On a Nonlinear System of Reaction-Diffusion Equations

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Abstract. The aim of this article is to study the existence of positive weak solution for a quasilinear reaction-diffusion system with Dirichlet boundary conditions,

$$\begin{cases} -\operatorname{div}\left(|\nabla u_1|^{p_1-2}\nabla u_1\right) = \lambda\, u_1^{\alpha_{11}}\, u_2^{\alpha_{12}} \dots u_n^{\alpha_{1n}}, & x \in \Omega, \\ -\operatorname{div}\left(|\nabla u_2|^{p_2-2}\nabla u_2\right) = \lambda\, u_1^{\alpha_{21}}\, u_2^{\alpha_{22}} \dots u_n^{\alpha_{2n}}, & x \in \Omega, \\ \quad \dots \\ \quad \operatorname{div}\left(|\nabla u_n|^{p_n-2}\nabla u_n\right) = \lambda\, u_1^{\alpha_{n1}}\, u_2^{\alpha_{n2}} \dots u_n^{\alpha_{nn}}, & x \in \Omega, \\ \quad u_i = 0, \quad x \in \partial\Omega, \quad i = 1, 2, \dots, n, \end{cases}$$

where λ is a positive parameter, Ω is a bounded domain in R^N $(N > 1)$ with smooth boundary $\partial\Omega$. In addition, we assume that $1 < p_i < N$, for $i = 1, 2, \dots, n$. For λ large by applying the method of sub-super solutions the existence of a large positive weak solution is established for the above nonlinear elliptic system.

Keywords: reaction-diffusion system, p-Laplacian, positive weak solutions.

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1 Introduction

In this paper we consider the existence of positive weak solution to the system

$$\begin{cases} -\Delta_{p_1} u_1 = \lambda\, u_1^{\alpha_{11}}\, u_2^{\alpha_{12}} \dots u_n^{\alpha_{1n}}, & x \in \Omega, \\ -\Delta_{p_2} u_2 = \lambda\, u_1^{\alpha_{21}}\, u_2^{\alpha_{22}} \dots u_n^{\alpha_{2n}}, & x \in \Omega, \\ \quad \dots \\ -\Delta_{p_n} u_n = \lambda\, u_1^{\alpha_{n1}}\, u_2^{\alpha_{n2}} \dots u_n^{\alpha_{nn}}, & x \in \Omega, \\ \quad u_i = 0, & x \in \partial\Omega, \quad i = 1, 2, \dots, n, \end{cases} \tag{1}$$

where $\lambda > 0$ is a parameter, Δ_p denotes the p-Laplacian operator defined by $\Delta_p z = \operatorname{div}(|\nabla z|^{p-2}\nabla z)$, and Ω is a bounded domain in $R^N (N > 1)$ with smooth boundary $\partial\Omega$. In addition, we assume that $1 < p_i < N$, for $i = 1, 2, \ldots, n$.

Problems involving the p-*Laplacian* arise from many branches of pure mathematics as in the theory of quasiregular and quasiconformal mapping (see [1]) as well as from various problems in mathematical physics notably the flow of non-Newtonian fluids.

The structure of positive solutions for quasilinear reaction-diffusion systems (nonlinear Newtonian filtration systems) and semilinear reaction-diffusion systems (Newtonian filtration systems) is a front topic in the study of static electric fields in dielectric media, in which the potential is described by the boundary value problem of a static non-Newtonian filtration system, called the Poisson-Boltzmann problem. This kind of problems also appears in the study of the non-Newtonian or Newtonian turbulent filtration in porous media and so on, which have extensive engineering background.

In recent years, many authors have investigated the following initial boundary value problem of a class of quasilinear reaction-diffusion system

$$\begin{cases} u_t = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + u^{\alpha_1}v^{\beta_1}, \\ v_t = \operatorname{div}\left(|\nabla v|^{q-2}\nabla v\right) + u^{\alpha_2}v^{\beta_2}, \quad (x,t) \in \Omega \times (0,T), \end{cases} \tag{2}$$

where Ω is as above, $p, q > 1$ (see e.g. [2]). For $p = q = 2$, (2) is the classical reaction-diffusion system of Fujita type. If $p \neq 2,\ q \neq 2$, (2) appears in the theory of non-Newtonian fluids [3] and in nonlinear filtration theory [4]. In the non-Newtonian fluids theory, the pair (p, q) is a characteristic quantity of the medium. Media with $(p, q) > (2, 2)$ are called dilatant fluids and those with $(p, q) < (2, 2)$ are called pseudoplastics. If $(p, q) = (2, 2)$, they are Newtonian fluids.

