Chaos Synchronization Using Active Control and Backstepping Control: A Comparative Analysis

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Abstract. This paper examines the synchronization performance of two widely used chaos synchronization techniques: active control and backstepping control. It is shown that the two methods have excellent performance, with the active control marginally outperforming the backstepping control in terms of transient analysis. However, the complexity of active controllers suggests that the backstepping control would be more attainable in engineering applications.

Keywords: chaos, synchronization, backstepping, active control.

1 Introduction

The control and synchronization of chaotic systems are extensively studied fields in nonlinear dynamics that were introduced in 1990 by Ott et al. [1] using a scheme now known as OGY closed-loop method; and Pecora and Carroll [2] using a scheme called APD method respectively. Various linear and nonlinear methods have emerged thereafter in search of more efficient algorithms for controlling and synchronizing identical and non-identical chaotic systems. Two of the most recently proposed methods are the active control [3, 4] and backstepping control [5, 6].

Chaos synchronization using active control was proposed by Bai and Lonngren [3, 4] and has recently been widely accepted as an efficient technique for the synchronization of chaotic systems, because it can be used to synchronize non-identical systems as well; a feature that gives it an advantage over other synchronization methods. This method has been applied to many practical systems such as spatiotemporal dynamical systems [7], the Rikitake two-disc dynamo – a geophysical system [8], nonlinear Bloch equations modeling nuclear magnetic resonance [9], Van der Pol Duffing oscillators [10], electric circuits modeling "jerk" equation [11], Chua's circuits [12], complex dynamos [13], nonlinear equations of acoustic gravity waves [14], Qi system [14, 15], parametrically excited systems [16] and RCL-shunted Josephson junctions [17]. In addition, we have very recently presented a novel technique using active control based synchronization scheme for controlling directed transport arising from co-existing attractors in non-equilibrium physics – the so-called ratchets [18].

Backstepping design on the other hand was originally used as a building block for adaptive control of chaotic systems [19] and was recently extended to the synchronization of chaotic systems [5, 6]. Backstepping design has been employed recently to control a third-order phase-locked loops [20], permanent magnet reluctance machine [21], a hydraulic servo system [22]; and to synchronize the Genesio chaotic systems [23] as well as demonstrate the control of directed transport in inertial ratchets [24]; among other applications. The method is a systematic design approach and consists in a recursive procedure that skillfully interlaces the choice of a Lyapunov function with the control. Indeed, backstepping control can guarantee global stability, tracking and transient performance for a broad class of strict-feedback nonlinear systems [19, 25].

In this paper, we design active controllers and a recursive backstepping control that will guarantee global synchronization between two identical chaotic systems and compare simulation results of the two methods. The rest of the paper is organized as follows: In Section 2, synchronization via active control is presented; while in Section 3, backstepping control design for chaos synchronization is presented. Section 4 deals with comparison of numerical simulation results. Finally, the paper is concluded in Section 5.

2 Design of active control for synchronization

To present the synchronization problem, we consider the same system used by Harb and Zohdy to study bifurcation and chaos control based on backstepping [26]. The drive-response system is given by:

$$\begin{aligned} \dot{x}_1 &= -z_1, \\ \dot{y}_1 &= x_1 - y_1, \\ \dot{z}_1 &= ax_1 + y_1^2 + bz_1 \end{aligned} \tag{1}$$

for the drive system and

$$\begin{aligned} \dot{x}_2 &= -z_2 + u_1, \\ \dot{y}_2 &= x_2 - y_2 + u_2, \\ \dot{z}_2 &= ax_2 + y_2^2 + bz_2 + u_3 \end{aligned} \tag{2}$$

for the response system, where u_i $(i = 1, 2, 3)$ are active control functions to be determined; a and b are the parameters of the system. Defining the error states for the state variables as

$$e_x = x_2 - x_1; \quad e_y = y_2 - y_1; \quad e_z = z_2 - z_1, \tag{3}$$

and following the procedures of active control design, we subtract equation (1) from equation (2) and using the definitions in equation (3), we obtain the error dynamics equation given by:

$$\begin{aligned} \dot{e}_x &= -e_z + u_1, \\ \dot{e}_y &= e_x - e_y + u_2, \\ \dot{e}_z &= ae_x + \left(y_2^2 - x_2^2\right) + be_z + u_3. \end{aligned} \tag{4}$$

