\\
&\quad = \big((\mathbf{f}_T, \operatorname{curl}\mathbf{j}_T), (\mathbf{w}, \boldsymbol{\Upsilon})\big)_{\mathbf{H}}, \quad \forall(\mathbf{w}, \boldsymbol{\Upsilon}) \in \mathbf{V}, \qquad (28)\\
&\big(\mathbf{u}_T(0), \mathbf{B}_T(0)\big) = (\mathbf{u}_0, \mathbf{B}_0) \quad \text{in} \quad \mathbf{H}. \qquad (29)
\end{aligned}$$

For each finite T, we obviously have $[\mathbf{U}_{\mathrm{ad}}(\infty)]|_{(0,T)} \subset \mathbf{U}_{\mathrm{ad}}(T)$. Therefore,

$$\begin{aligned}
\mathscr{J}_T(\mathbf{u}_T, \mathbf{B}_T, \mathbf{f}_T, \operatorname{curl}\mathbf{j}_T) \le \mathscr{J}_T(\mathbf{w}, \boldsymbol{\Upsilon}, \boldsymbol{\phi}, \operatorname{curl}\boldsymbol{\psi}) \le \mathscr{J}_\infty(\mathbf{w}, \boldsymbol{\Upsilon}, \boldsymbol{\phi}, \operatorname{curl}\boldsymbol{\psi})\\
\forall(\mathbf{w}, \boldsymbol{\Upsilon}, \boldsymbol{\phi}, \operatorname{curl}\boldsymbol{\psi}) \in \mathbf{U}_{\mathrm{ad}}(\infty).
\end{aligned} \qquad (30)$$

Since $\mathscr{J}_\infty(\widetilde{\mathbf{u}}, \widetilde{\mathbf{B}}, \widetilde{\mathbf{f}}, \operatorname{curl}\widetilde{\mathbf{j}}) < \infty$, $(\widetilde{\mathbf{u}}, \widetilde{\mathbf{B}}, \widetilde{\mathbf{f}}, \operatorname{curl}\widetilde{\mathbf{j}}) \in \mathbf{U}_{\mathrm{ad}}(\infty)$, where $(\widetilde{\mathbf{u}}, \widetilde{\mathbf{B}}, \widetilde{\mathbf{f}}, \operatorname{curl}\widetilde{\mathbf{j}})$ is a quasi-optimizer constructed in Section 2. It then follows that

$$\mathscr{J}_T(\mathbf{u}_T, \mathbf{B}_T, \mathbf{f}_T, \operatorname{curl}\mathbf{j}_T) \le \inf_{(\mathbf{w},\boldsymbol{\Upsilon},\boldsymbol{\phi},\operatorname{curl}\boldsymbol{\psi})\in\mathbf{U}_{\mathrm{ad}}(\infty)} \mathscr{J}_\infty(\mathbf{w}, \boldsymbol{\Upsilon}, \boldsymbol{\phi}, \operatorname{curl}\boldsymbol{\psi}) < \infty.$$

For each integer $k > 0$, we denote by $(\mathbf{u}_k, \mathbf{B}_k, \mathbf{f}_k, \operatorname{curl}\mathbf{j}_k)$ a solution of (28), (29) for $T = k$. We set $(\mathbf{v}_k, \boldsymbol{\Xi}_k, \mathbf{h}_k, \mathbf{g}_k) = (\mathbf{u}_k - \mathbf{u}^d, \mathbf{B}_k - \mathbf{B}^d, \mathbf{f}_k - \mathbf{f}^d, \operatorname{curl}\mathbf{j}_k - \operatorname{curl}\mathbf{j}^d)$. Then $(\mathbf{v}_k, \boldsymbol{\Xi}_k, \mathbf{h}_k, \mathbf{g}_k)$ satisfies (6), (7) with $T = k$. Using the standard estimates of the MHD equations on the finite time interval and (30), we obtain that $\|\mathbf{f}_k\|_{\mathrm{L}^2(0,k;\mathbf{L}^2(\Omega))}$, $\|\operatorname{curl}\mathbf{j}_k\|_{\mathrm{L}^2(0,k;\mathbf{L}^2(\Omega))}$, $\|(\mathbf{u}_k, \mathbf{B}_k)\|_{\mathbf{W}^{(1)}(Q_T)}$ and $\|(\mathbf{u}_k, \mathbf{B}_k)\|_{\mathrm{L}^\infty(0,k;\mathbf{H})}$ are uniformly bounded for all k. By induction, we can choose successive subsequences of positive integers $\{k_n^{(m)}\}_{n=1}^\infty$ for $m = 1, 2, \ldots$ such that $\{k_n^{(1)}\}_{n=1}^\infty \supset \{k_n^{(2)}\}_{n=1}^\infty \supset \ldots$ and

$$\begin{aligned}
&(\mathbf{v}_{k_n^{(m)}}, \boldsymbol{\Xi}_{k_n^{(m)}}) \rightharpoonup \big(\mathbf{v}^{(m)}, \boldsymbol{\Xi}^{(m)}\big) \quad \text{in} \ \ \mathbf{W}^{(1)}(Q_m) \ \ \text{as} \ \ n \to \infty,\\
&(\mathbf{v}_{k_n^{(m)}}, \boldsymbol{\Xi}_{k_n^{(m)}}) \overset{*}{\rightharpoonup} \big(\mathbf{v}^{(m)}, \boldsymbol{\Xi}^{(m)}\big) \quad \text{in} \ \ \mathrm{L}^\infty(0, m; \mathbf{H}_{\mathbf{u}}) \ \ \text{as} \ \ n \to \infty,\\
&\mathbf{h}_{k_n^{(m)}} \rightharpoonup \mathbf{h}^{(m)} \quad \text{in} \ \ \mathrm{L}^2\big(0, m; \mathrm{L}^2(\Omega)\big) \ \ \text{as} \ \ n \to \infty,
\end{aligned}$$

and

$$\mathbf{g}_{k_n^{(m)}} \rightharpoonup \mathbf{g}^{(m)} \quad \text{in } \mathrm{L}^2\big(0,m;\mathrm{L}^2(\Omega)\big) \text{ as } n \to \infty$$

for some

$$\big(\mathbf{v}^{(m)},\boldsymbol{\Xi}^{(m)}\big) \in \mathbf{W}^{(1)}(Q_m), \quad \mathbf{h}^{(m)} \in \mathrm{L}^2\big(0,m;\mathrm{L}^2(\Omega)\big) \text{ and } \mathbf{g}^{(m)} \in \mathrm{L}^2\big(0,m;\mathrm{L}^2(\Omega)\big).$$