Yang and Lu [2] studied the nonexistence of positive solutions to the system (2). We refer to [5–7] for additional results on elliptic systems. In this paper, we shall prove that if $\sum_{j=1}^n \alpha_{1j} < p_1 - 1,\ \sum_{j=1}^n \alpha_{2j} < p_2 - 1, \ldots, \sum_{j=1}^n \alpha_{nj} < p_n - 1$, (1) admits a positive weak solution for each $\lambda > 0$. Our approach is based on the method of sub- and supersolutions, see [8].

2 Existence results

Let $W_0^{1,s} = W_0^{1,s}(\Omega)$, $s > 1$, denote the usual Sobolev space. We first give the definition of weak solution of (1).

Definition 1. *A pair of nonnegative functions* $(\psi_1, \psi_2, \dots, \psi_n)$, $(z_1, z_2, \dots, z_n)$ *in* $W_0^{1,p_1} \times W_0^{1,p_2} \times \dots \times W_0^{1,p_n}$ *are called a weak subsolution and supersolution of* (1) *if they satisfy* $\psi_i(x) \le z_i(x)$ *in* Ω *for* $i = 1, 2, \dots, n$*, and*

$$\int_\Omega |\nabla \psi_i|^{p_i - 2} \nabla \psi_i \nabla w_i \, dx \le \lambda \int_\Omega \prod_{j=1}^{n} \psi_j^{\alpha_{ij}} \, w_i \, dx,$$

for $i = 1, 2, \dots, n$ *and*

$$\int_\Omega |\nabla z_i|^{p_i - 2} \nabla z_i \nabla w_i \, dx \ge \lambda \int_\Omega \prod_{j=1}^{n} z_j^{\alpha_{ij}} \, w_i \, dx,$$

for $i = 1, 2, \dots, n$ *and for all* $w_i(x) \in W_0^{1,p_i}$*, with* $w_i \ge 0$*.*

We shall obtain the existence of positive weak solution to system (1) by constructing a positive weak subsolution $(\psi_1, \psi_2, \dots, \psi_n)$ and supersolution $(z_1, z_2, \dots, z_n)$.

Our main result is formulated in the following theorem.

Theorem 1. *Suppose that* $\alpha_{ii} \ge 0$, $\alpha_{ij} > 0$ $(i \ne j)$*, and* $\sum_{j=1}^{n} \alpha_{1j} < p_1 - 1$, $\sum_{j=1}^{n} \alpha_{2j} < p_2 - 1, \dots, \sum_{j=1}^{n} \alpha_{nj} < p_n - 1$*. Then system* (1) *has a positive weak solution for each* $\lambda > 0$*.*

Proof. Let $\lambda_1^{(i)}$ $(i = 1, 2, \dots, n)$ be the first eigenvalue of the problems, respectively,

$$\begin{cases} -\Delta_{p_i} \phi_1^{(i)} = \lambda_1^{(i)} \, |\phi_1^{(i)}|^{p_i - 2} \, \phi_1^{(i)}, & x \in \Omega, \\ \ \phi_1^{(i)} = 0, & x \in \partial\Omega, \quad i = 1, 2, \dots, n, \end{cases}$$

where $\phi_1^{(i)}$, denote the corresponding eigenfunctions, respectively, satisfying $\phi_1^{(i)}(x) > 0$ in Ω, $|\nabla \phi_1^{(i)}| > 0$ on $\partial\Omega$ (this is possible since by the Maximum principle $\partial\phi_1^{(i)}/\partial n < 0$ for $x \in \partial\Omega$ where n denotes the outward normal, see [9]), and $||\phi_1^{(i)}||_\infty = 1$ for $i = 1, 2, \ldots, n$. We shall verify that

$$(\psi_1, \psi_2, \ldots, \psi_n) = \left(k \left(\frac{(p_1 - 1)}{p_1} \right) (\phi_1^{(1)})^{\frac{p_1}{p_1 - 1}}, \ldots, k \left(\frac{(p_n - 1)}{p_n} \right) (\phi_1^{(n)})^{\frac{p_n}{p_n - 1}} \right),$$

is a subsolution of (1), where $k > 0$ is small and specified later. Let $w_i \in W_0^{1,p_i}$ with $w_i \geq 0\,(i = 1, 2, \ldots, n)$. A calculation shows that