Re-defining the control functions as follows:

$$\begin{aligned} u_1 &= v_1, \\ u_2 &= v_2 \\ u_3 &= v_3 - \left(y_2^2 - x_2^2\right) \end{aligned} \tag{5}$$

the error dynamics equation (4) becomes

$$\begin{aligned} \dot{e}_x &= -e_z + v_1, \\ \dot{e}_y &= e_x - e_y + v_2, \\ \dot{e}_z &= ae_x + be_z + v_3. \end{aligned} \tag{6}$$

In the active control method, we choose a constant matrix A which will control the error dynamics (6) such that

$$[v_1, v_2, v_3]^T = A[e_1, e_2, e_3]^T. \tag{7}$$

Several choice of A can satisfy system (7). Here, we choose the following matrix that satisfy the Routh-Hurwtiz criteria for the stability of the synchronized state:

$$A = \begin{pmatrix} \lambda_1 & 0 & 1 \\ -1 & \lambda_2 + 1 & 0 \\ -a & 0 & \lambda_3 - b \end{pmatrix}, \tag{8}$$

which immediately yields the control functions

$$\begin{aligned} u_1 &= \lambda_1 e_x + e_z, \\ u_2 &= -e_x + (\lambda_2 + 1)e_y, \\ u_3 &= -ae_x + (\lambda_3 - b)e_z - e_y(e_y + 2y_1) \end{aligned} \tag{9}$$

provided the eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ are negative. Here, we have chosen $(\lambda_1, \lambda_2, \lambda_3) = (-1, -1, -1)$ for simplicity.

3 Design of backstepping control for synchronization

In [26], the author presented a recursive backstepping control for chaos control. Here, we present backstepping control for chaos synchronization. This will enable us to give a reliable performance comparison of the two control methods for chaos synchronization.

To treat this problem, we follow the method used by Tan et al. [5]. Consider the drive-response systems given below:

$$\begin{aligned} \dot{x}_1 &= -z_1, \\ \dot{y}_1 &= x_1 - y_1, \\ \dot{z}_1 &= ax_1 + y_1^2 + bz_1 \end{aligned} \tag{10}$$

for the drive system and

$$\begin{aligned} \dot{x}_2 &= -z_2, \\ \dot{y}_2 &= x_2 - y_2, \\ \dot{z}_2 &= ax_2 + y_2^2 + bz_2 + u, \end{aligned} \tag{11}$$

where u is a control function to be determined. In backstepping, only one controller is required. Subtracting equation (5) from (6) and using the error states definition (3), we obtain

$$\begin{aligned} \dot{e}_x &= -e_z, \\ \dot{e}_y &= e_x - e_y, \\ \dot{e}_z &= ae_x + be_z + e_y(e_y + 2y_1) + u. \end{aligned} \tag{12}$$

In the absence of the control u, equation (12) would have an equilibrium $(0, 0, 0)$. If u were chosen such that the equilibrium $(0, 0, 0)$ is unchanged, then the problem of synchronization of the drive-response system would be reduced to that of asymptotic stability of system (12). Thus, the goal is to find a control law u such that system (12) is stabilized at the origin. Considering the stability of system (13) given below:

$$\dot{e}_x = -e_z \tag{13}$$

and regarding e_z as a virtual control, an estimate stabilizing function $\alpha_1(e_x)$ can be designed for the virtual control e_z. Choosing a Lyapunov function

$$V_1(e_x) = \frac{1}{2}e_x^2. \tag{14}$$

The derivative is

$$\dot{V}_1(e_x) = e_x\dot{e}_x. \tag{15}$$

For $V_1(e_x)$ to be negative definite, then, $\dot{e}_x = -e_x$, so that

$$\dot{V}_1(e_x) = -e_x^2 < 0. \tag{16}$$

Thus $\alpha_1(e_x) = -e_x$. Note that the function $\alpha_1(e_x)$ is an estimate control function when e_z is considered as a controller. Let

$$w_2 = e_y - \alpha_1(e_x) \tag{17}$$

and consider the subspace (e_x, w_2) given by

$$\begin{aligned} \dot{e}_x &= -e_z, \\ \dot{e}_y &= 2e_x - w_2. \end{aligned} \tag{18}$$