Hence, by extracting the diagonal subsequence, we have that for each m',

$$(\mathbf{v}_{k_n^{(m)}},\boldsymbol{\Xi}_{k_n^{(m)}}) \rightharpoonup \big(\mathbf{v}^{(m')},\boldsymbol{\Xi}^{(m')}\big) \quad \text{in } \mathbf{W}^{(1)}(Q_{m'}) \text{ as } m \to \infty, \tag{31}$$

$$(\mathbf{v}_{k_n^{(m)}},\boldsymbol{\Xi}_{k_n^{(m)}}) \overset{*}{\rightharpoonup} \big(\mathbf{v}^{(m')},\boldsymbol{\Xi}^{(m')}\big) \quad \text{in } \mathrm{L}^\infty(0,m';\mathbf{H}_\mathbf{u} \text{ as } m \to \infty, \tag{32}$$

$$\mathbf{h}_{k_m}^{(m)} \rightharpoonup \mathbf{h}^{(m')} \text{ and } \mathbf{g}_{k_m}^{(m)} \rightharpoonup \mathbf{g}^{(m')} \quad \text{in } \mathrm{L}^2\big(0,m';\mathrm{L}^2(\Omega)\big) \text{ as } m \to \infty. \tag{33}$$

For each $m' > 0$, using (31)–(33), standard techniques for the MHD equations, compactness results and density arguments (see [19]) allow us to pass to the limit as $m \to \infty$ in the equation below

$$\begin{aligned}
&\int_0^\infty \Bigg[\bigg(\frac{\partial}{\partial t}(\mathbf{v}_{k_m{}^{(m)}},\boldsymbol{\Xi}_{k_m{}^{(m)}}),(\mathbf{w},\boldsymbol{\Upsilon}) \bigg)_{\mathbf{H}} \chi(t) + a\big((\mathbf{v}_{k_m{}^{(m)}},\boldsymbol{\Xi}_{k_m{}^{(m)}}), S(\mathbf{w},\boldsymbol{\Upsilon})\big)\chi(t) \\
&\quad + b\big((\mathbf{v}_{k_m{}^{(m)}},\boldsymbol{\Xi}_{k_m{}^{(m)}}),(\mathbf{v}_{k_m{}^{(m)}},\boldsymbol{\Xi}_{k_m{}^{(m)}}),(\mathbf{w},\boldsymbol{\Upsilon})\big)\chi(t) \\
&\quad + b\big((\mathbf{v}_{k_m{}^{(m)}},\boldsymbol{\Xi}_{k_m{}^{(m)}}),\big(\mathbf{u}^d,\mathbf{B}^d\big),(\mathbf{w},\boldsymbol{\Upsilon})\big)\chi(t) \\
&\quad + b\big(\big(\mathbf{u}^d,\mathbf{B}^d\big),(\mathbf{v}_{k_m{}^{(m)}},\boldsymbol{\Xi}_{k_m{}^{(m)}}),(\mathbf{w},\boldsymbol{\Upsilon})\big)\chi(t) \Bigg] \,\mathrm{d}t \\
&= \int_0^\infty (\mathbf{h}_{k_m{}^{(m)}},\mathbf{g}_{k_m{}^{(m)}}),(\mathbf{w},\boldsymbol{\Upsilon})_{\mathbf{H}}\chi(t)\,\mathrm{d}t \quad \forall(\mathbf{w},\boldsymbol{\Upsilon}) \in \mathbf{V}, \;\; \chi \in C_0^\infty\big((0,m')\big)
\end{aligned} \tag{34}$$

to obtain

$$\begin{aligned}
&\bigg(\frac{\partial}{\partial t}\big(\mathbf{u}^{(m')},\mathbf{B}^{(m')}\big),(\mathbf{w},\boldsymbol{\Upsilon}) \bigg)_{\mathbf{H}} + a\big(\big(\mathbf{u}^{(m')},\mathbf{B}^{(m')}\big),\mathbf{w},\boldsymbol{\Upsilon}\big) \\
&\quad + b\big(\big(\mathbf{u}^{(m')},\mathbf{B}^{(m')}\big),\big(\mathbf{u}^{(m')},\mathbf{B}^{(m')}\big),(\mathbf{w},\boldsymbol{\Upsilon})\big) \\
&\qquad = \big(\big(\mathbf{f}^{(m')},\operatorname{curl}\mathbf{j}^{(m')}\big),(\mathbf{w},\boldsymbol{\Upsilon})\big)_{\mathbf{H}} \quad \forall(\mathbf{w},\boldsymbol{\Upsilon}) \in \mathbf{V}, \;\; t \in (0,m'),
\end{aligned} \tag{35}$$

