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Controllability of Hilfer fractional noninstantaneous impulsive semilinear differential inclusions with nonlocal conditions*

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Abstract. In this paper, we investigate the controllability of nonlocal Hilfer-type fractional differential inclusions with noninstantaneous impulsive conditions in Banach spaces.

Keywords: controllability, Hilfer fractional derivatives, differential inclusions, noninstantaneous impulsive, nonlocal conditions.

1 Introduction

Fractional differential equations and inclusions arise naturally in various fields, and there are many papers in the literature on existence and controllability results (see, for example, [4, 9, 11, 12, 14, 17, 23]). Impulsive differential equations and inclusions arise in applications in physics, biology, engineering, medical fields, industry and technology. Mild solutions to impulsive differential equations and inclusions were studied in [2] and

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reference therein. Note that the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous processes. To characterize this process, Hernándaz and O'Regan [8] introduced noninstantaneous impulsive differential equations, and for recent contributions, we refer the reader to [13, 18–21].

Hilfer fractional differential equations were studied in [3–5, 7, 9, 10, 22]. However, there are only a few papers on controllability of Hilfer-type fractional noninstantaneous impulsive differential inclusions. In this paper, by avoiding any condition on the invertibility of the linear controllability operator expressed in terms of measures of noncompactness, we study the controllability of the Hilfer-type fractional noninstantaneous impulsive differential inclusions with nonlocal conditions

$$\begin{aligned} &D^{\alpha,\beta}_{s_i^+}x(t) \in Ax(t) + F\big(t, x(t)\big) + B\big(u(t)\big), \quad \text{a.e. } t \in (s_i, t_{i+1}],\ i = 0, 1, \dots, m, \\ &x\big(t_i^+\big) = g_i\big(t_i, x\big(t_i^-\big)\big), \quad x(t) = g_i\big(t, x\big(t_i^-\big)\big), \quad t \in (t_i, s_i],\ i = 1, \dots, m, \\ &I^{1-\gamma}_{0^+}x(0) = x_0 + g(x), \qquad I^{1-\gamma}_{s_i^+}x\big(s_i^+\big) = g_i\big(s_i, x\big(t_i^-\big)\big), \quad i = 1, \dots, m, \end{aligned} \tag{1}$$

where $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$, $\gamma = \alpha + \beta - \alpha\beta$, $D^{\alpha,\beta}_{s_i+}x(t)$ is the left-sided Hilfer derivative [9] with lower limit at s_i of order α and type β. Let $J = [0, b]$, $b > 0$, E be a real Banach space and A be the infinitesimal of strongly continuous semigroup $T(t)$, $t > 0$. In addition, $0 = s_0 < t_1 < s_1 < t_2 < \cdots < t_m < s_m < t_{m+1} = b$, $x(t_i^+)$, $x(t_i^-)$ are the right and left limits of x at the point t_i, respectively, $I^{1-\gamma}_{s_i^+}$ is the left-sided Riemann–Liouville integral of order $1 - \gamma$ [11] with lower limit at s_i, and $I^{1-\gamma}_{s_i^+}x(s_i^+) = \lim_{t\to s_i^+} I^{1-\gamma}_{s_i^+}(t)$. Moreover, $F : J \times E \to 2^E - \{\emptyset\}$ is a multifunction, $g : PC_{1-\gamma}(J, E) \to E$ and $g_i : [t_i, s_i] \times E \to E$, $i = 1, 2, \dots, m$, are functions. The control function u is given in $L^p(J, X)$, $p > 1/\alpha$, a Banach space of admissible control functions, with X being a real Banach space, B is a bounded linear operator from X into E, and x_0 is a fixed point of E. The space $PC_{1-\gamma}$ will be discussed in the next section.

The paper is organized as follows. In Section 2, we collect some background material concerning multifunctions and fractional calculus, and we discuss a measure of noncompactness on the space of piecewise weighted continuous functions. In Section 3, we consider the controllability of (1), and in Section 4, an example is given to illustrate our theory.

2 Preliminaries and notation

Let $P_b(E) = \{B \subseteq E\colon B \text{ is nonempty and bounded}\}$, $P_{cl}(E) = \{B \subseteq E\colon B \text{ is nonempty, convex and closed}\}$, $P_{\mathrm{ck}}(E) = \{B \subseteq E\colon B \text{ is nonempty, convex and compact}\}$, $\mathrm{conv}(B)$ (respectively, $\overline{\mathrm{conv}}(B)$) be the convex hull (respectively, convex closed hull in E) of a subset B, and $C(J, E)$ be Banach space of all E-valued continuous functions from J to E with the norm $\|x\|_{C(J,E)} = \sup_{t\in J}\|x(t)\|$. Let $L^p(J, E) = \{v\colon J \to E$ is Bochner integrable$\}$ endowed with the norm $\|v\|_{L^p(J,E)} = (\int_J \|v(t)\|^p\,\mathrm{d}t)^{1/p}$, $p \in [1, \infty)$. For $a \in [0, b)$ and $0 \leqslant \gamma \leqslant 1$, consider the weighted spaces of continuous functions $C_\gamma([a, b], E) = \{x \in C((a, b], E)\colon (t-a)^\gamma x(t) \in C([a, b], E)\}$. Now $C_\gamma([a, b], E)$ is a Banach space with norm $\|x\|_{C_\gamma([a,b],E)} = \sup_{t\in(a,b]}(t - a)^\gamma\|x(t)\|$.