$$\begin{aligned}
&\int\limits_\Omega |\nabla\psi_i|^{p_i - 2}\nabla\psi_i\nabla w_i\,dx \\
&\qquad = k^{p_i - 1}\int\limits_\Omega \phi_1^{(i)}|\nabla\phi_1^{(i)}|^{p_i - 2}\nabla\phi_1^{(i)}\nabla w_i dx \\
&\qquad = k^{p_i - 1}\left\{ \int\limits_\Omega |\nabla\phi_1^{(i)}|^{p_i - 2}\nabla\phi_1^{(i)}\nabla(\phi_1^{(i)} w_i) dx - \int\limits_\Omega |\nabla\phi_1^{(i)}|^{p_i} w_i dx \right\} \\
&\qquad = k^{p_i - 1}\int\limits_\Omega \left(\lambda_1^{(i)}\,(\phi_1^{(i)})^{p_i} - |\nabla\phi_1^{(i)}|^{p_i}\right) w_i dx,
\end{aligned}$$

for $i = 1, 2, \ldots, n$. Since $\phi_1^{(i)} = 0$ and $|\nabla\phi_1^{(i)}| > 0$ on $\partial\Omega$, for $i = 1, 2, \ldots, n$, there is $\delta > 0$ such that for $i = 1, 2, \ldots, n$, we have

$$\lambda_1^{(i)}(\phi_1^{(i)})^{p_i} - |\nabla\phi_1^{(i)}|^{p_i} \leq 0, \quad x \in \bar{\Omega}_\delta,$$

with $\bar{\Omega}_\delta = \{x \in \Omega \mid d(x, \partial\Omega) \leq \delta\}$. Now on $\bar{\Omega}_\delta$ we have

$$k^{p_i - 1}\left(\lambda_1^{(i)}\,(\phi_1^{(i)})^{p_i} - |\nabla\phi_1^{(i)}|^{p_i}\right) \leq 0 \leq \lambda\prod_{j=1}^n \psi_j^{\alpha_{ij}} \quad (i = 1, 2, \ldots, n).$$

Next, we note that $\phi_1^{(i)}(x) \geq \eta > 0$ in $\Omega_0 = \Omega \setminus \bar{\Omega}_\delta$ for some $\eta > 0$, and $i = 1, 2, \ldots, n$. Since for $i = 1, 2, \ldots, n$ we have $\sum_{j=1}^n \alpha_{ij} < p_i - 1$, then there is $k_0 > 0$ such that if $k \in (0, k_0)$ we have

$$\begin{aligned}
&k^{p_i - 1}\,\lambda_1^{(i)}\,(\phi_1^{(i)})^{p_i - \alpha_{ii}\,p_i/(p_i - 1)} \\
&\qquad \leq \lambda\, k^{\sum_{j=1}^n \alpha_{ij}} \left(\prod_{j=1}^n \left(\frac{p_j - 1}{p_j} \right)^{\alpha_{ij}} \right) \left(\eta^{\sum_{j=2}^n \frac{(\alpha_{ij})p_j}{p_j - 1}} \right) \\
&\qquad \leq \lambda \left(\prod_{j=1}^n \left(\frac{p_j - 1}{p_j} \right)^{\alpha_{ij}} \right) \left(\prod_{j=2}^n (\phi_1^{(i)})^{\frac{(\alpha_{ij})p_j}{p_j - 1}} \right), \quad x \in \Omega_0,
\end{aligned}$$

for $i = 1, 2, \ldots, n$. Then in Ω_0

$$k^{p_i-1}\left(\lambda_1^{(i)}\,(\phi_1^{(i)})^{p_i} - |\nabla\phi_1^{(i)}|^{p_i}\right) \leq \lambda \prod_{j=1}^{n} \psi_j^{\alpha_{ij}},$$