where $(\mathbf{u}^{(m')},\mathbf{B}^{(m')}) := (\mathbf{v}^{(m')},\boldsymbol{\Xi}^{(m')}) + (\mathbf{u}^d,\mathbf{B}^d)$. For all m_1, m_2 with $m_1 < m_2$, we have that $(\mathbf{v}^{(m_1)},\boldsymbol{\Xi}^{(m_1)})|_{(0,m_2)} = (\mathbf{v}^{(m_2)},\boldsymbol{\Xi}^{(m_2)})$, $\mathbf{f}^{(m_1)}|_{(0,m_2)} = \mathbf{f}^{(m_2)}$ and $\operatorname{curl}\mathbf{j}^{(m_1)}|_{(0,m_2)} = \operatorname{curl}\mathbf{j}^{(m_2)}$ because of the uniqueness of weak limits. Therefore the functions $(\widehat{\mathbf{v}},\widehat{\boldsymbol{\Xi}}) := (\mathbf{u}^{(m)}(t),\mathbf{B}^{(m)}(t))$ if $t \le m$ and $(\widehat{\mathbf{f}},\operatorname{curl}\widehat{\mathbf{j}}) := (\mathbf{f}^{(m)}(t),\operatorname{curl}\mathbf{j}^{(m)}(t))$ if $t \le m$ are well defined on $(0,\infty)$, and furthermore, $(\widehat{\mathbf{u}},\widehat{\mathbf{B}}) \in \mathbf{W}^{(1)}_{\mathrm{loc}}(Q)$ and $(\widehat{\mathbf{f}},\operatorname{curl}\widehat{\mathbf{j}}) \in$

$\mathrm{L}^2(0,\infty;\mathbf{L}^2(\Omega))^2$. Upon setting $(\widehat{\mathbf{u}},\widehat{\mathbf{B}})=(\widehat{\mathbf{v}},\widehat{\boldsymbol{\Xi}})+(\mathbf{u}^d,\mathbf{B}^d)$, $\widehat{\mathbf{h}}+\mathbf{f}^d=\widehat{\mathbf{f}}$ and $\widehat{\mathbf{g}}+\operatorname{curl}\mathbf{j}^d=\operatorname{curl}\widehat{\mathbf{j}}$ and noting that m' is arbitrary in (35), we have

$$\begin{aligned}&\left(\frac{\partial}{\partial t}(\widehat{\mathbf{u}},\widehat{\mathbf{B}}),(\mathbf{w},\boldsymbol{\Upsilon})\right)_{\mathbf{H}}+a\big((\widehat{\mathbf{u}},\widehat{\mathbf{B}}),(\mathbf{w},\boldsymbol{\Upsilon})\big)+b\big((\widehat{\mathbf{u}},\widehat{\mathbf{B}}),(\widehat{\mathbf{u}},\widehat{\mathbf{B}}),(\mathbf{w},\boldsymbol{\Upsilon})\big)\\&\quad=\big((\widehat{\mathbf{f}},\operatorname{curl}\widehat{\mathbf{j}}),(\mathbf{w},\boldsymbol{\Upsilon})\big)_{\mathbf{H}}\ \ \forall(\mathbf{w},\boldsymbol{\Upsilon})\in\mathbf{V}\ \ \text{almost everywhere}\ \ t\in(0,\infty).\end{aligned}\tag{36}$$

We next examine the initial condition for $(\widehat{\mathbf{u}},\widehat{\mathbf{B}})$. The continuous embedding $\mathbf{W}^{(1)}(Q)\hookrightarrow\mathrm{C}([0,T];\mathbf{H})$ implies that $(\widehat{\mathbf{u}}(0),\widehat{\mathbf{B}}(0))$ is well-defined in $\mathbf{H}$. Replacing χ in (34) by a continuously differentiable function in $(0,\infty)$ with a bounded support, integrating by parts using the fact that $(\mathbf{u}_{k_m}^{(m)}(0),\mathbf{B}_{k_m}^{(m)}(0))=(\mathbf{u}_0,\mathbf{B}_0)$ and then passing to the limit, we obtain

$$\begin{aligned}&\int_0^\infty\Big[-\big((\widehat{\mathbf{u}},\widehat{\mathbf{B}}),(\mathbf{w},\boldsymbol{\Upsilon})\big)_{\mathbf{H}}\chi'(t)+a\big((\widehat{\mathbf{u}},\widehat{\mathbf{B}}),(\mathbf{w},\boldsymbol{\Upsilon})\big)\chi(t)\\&\qquad\qquad+b\big((\widehat{\mathbf{u}},\widehat{\mathbf{B}}),(\widehat{\mathbf{u}},\widehat{\mathbf{B}}),(\mathbf{w},\boldsymbol{\Upsilon})\big)\chi(t)\Big]\,\mathrm{d}t\\&\quad=\int_0^\infty\big((\widehat{\mathbf{f}},\operatorname{curl}\widehat{\mathbf{j}}),(\mathbf{w},\boldsymbol{\Upsilon})\big)_{\mathbf{H}}\chi(t)\,\mathrm{d}t+\big((\mathbf{u}_0,\mathbf{B}_0),(\mathbf{w},\boldsymbol{\Upsilon})\big)_{\mathbf{H}}\chi(0)\\&\qquad\qquad\forall(\mathbf{w},\boldsymbol{\Upsilon})\in\mathbf{V}.\end{aligned}\tag{37}$$

On the other hand, multiplying (36) by $\chi(t)$ and integrating by parts, we obtain

$$\begin{aligned}&\int_0^\infty\Big[-\big((\widehat{\mathbf{u}},\widehat{\mathbf{B}}),(\mathbf{w},\boldsymbol{\Upsilon})\big)_{\mathbf{H}}\chi'(t)+a\big((\widehat{\mathbf{u}},\widehat{\mathbf{B}}),(\mathbf{w},\boldsymbol{\Upsilon})\big)\chi(t)\\&\qquad\qquad+b\big((\widehat{\mathbf{u}},\widehat{\mathbf{B}}),(\widehat{\mathbf{u}},\widehat{\mathbf{B}}),(\mathbf{w},\boldsymbol{\Upsilon})\big)\chi(t)\Big]\,\mathrm{d}t\\&\quad=\int_0^\infty\big((\widehat{\mathbf{f}},\operatorname{curl}\widehat{\mathbf{j}}),(\mathbf{w},\boldsymbol{\Upsilon})\big)_{\mathbf{H}}\chi(t)\,\mathrm{d}t+\big((\widehat{\mathbf{u}}(0),\widehat{\mathbf{B}}(0)),(\mathbf{w},\boldsymbol{\Upsilon})\big)_{\mathbf{H}}\chi(0)\\&\qquad\qquad\forall(\mathbf{w},\boldsymbol{\Upsilon})\in\mathbf{V}.\end{aligned}\tag{38}$$