Let e_z be a virtual controller in system (18) and assume that when $e_z = \alpha_1(e_x, w_2)$, system (18) is made asymptotically stable. Choose the Lyapunov function

$$V_2(e_x, w_2) = V_1(e_x) + \frac{1}{2}w_x^2 \tag{19}$$

for subspace above. The derivative of (19) is given by

$$\dot{V}_2(e_x, w_2) = \dot{V}_1(e_x) + w_2\dot{w}_2 = -e_x^2 - w_2^2 + w_2(2e_x + e_z). \tag{20}$$

If $\alpha_1(e_x, w_2) = -2e_x$, then $e_z = -2e_x$ and

$$\dot{V}_2(e_x, w_2) = -e_x^2 - w_2^2 < 0 \tag{21}$$

is negative definite.

Define the error dynamics w_3 as

$$w_3 = e_z - \alpha_2(e_x w_2). \tag{22}$$

We now study the full dimension or the complete space (e_x, w_2, w_3)

$$\begin{aligned} \dot{e}_x &= 2e_x, \\ \dot{w}_2 &= 2e_x - w_2, \\ \dot{w}_3 &= (a+2)e_x + b(w_2 - 2e_x) + e_y(e_y + 2y_1) + u. \end{aligned} \tag{23}$$

Choose a Lyapunov function

$$V_3(e_x, w_2) = V_2(e_x, w_2) + \frac{1}{2}w_3^2. \tag{24}$$

If

$$u = -(a+2)e_x - b(w_2 - 2e_x) - e_y(e_y + 2y_1) - w_3, \tag{25}$$

then,

$$\dot{V}_3(e_x, w_2) = -e_x^2 - w_2^2 - w_3^2 < 0 \tag{26}$$

is negative definite and according to LaSalle-Yoshizawa theorem [19], the error dynamics (e_x, e_y, e_z) will converge to zero as $t \to \infty$, while the equilibrium $(0, 0, 0)$ remains asymptotically stable. Thus, the synchronization of the drive-response system is achieved.

4 Numerical results

In the following numerical results, we employed the 4th order Runge-Kutta algorithm with time grid of 0.05, $a = 3.1$, $b = 0.5$ and the initial conditions $x_1(0) = y_1(0) = z_1(0) = 0$, $x_2(0) = 0.05$, $y_2(0) = 0.01$, $z_2(0) = 0.06$. In Fig. 1 we display the same chaotic attractor as obtained in [26] as well as the synchronization errors when control is de-activated.

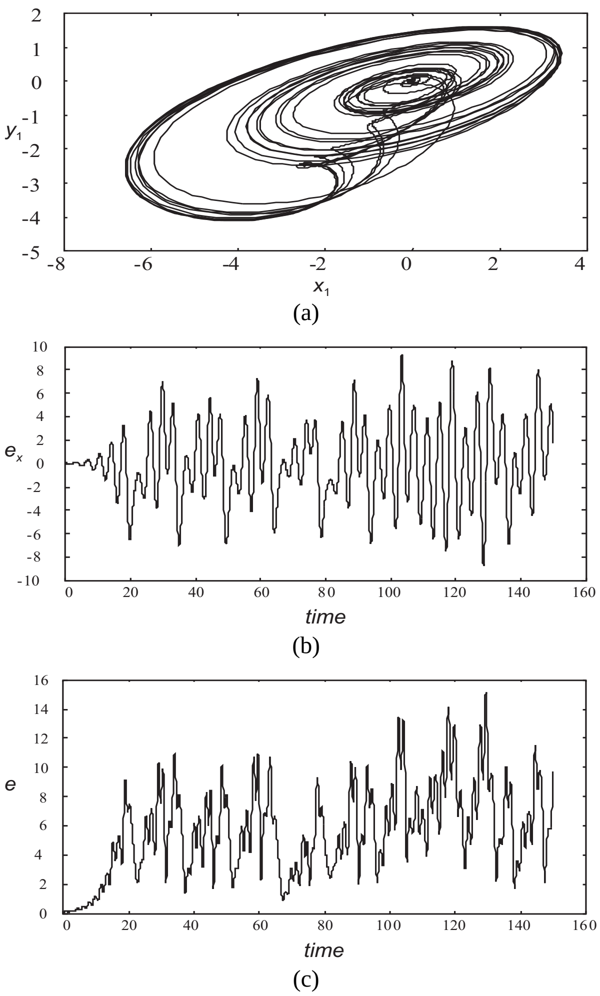

Fig. 1. Dynamics of the chaotic system when control is de-activated: (a) chaotic attractor in the (x_1-y_1) plane; (b) error dynamics (e_x) for the state variable x and (c) average error dynamics (e).