for $i = 1, 2, \ldots, n$. Hence

$$\begin{aligned}
&\int\limits_{\Omega} |\nabla\psi_i|^{p_i-2}\nabla\psi_i\nabla w_i\,dx \\
&\qquad= \int\limits_{\bar{\Omega}_\delta} |\nabla\psi_i|^{p_i-2}\nabla\psi_i\nabla w_i\,dx + \int_{\Omega_0} |\nabla\psi_i|^{p_i-2}\nabla\psi_i\nabla w_i\,dx \\
&\qquad= k^{p_i-1}\int\limits_{\bar{\Omega}_\delta} \left(\lambda_1^{(i)}\,(\phi_1^{(i)})^{p_i} - |\nabla\phi_1^{(i)}|^{p_i}\right) w_i\,dx \\
&\qquad\quad + k^{p_i-1}\int\limits_{\Omega_0} \left(\lambda_1^{(i)}\,(\phi_1^{(i)})^{p_i} - |\nabla\phi_1^{(i)}|^{p_i}\right) w_i\,dx \\
&\qquad\leq \lambda \int\limits_{\bar{\Omega}_\delta} \prod_{j=1}^{n} \psi_j^{\alpha_{ij}}\, w_i\,dx + \lambda \int\limits_{\Omega_0} \prod_{j=1}^{n} \psi_j^{\alpha_{ij}}\, w_i\,dx \\
&\qquad= \lambda \int\limits_{\Omega} \prod_{j=1}^{n} \psi_j^{\alpha_{ij}}\, w_i\,dx,
\end{aligned}$$

for $i = 1, 2, \ldots, n$, i.e. $(\psi_1, \psi_2, \ldots, \psi_n)$ is a subsolution of (1).

Next, let $\zeta_i(x)$ ($i = 1, 2, \ldots, n$) be the positive solution of

$$\begin{cases} -\Delta_{p_i}\zeta_i = 1, & x \in \Omega, \\ \quad \zeta_i = 0, & x \in \partial\Omega, \quad i = 1, 2, \ldots, n. \end{cases}$$

For existence results of positive solutions for above boundary value problems see [9]. Let

$$(z_1, z_2, \ldots, z_n) = \big(C_1\,\zeta_1(x),\, C_2\,\zeta_2(x), \ldots, C_n\,\zeta_n(x)\big),$$

where $C_i > 0$ are large numbers to be chosen later. We shall verify that $(z_1, z_2, \ldots, z_n)$ is a supersolution of (1). To this end, let $w_i(x) \in W_0^{1,p_i}$, with

$w_i \geq 0$, for $i = 1, 2, \ldots, n$. Then we have

$$\int_\Omega |\nabla z_i|^{p_i-2}\nabla z_i \nabla w_i \, dx = C_i^{p_i-1} \int_\Omega |\nabla \zeta_i|^{p_i-2}\nabla \zeta_i \nabla w_i \, dx$$
$$= C_i^{p_i-1} \int_\Omega w_i dx,$$

for $i = 1, 2, \ldots, n$. Let $l_i = ||\zeta_i||_\infty$, $i = 1, 2, \ldots, n$. It is easy to prove that there exist positive large constants $C_1, C_2, \ldots, C_n$ such that

$$\begin{aligned} C_1^{p_1-1-\alpha_{11}} &\geq \lambda \Big(\prod_{j=2}^n C_j^{\alpha_{1j}} \Big) \Big(\prod_{j=1}^n l_j^{\alpha_{1j}} \Big) \\ &\cdots \\ C_n^{p_n-1-\alpha_{nn}} &\geq \lambda \Big(\prod_{j=1}^{n-1} C_j^{\alpha_{nj}} \Big) \Big(\prod_{j=1}^n l_j^{\alpha_{nj}} \Big). \end{aligned}$$

Then for $i = 1, 2, \ldots, n$ we have

$$C_i^{p_i-1} \geq \lambda \Big(\prod_{j=1}^n (C_j \, l_j)^{\alpha_{ij}} \Big) \geq \lambda \Big(\prod_{j=1}^n (C_j \, \zeta_j)^{\alpha_{ij}} \Big) = \lambda \Big(\prod_{j=1}^n z_j^{\alpha_{ij}} \Big)$$

and therefore

$$\int_\Omega |\nabla z_i|^{p_i-2}\nabla z_i \nabla w_i \, dx \geq \lambda \int_\Omega \prod_{j=1}^n z_j^{\alpha_{ij}} \, w_i \, dx,$$

for $i = 1, 2, \ldots, n$, i.e. $(z_1, z_2, \ldots, z_n)$ is a supersolution of (1) with $z_i \geq \psi_i$ in Ω for large C_i, $i = 1, 2, \ldots, n$. Thus, by the comparison principle, there exists a solution $(u_1, u_2, \ldots, u_n)$ of (1) with $\psi_i \leq u_i \leq z_i$, for $i = 1, 2, \ldots, n$. This completes the proof of Theorem 1. $\square$

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