By comparing (37) with (38) and then choosing χ with $\chi(0)=1$, we have $(\widehat{\mathbf{u}}(0),\widehat{\mathbf{B}}(0))=(\mathbf{u}_0,\mathbf{B}_0)$ in $\mathbf{H}$. Finally, using the lower semi-continuity of the functional $\mathscr{J}_T(\cdot,\cdot)$ and the fact that $(\widehat{\mathbf{v}},\widehat{\boldsymbol{\Xi}})=(\widehat{\mathbf{u}},\widehat{\mathbf{B}})-(\mathbf{u}^d,\mathbf{B}^d)\in\mathrm{L}^2(0,\infty;\mathbf{V})$, $\widehat{\mathbf{h}}=\widehat{\mathbf{f}}-\mathbf{f}^d\in\mathrm{L}^2(0,\infty;\mathbf{L}^2(\Omega))$, and $\widehat{\mathbf{g}}=\operatorname{curl}\widehat{\mathbf{j}}-\operatorname{curl}\mathbf{j}^d\in\mathrm{L}^2(0,\infty;\mathbf{L}^2(\Omega))$, we obtain

$$\begin{aligned}\mathscr{J}_{k_m^{(m)}}\big(\widehat{\mathbf{u}},\widehat{\mathbf{B}},\widehat{\mathbf{f}},\operatorname{curl}\widehat{\mathbf{j}}\big)&\le\liminf_{m\to\infty}\mathscr{J}_{k_m^{(m)}}\big(\mathbf{u}_{k_m^{(m)}},\mathbf{B}_{k_m^{(m)}},\mathbf{f}_{k_m^{(m)}}^{(m)},\operatorname{curl}\mathbf{j}_{k_m^{(m)}}^{(m)}\big)\\&\le\mathscr{J}_\infty(\mathbf{w},\boldsymbol{\Upsilon},\phi,\operatorname{curl}\boldsymbol{\psi})\quad\forall(\mathbf{w},\boldsymbol{\Upsilon},\phi,\operatorname{curl}\boldsymbol{\psi})\in\mathbf{U}_{\mathrm{ad}}(\infty)\end{aligned}$$

so that by letting $m \to \infty$,

$$\mathscr{J}_\infty\big(\widehat{\mathbf{u}}, \widehat{\mathbf{B}}, \widehat{\mathbf{f}}, \operatorname{curl}\widehat{\mathbf{j}}\big) \le \mathscr{J}_\infty(\mathbf{w}, \boldsymbol{\Upsilon}, \boldsymbol{\phi}, \operatorname{curl}\boldsymbol{\psi}) \quad \forall(\mathbf{w}, \boldsymbol{\Upsilon}, \boldsymbol{\phi}, \operatorname{curl}\boldsymbol{\psi}) \in \mathbf{U}_{\mathrm{ad}}(\infty).$$

Hence we have proved that $(\widehat{\mathbf{u}}, \widehat{\mathbf{B}}, \widehat{\mathbf{f}}, \operatorname{curl}\widehat{\mathbf{j}})$ is the desired optimizer for (OP-CON) with $T = \infty$. □

4 Dynamics of optimal control solutions on the infinite time interval

Using the preliminary estimate (14) that $\|(\mathbf{u}(t), \mathbf{B}(t)) - (\mathbf{u}^d(t), \mathbf{B}^d(t))\|_{\mathbf{H}}$ stays bounded, we will prove much stronger result: $\|(\mathbf{u}(t), \mathbf{B}(t)) - (\mathbf{u}^d(t), \mathbf{B}^d(t))\|_{\mathbf{H}}$ approaches zero as $t \to \infty$.

Lemma 3. *Let $T \in (0, \infty]$. Assume that $(\mathbf{u}, \mathbf{B}, \mathbf{f}, \operatorname{curl}\mathbf{j}) \in \mathbf{U}_{\mathrm{ad}}(T)$. If $\|(\mathbf{u}(t), \mathbf{B}(t)) - (\mathbf{u}^d(t), \mathbf{B}^d(t))\|_{\mathbf{H}} > 0$ for all $t \in (t_1, t_2) \subset [0, T]$, then*

$$\begin{aligned} &\big\|\big(\mathbf{u}(t_2), \mathbf{B}(t_2)\big) - \big(\mathbf{u}^d(t_2), \mathbf{B}^d(t_2)\big)\big\|_{\mathbf{H}} \\ &\quad \le \big\|\big(\mathbf{u}(t_1), \mathbf{B}(t_1)\big) - \big(\mathbf{u}^d(t_1), \mathbf{B}^d(t_1)\big)\big\|_{\mathbf{H}} + c_6\sqrt{t_2 - t_1}\big(\mathscr{J}_T(\mathbf{u}, \mathbf{B}, \mathbf{f}, \operatorname{curl}\mathbf{j})\big)^{\frac{1}{2}}, \end{aligned} \tag{39}$$

where

$$c_6 := \left[\left(\frac{1}{\alpha_1} + \frac{S}{\alpha_2}\right)\frac{c_b^4}{4}\big\|\big(\mathbf{u}^d, \mathbf{B}^d\big)\big\|_{\mathbf{V}}^4 + \frac{1}{\beta_1} + \frac{S^2}{\beta_2}\right]^{\frac{1}{2}}.$$