To examine and compare the synchronization performance, we compute the synchronization quantity, defined as the average error propagation on the system state variables [8, 27, 28] given by

$$e = \sqrt{e_x^2 + e_y^2 + e_z^2}. \tag{27}$$

Fig. 2 shows the synchronization error (e) for the two methods when controls are activated at $t = 0$ for both methods. We find that at $t = 5$ s, the synchronization was already attained for active control while synchronization was attained at a later time ($t = 13$ s) for backstepping control, the time delay being 8 s. Though it is clear that the active control performs better and is much easier to design, there are three controllers required in the design, while only one controller is needed for the backstepping. Thus, the controllers in active control are more complex for practical implementation. Comparing equation (1) defining the chaotic system and equation (4) defining the active controllers, one can readily observe that the controller is more complex than the system itself.

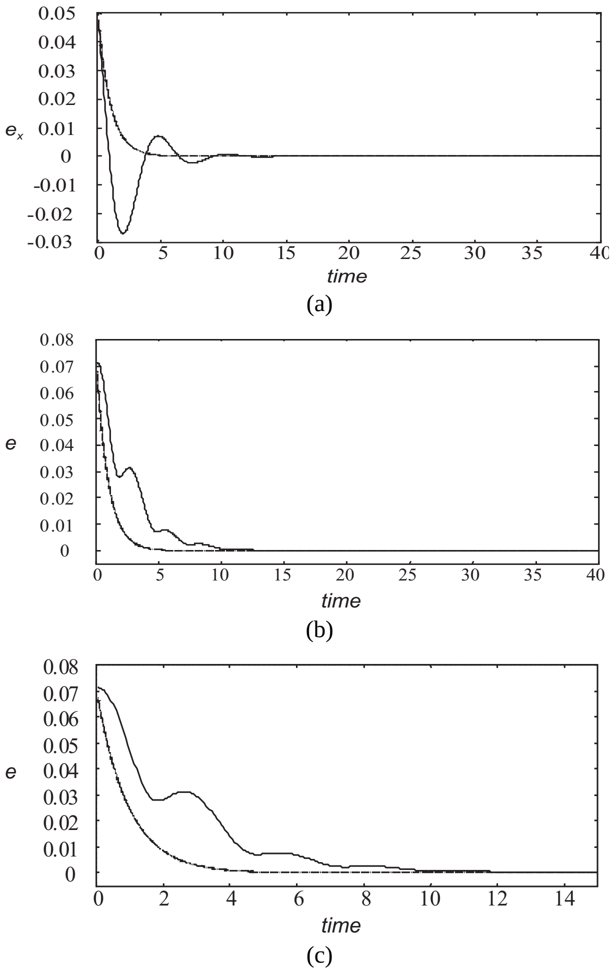

Fig. 2. Error dynamics in the synchronized state when controls have been activated at $t = 0$. Solid (backstepping method), dashed line (active control method): (a) e_x; (b) average error e and (c) zoom of the initial transient in (b).

The problem of controller complexity is a very crucial issue in the practical implementation [29], for example in electronics and engineering applications. Two fundamental issues in this direction are (a) the cost implication and density requirement for designing controllers and (b) the need to make the complexity of the controller to be comparable with, or less than, the device being controlled, if the controlling technique is desired to achieve a useful end and not merely a scientific curiosity.

5 Conclusions

This paper has examined the performance of two control schemes for chaos synchronization: active control and recursive backstepping control. It has been shown that the two schemes have excellent transient performance; with the active control marginally outperforming the backstepping. However, the complexity of the active controllers with regard to the system being synchronized suggests that the backstepping controller, with one control function would be more attainable in applications; owing to cost effectiveness and density requirement.

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