If, in addition, $\mathscr{J}_T(\mathbf{u}, \mathbf{B}, \mathbf{f}, \operatorname{curl}\mathbf{j}) \le \mathscr{J}_T(\widetilde{\mathbf{u}}, \widetilde{\mathbf{B}}, \widetilde{\mathbf{f}}, \operatorname{curl}\widetilde{\mathbf{j}})$, where $(\widetilde{\mathbf{u}}, \widetilde{\mathbf{B}}, \widetilde{\mathbf{f}}, \operatorname{curl}\widetilde{\mathbf{j}})$, is as defined in Theorem 5, then

$$\begin{aligned} &\big\|\big(\mathbf{u}(t_2), \mathbf{B}(t_2)\big) - \big(\mathbf{u}^d(t_2), \mathbf{B}^d(t_2)\big)\big\|_{\mathbf{H}} \\ &\quad \le \big\|\big(\mathbf{u}(t_1), \mathbf{B}(t_1)\big) - \big(\mathbf{u}^d(t_1), \mathbf{B}^d(t_1)\big)\big\|_{\mathbf{H}} + c_6\sqrt{t_2 - t_1}\big\|(\mathbf{u}_0, \mathbf{B}_0) - \big(\mathbf{u}_0^d, \mathbf{B}_0^d\big)\big\|_{\mathbf{H}}, \end{aligned} \tag{40}$$

where ω is as defined in Theorem 1.

Proof. Setting $(\mathbf{w}, \boldsymbol{\Upsilon}) = (\mathbf{v}, \boldsymbol{\Xi})$ in (6) and using the skew symmetric property of the trilinear form yields

$$\begin{aligned} &\|(\mathbf{v}, \boldsymbol{\Xi})\|_{\mathbf{H}}\frac{\mathrm{d}}{\mathrm{d}t}\|(\mathbf{v}, \boldsymbol{\Xi})\|_{\mathbf{H}} + \|\mathbf{v}, \boldsymbol{\Xi}\|_{\mathbf{V}}^2 + b\big((\mathbf{v}, \boldsymbol{\Xi}), \big(\mathbf{u}^d, \mathbf{B}^d\big), (\mathbf{v}, \boldsymbol{\Xi})\big) \\ &\qquad = \big((\mathbf{h}, \mathbf{g}), (\mathbf{v}, \boldsymbol{\Xi})\big)_{\mathbf{H}} \end{aligned} \tag{41}$$

Using the continuity property of the trilinear form and Young's inequality yields

$$\begin{aligned} &\|(\mathbf{v}, \boldsymbol{\Xi})\|_{\mathbf{H}}\frac{\mathrm{d}}{\mathrm{d}t}\|(\mathbf{v}, \boldsymbol{\Xi})\|_{\mathbf{H}} + \frac{1}{2}\|\mathbf{v}, \boldsymbol{\Xi}\|_{\mathbf{V}}^2 \\ &\qquad \le (\|\mathbf{h}\| + S\|\mathbf{g}\|)\|(\mathbf{v}, \boldsymbol{\Xi})\|_{\mathbf{H}} + \frac{c_b^2}{2}\|(\mathbf{v}, \boldsymbol{\Xi})\|_{\mathbf{H}}^2\big\|\big(\mathbf{u}^d, \mathbf{B}^d\big)\big\|_{\mathbf{V}}^2. \end{aligned}$$

If $\|(\mathbf{v}(t), \Xi(t))\|_{\mathbf{H}} > 0$ for all $t \in (t_1, t_2)$, then we may divide this inequality by $\|(\mathbf{v}(t), \Xi(t))\|_{\mathbf{H}}$ to obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|(\mathbf{v}, \Xi)\|_{\mathbf{H}} + \frac{\kappa}{2}\|(\mathbf{v}, \Xi)\|_{\mathbf{H}} \\
&\quad \le (\|\mathbf{h}\| + S\|\mathbf{g}\|) + \frac{c_b^2}{2}\|(\mathbf{v}, \Xi)\|_{\mathbf{H}} \big\|\big(\mathbf{u}^d, \mathbf{B}^d\big)\big\|_{\mathbf{V}}^2 \\
&\quad \le \left(\left(\frac{1}{\alpha_1} + \frac{S}{\alpha_2}\right)\left(\frac{c_b^2}{2}\big\|\big(\mathbf{u}^d, \mathbf{B}^d\big)\big\|_{\mathbf{V}}^2\right)^2 + \frac{1}{\beta_1} + \frac{S^2}{\beta_2}\right)^{\frac{1}{2}} \\
&\qquad \times \big(\alpha_1\|\mathbf{v}(t)\|^2 + S\alpha_2\|\Xi(t)\|^2 + \beta_1\|\mathbf{h}\|^2 + \beta_2\|\mathbf{g}\|^2\big)^{\frac{1}{2}} \quad \forall t \in (t_1, t_2).
\end{aligned} \tag{42}$$

Multiplying the last inequality by $e^{\frac{\kappa t}{2}}$ and integrating over (t_1, t_2) yields

$$\begin{aligned}
&\big\|\big(\mathbf{v}(t_2), \Xi(t_2)\big)\big\|_{\mathbf{H}} \\
&\quad \le \big\|\big(\mathbf{v}(t_1), \Xi(t_1)\big)\big\|_{\mathbf{H}} e^{-\frac{\kappa}{2}(t_2 - t_1)} \\
&\qquad + c_6 \int_{t_1}^{t_2} \big[\alpha_1\|\mathbf{v}(t)\|^2 + S\alpha_2\|\Xi(t)\|^2 + \beta_1\|\mathbf{h}\|^2 + \beta_2\|\mathbf{g}\|^2\big]^{\frac{1}{2}} e^{-\frac{\kappa}{2}(t_2 - s)}\,\mathrm{d}s \\
&\quad \le \big\|\big(\mathbf{v}(t_1), \Xi(t_1)\big)\big\|_{\mathbf{H}} + c_6 \mathscr{J}_T(\mathbf{u}, \mathbf{B}, \mathbf{f}, \operatorname{curl}\mathbf{j})^{\frac{1}{2}} \left[\int_{t_1}^{t_2} e^{-\kappa(t_2 - s)}\,\mathrm{d}s\right]^{\frac{1}{2}} \\
&\quad \le \big\|\big(\mathbf{v}(t_1), \Xi(t_1)\big)\big\|_{\mathbf{H}} + c_6 \mathscr{J}_T(\mathbf{u}, \mathbf{B}, \mathbf{f}, \operatorname{curl}\mathbf{j})^{\frac{1}{2}} \left[\frac{1 - e^{-\kappa(t_2 - t_1)}}{\kappa}\right]^{\frac{1}{2}} \\
&\quad \le \big\|\big(\mathbf{v}(t_1), \Xi(t_1)\big)\big\|_{\mathbf{H}} + \sqrt{t_2 - t_1}\, c_6 \big(\mathscr{J}_T(\mathbf{u}, \mathbf{B}, \mathbf{f}, \operatorname{curl}\mathbf{j})\big)^{\frac{1}{2}},
\end{aligned}$$

where we used the inequality $1 - e^{-y} \le y$ for $y \ge 0$. This proves inequality (39). Finally the inequality (40) follows from the bound (9). □

Theorem 10. *Assume that* $(\mathbf{u}, \mathbf{B}, \mathbf{f}, \operatorname{curl}\mathbf{j}) \in \mathbf{U}_{\mathrm{ad}}(\infty)$. *Then*

$$\lim_{t\to\infty} \big\|\big(\mathbf{u}(t), \mathbf{B}(t)\big) - \big(\mathbf{u}^d(t), \mathbf{B}^d(t)\big)\big\|_{\mathbf{H}} = 0. \tag{43}$$

Proof. The theorem is trivially proved if $\mathscr{J}_\infty(\mathbf{u}, \mathbf{B}, \mathbf{f}, \operatorname{curl}\mathbf{j}) = 0$. Therefore we assume $\mathscr{J}_\infty(\mathbf{u}, \mathbf{B}, \mathbf{f}, \operatorname{curl}\mathbf{j}) \ge 0$ and proceed to prove (43) by contradiction. Assume that $\lim_{t\to\infty} \|(\mathbf{u}, \mathbf{B}) - (\mathbf{u}^d, \mathbf{B}^d)\|_{\mathbf{H}} \ne 0$. Then for any given $\epsilon > 0$ we define $\delta := \frac{\epsilon^2}{4c_6^2 \mathscr{J}_T(\mathbf{u}, \mathbf{B}, \mathbf{f}, \operatorname{curl}\mathbf{j})} > 0$. This allows us to choose a sequence $\{t_n\}$ such that $t_n \to \infty$, $t_{n+1} - t_n \ge \delta$ and

$$\big\|\big(\mathbf{u}(t_n), \mathbf{B}(t_n)\big) - \big(\mathbf{u}^d(t_n), \mathbf{B}^d(t_n)\big)\big\|_{\mathbf{H}} \ge \epsilon > 0.$$

We claim that for each n

$$\big\|\big(\mathbf{u}(t), \mathbf{B}(t)\big) - \big(\mathbf{u}^d(t), \mathbf{B}^d(t)\big)\big\|_{\mathbf{H}} > 0 \quad \forall t \in (t_n - \delta, t_n). \tag{44}$$

We prove this as follows. We set

$$\bar{t} := \sup\Big\{t \in (t_{n-1}, t_n)\colon \big\|\big(\mathbf{u}(t), \mathbf{B}(t)\big) - \big(\mathbf{u}^d(t), \mathbf{B}^d(t)\big)\big\|_{\mathbf{H}} = 0\Big\}$$

and assume that $t_n - \bar{t} < \delta$, that is, $\bar{t} \in (t_n - \delta, t_n)$.Then it is clear that

$$\big\|\big(\mathbf{u}(t), \mathbf{B}(t)\big) - \big(\mathbf{u}^d(t), \mathbf{B}^d(t)\big)\big\|_{\mathbf{H}} > 0 \quad \forall t \in \big(\bar{t}, t_n\big)$$

so that by (39)

$$\begin{aligned}&\big\|\big(\mathbf{u}(\bar{t}), \mathbf{B}(\bar{t})\big) - \big(\mathbf{u}^d(\bar{t}), \mathbf{B}^d(\bar{t})\big)\big\|_{\mathbf{H}}\\ &\qquad \geq \big\|\big(\mathbf{u}(t_n), \mathbf{B}(t_n)\big) - \big(\mathbf{u}^d(t_n), \mathbf{B}^d(t_n)\big)\big\|_{\mathbf{H}} - c_6\delta^{\frac{1}{2}}\mathscr{J}^{\frac{1}{2}} \geq \frac{\epsilon}{2}\end{aligned}$$

which contradicts $\|(\mathbf{u}(\bar{t}), \mathbf{B}(\bar{t})) - (\mathbf{u}^d(\bar{t}), \mathbf{B}^d(\bar{t}))\|_{\mathbf{H}} = 0$. This proves the claim in (44). Now using (39) and (44), we have $\|(\mathbf{u}(t), \mathbf{B}(t)) - (\mathbf{u}^d(t), \mathbf{B}^d(t))\|_{\mathbf{H}} \geq \frac{\epsilon}{2} \quad \forall t \in (t_n - \delta, t_n)$ and we are led to

$$\begin{aligned}&\mathscr{J}_\infty(\mathbf{u}, \mathbf{B}, \mathbf{f}, \operatorname{curl}\mathbf{j})\\ &\qquad \geq \frac{\alpha}{2}\sum_{n=2}^{\infty}\int_{t_n-\delta}^{t_n} \big\|\big(\mathbf{u}(t), \mathbf{B}(t)\big) - \big(\mathbf{u}^d(t), \mathbf{B}^d(t)\big)\big\|_{\mathbf{H}}^2\,\mathrm{d}t \geq \frac{\alpha\epsilon^2}{8}\sum_{n=2}^{\infty}\delta = \infty.\end{aligned}$$

which is a contradiction and thus (43) is proved. ☐

We next study the asymptotic behavior of $\|(\mathbf{u}(t), \mathbf{B}(t)) - (\mathbf{u}^d(t), \mathbf{B}^d(t))\|_{\mathbf{V}}$, where $(\mathbf{u}(t), \mathbf{B}(t))$ is the optimizer of (OP-CON) with $T = \infty$.

Lemma 4. *Let $T \in (0, \infty]$. Assume that $(\mathbf{u}, \mathbf{B}, \mathbf{f}, \operatorname{curl}\mathbf{j}) \in U_{ad}(T)$ is a solution of (OP-CON). Assume further that $\|(\mathbf{u}(t), \mathbf{B}(t)) - (\mathbf{u}^d(t), \mathbf{B}^d(t))\|_{\mathbf{V}} > 0$ for all $t \in (t_1, t_2) \subset [\epsilon, T]$. Then*

$$\begin{aligned}&\big\|\big(\mathbf{u}(t_2), \mathbf{B}(t_2)\big) - \big(\mathbf{u}^d(t_2), \mathbf{B}^d(t_2)\big)\big\|_{\mathbf{V}}\\ &\quad \leq \big\|\big(\mathbf{u}(t_1), \mathbf{B}(t_1)\big) - \big(\mathbf{u}^d(t_1), \mathbf{B}^d(t_1)\big)\big\|_{\mathbf{V}} + c_{12}(t_2 - t_1) + \frac{1}{\beta}(2M\mathscr{J}_T)^{\frac{1}{2}}\sqrt{t_2 - t_1},\end{aligned}$$

where M is defined in Theorem 8 *and*

$$\begin{aligned}c_{12} &:= c_7 c_{10}(\epsilon)^3 c_{11}^2 + c_8 c_{11}\big\|\big(\mathbf{u}^d, \mathbf{B}^d\big)\big\|_{L^\infty(0,\infty;\mathbf{V})}\big\|\mathscr{A}\big(\mathbf{u}^d, \mathbf{B}^d\big)\big\|_{L^\infty(0,\infty;\mathbf{H})}\\ &\quad + c_9 c_{10}(\epsilon)\big\|\big(\mathbf{u}^d, \mathbf{B}^d\big)\big\|_{L^\infty(0,\infty;\mathbf{H})}^2\big\|\big(\mathbf{u}^d, \mathbf{B}^d\big)\big\|_{L^\infty(0,\infty;\mathbf{V})}^2.\end{aligned}$$

Proof. Setting $(\mathbf{w}, \boldsymbol{\Upsilon}) = \mathscr{A}(\mathbf{v}, \boldsymbol{\Xi})$ in (6) yields as in the proof of Theorem 3 that

$$\begin{aligned}&\frac{d}{dt}\|(\mathbf{v}, \boldsymbol{\Xi})\|_{\mathbf{V}} + \frac{1}{2}\|(\mathbf{v}, \boldsymbol{\Xi})\|_{\mathbf{V}}\\ &\quad \leq c_7\|(\mathbf{v}, \boldsymbol{\Xi})\|_{\mathbf{V}}^3\|(\mathbf{v}, \boldsymbol{\Xi})\|_{\mathbf{H}}^2 + c_8\|(\mathbf{v}, \boldsymbol{\Xi})\|_{\mathbf{H}}\big\|\big(\mathbf{u}^d, \mathbf{B}^d\big)\big\|_{\mathbf{V}}\big\|A\big(\mathbf{u}^d, \mathbf{B}^d\big)\big\|_{\mathbf{H}}\\ &\qquad + c_9\big\|\big(\mathbf{u}^d, \mathbf{B}^d\big)\big\|_{\mathbf{H}}^2\big\|\big(\mathbf{u}^d, \mathbf{B}^d\big)\big\|_{\mathbf{V}}^2\|(\mathbf{v}, \boldsymbol{\Xi})\|_{\mathbf{V}} + \frac{1}{\beta}\big\|(\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\Pi}})\big\|_{\mathbf{V}},\end{aligned}$$

where $\frac{1}{\beta} := \max\{\frac{1}{\beta_1}, \frac{1}{\beta_2}\}$. By (14) and (18),

$$\sup \|(\mathbf{v}, \mathbf{\Xi})\|_{\mathbf{V}} < \sqrt{\widetilde{c_1}(\epsilon)}\|(\mathbf{v}_0, \mathbf{\Xi}_0)\|_{\mathbf{H}} := c_{10}(\epsilon),$$
$$\sup \|(\mathbf{v}, \mathbf{\Xi})\|_{\mathbf{H}} < \sqrt{c_1}\|(\mathbf{v}_0, \mathbf{\Xi}_0)\|_{\mathbf{H}} := c_{11}.$$

Therefore we have

$$\frac{d}{dt}\|(\mathbf{v}, \mathbf{\Xi})\|_{\mathbf{V}} + \frac{1}{2}\|(\mathbf{v}, \mathbf{\Xi})\|_{\mathbf{V}} \leq c_{12} + \frac{1}{\beta}\big\|(\widehat{\boldsymbol{\zeta}}, \widehat{\mathbf{\Pi}})\big\|_{\mathbf{V}} \quad \forall t \in (t_1, t_2).$$

Multiplying both sides by $e^{t/2}$ and integrating over (t_1, t_2) yields

$$\begin{aligned}
&\big\|\big(\mathbf{v}(t_2), \mathbf{\Xi}(t_2)\big)\big\|_{\mathbf{V}} \\
&\qquad \leq \big\|\big(\mathbf{v}(t_1), \mathbf{\Xi}(t_1)\big)\big\|_{\mathbf{V}} \exp\{-(t_2 - t_1)/2\} + c_{12}\big(1 - \exp\{-(t_2 - t_1)/2\}\big) \\
&\qquad\quad + \frac{1}{\beta} e^{-t_2/2} \int_{t_1}^{t_2} \big\|(\widehat{\boldsymbol{\zeta}}, \widehat{\mathbf{\Pi}})\big\|_{\mathbf{V}} e^{s/2} \, \mathrm{d}s.
\end{aligned}$$

By Cauchy-Schwartz inequality and by the estimate for the adjoint variables in Theorem 8,

$$\begin{aligned}
\int_{t_1}^{t_2} \big\|(\widehat{\boldsymbol{\zeta}}, \widehat{\mathbf{\Pi}})\big\|_{\mathbf{V}} e^{s/2} \, \mathrm{d}s &\leq \left(\int_{t_1}^{t_2} \big\|(\widehat{\boldsymbol{\zeta}}, \widehat{\mathbf{\Pi}})\big\|_{\mathbf{V}}^2 \, \mathrm{d}s \right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} e^s \, \mathrm{d}s \right)^{\frac{1}{2}} \\
&\leq (2M \mathscr{J}_T)^{\frac{1}{2}} [\exp(t_2) - \exp(t_1)]^{\frac{1}{2}}.
\end{aligned}$$

Combining the last two inequalities yields the desired inequality. □

We can now establish the long time behavior for $\|(\mathbf{u}(t), \mathbf{B}(t)) - (\mathbf{u}^d(t), \mathbf{B}^d(t))\|_{\mathbf{V}}$ based on Lemma 4.

Theorem 11. *Let $(\mathbf{u}, \mathbf{B}, \mathbf{f}, \mathrm{curl}\,\mathbf{j})$ be a solution for (OP-CON) with $T = \infty$. Then*

$$\lim_{t \to \infty} \big\|\big(\mathbf{u}(t), \mathbf{B}(t)\big) - \big(\mathbf{u}^d(t), \mathbf{B}^d(t)\big)\big\|_{\mathbf{V}} = 0.$$

The proof of this theorem is omitted as it is similar to the proof of Theorem 10 with the help of the bound in (17).

References

1. T. G. Cowling, *Magnetohydrodynamics*, 2nd edition, Adam Hilger, Bristol, 1976.
2. A. C. Eringen, G. A. Maugin, *Electrodynamics of Continua*, Vols. I, II, Springer Verlag, New York, 1990.

3. M. D. Gunzburger, A. J. Meir, J. S. Peterson, On the existence, uniqueness and finite element approximation of solutions of the equations of stationary, incompressible, magnetohydrodynamics, *Math. Comput.*, **56**(194), pp. 523–563, 1991.
4. J. D. Jackson, *Classical Electrodynamics*, Wiley, New York, 1975.
5. O. A. Ladyzenskaya, V. A. Solonnikov, On the solvability of unsteady motion problems in magnetohydrodynamics, *Dokl. Akad. Nauk SSSR*, **124**, pp. 26–28, 1959.
6. J. A. Shercliff, *A Textbook of Magnetohydrodynamics*, Pergamon Press, Oxford, 1965.
7. M. D. Gunzburger, *Perspectives in Flow Control and Optimization*, No. 5 in: Advances in Design and Control, SIAM, Philadelphia, PA, 2003.
8. L. S. Hou, S. S. Ravindran, A penalized Neumann control approach for solving an optimal Dirichlet control problem for the Navier-Stokes equations, *SIAM J. Control Optim.*, **36**(5), pp. 1795–1814, 1997.
9. L. S. Hou, S. S. Ravindran, Numerical approximation of optimal flow control problems by a penalty method: error estimates and numerical results, *SIAM J. Sci. Comput.*, **20**(5), pp. 1753–1777, 1999.
10. S. S. Sritharan (Ed.), *Optimal Control of Viscous Flows*, SIAM, Philadelphia, 1998.
11. L. S. Hou, Y. Yan, Dynamics for controlled Navier-Stokes systems with distributed controls, *SIAM J. Control Optim.*, **35**(2), pp. 654–677, 1997.
12. L. S. Hou, Y. Yan, Dynamics and approximations of a velocity tracking problem for the Navier-Stokes flows with piecewise distributed controls, *SIAM J. Control Optim.*, **35**(6), pp. 1847–1885, 1997.
13. V. Barbu, T. Havrineanu, C. Popa, S. S. Sritharan, Exact controllability for the magnetohydrodynamic equations, *Commun. Pur. Appl. Math.*, **56**(6), pp. 732–783, 2003.
14. R. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.
15. V. Girault, P. A. Raviart, *Finite Element Method for Navier-Stokes Equations*, Springer, Berlin, 1986.
16. S. Agmun, *Elliptic Boundary Value Problems*, Van Nostrand, Princeton, 1965.
17. F. Brauer, J. A. Nohel, *The Qualitative Theory of Ordinary Differential Equations: An Introduction*, Dover Publications, New York, 1989.
18. R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, New York, 1988.
19. M. Sermange, R. Temam, Some mathematical questions related to the MHD equations, *Commun. Pur. Appl. Math.*, **36**, pp. 635–664, 1983.
20. R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*, North-Holland, Amsterdam, 1984.
21. L. S. Hou, S. S. Ravindran, Computations of boundary optimal control problems for an electrically conducting fluid, *J. Comput. Phys.*, **128**, pp. 319–330, 1996.
22. W. F. Hughes, F. J. Young, *The Electromagnetohydrodynamics of Fluids*, John Wiley and Sons, New York, 1966.