Let $J_k = (s_k, t_{k+1}]$, $\overline{J_k} = [s_k, t_{k+1}]$ $(k = 0, 1, \ldots, m)$, $T_i = (t_i, s_i]$ and $\overline{T_i} = [t_i, s_i]$ $(i = 1, 2, \ldots, m)$, and consider the Banach space $PC_{1-\gamma}(J, E) = \{x\colon (t - s_k)^{1-\gamma} x \in C(J_k, E),\ \lim_{t \to s_k^+} (t - s_k)^{1-\gamma} x(t),\ x \in C(T_i, E), \text{ and } \lim_{t \to t_i^+} x(t) \text{ exist},\ k = 0, 1, \ldots, m,\ i = 1, 2, \ldots, m\}$, with

$$\|x\|_{PC_{1-\gamma}(J,E)} = \max\Big\{ \max_{k=0,1,\ldots,m} \sup_{t \in J_k} (t - s_k)^{1-\gamma} \big\|x(t)\big\|, \max_{i=1,2,\ldots,m} \sup_{t \in T_i} \big\|x(t)\big\| \Big\}.$$

Similar to the scalar case given in [5], we have

Remark 1. If $x \in PC_{1-\gamma}(J, E)$, then for any $k = 0, 1, \ldots, m$, the following hold:

(i) x is not necessarily defined at s_k, but $\lim_{t \to s_k^+} (t - s_k) x(t)$ and $x(s_{i+1}^-)$ exist.

(ii) $x(t_{k+1}) = x(t_{k+1}^-)$ and $x(t_{k+1}^+)$ exists. Moreover, $(t_{k+1} - s_k)^{1-\gamma} \|x(t_{k+1}^-)\| \leqslant \|x\|_{PC_{1-\gamma}(J,E)}$.

(iii) If $x_n \to x$ in $PC_{1-\gamma}(J, E)$, then $x_n(t) \to x(t)$, $t \in (t_i, s_i]$, $i = 1, \ldots, m$, and $(t - s_k)^{1-\gamma} x_n(t) \to (t - s_k)^{1-\gamma} x(t)$, $t \in (s_k, t_{k+1}]$. Consequently, $x_n(t) \to x(t)$, $t \in (s_i, t_{i+1}]$, and hence $x_n(t_{i+1}) = x_n(t_{i+1}^-) \to x(t_{i+1}) = x(t_{i+1}^-)$, $i = 0, 1, \ldots, m$. Then $x_n(t) \to x(t)$ a.e. for $t \in J$.

Next, the function $\chi_{PC_{1-\gamma}(J,E)} : P_b(PC_{1-\gamma}(J, E)) \to [0, \infty)$, defined by

$$\chi_{PC_{1-\gamma}(J,E)}(Z) = \max\Big\{ \max_{k=0,1,\ldots,m} \chi_{C(\overline{J_k},E)}(Z_{|\overline{J_k}}), \max_{i=1,\ldots,m} \chi_{C(\overline{T_i},E)}(Z_{|\overline{T_i}}) \Big\},$$

is a measure of noncompactness on $PC_{1-\gamma}(J, E)$, where $Z_{|\overline{J_k}} = \{y^* \in C(\overline{J_k}, E)\colon y^*(t) = (t - s_k)^{1-\gamma} y(t),\ t \in J_k,\ y^*(s_k) = \lim_{t \to s_k^+} (t - s_k)^{1-\gamma} y(t),\ y \in Z\}$ and $Z_{|\overline{T_i}} = \{y^* \in C(\overline{T_i}, E)\colon y^*(t) = y(t),\ t \in T_i,\ y^*(t_i) = y(t_i^+),\ y \in Z\}$.

Definition 1. (See [7, Def. 2.13].) Let $f : J \times E \to E$ be a function. By a mild solution of

$$D_{0+}^{\alpha,\beta} x(t) = Ax(t) + f\big(t, x(t)\big), \quad t \in (0, b], \qquad I_{0+}^{1-\gamma} x(0^+) = x_0 \tag{2}$$

we mean a function $x \in C((0, b], E)$, which satisfies

$$x(t) = S_{\alpha,\beta}(t) x_0 + \int_0^t K_\alpha(t - s) f\big(s, x(s)\big)\, \mathrm{d}s, \quad t \in (0, b],$$

where $K_\alpha(t) = t^{\alpha-1} P_\alpha(t)$, $P_\alpha(t) = \int_0^\infty \alpha\theta M_\alpha(\theta) T(t^\alpha \theta)\, \mathrm{d}\theta$, $t \geqslant 0$, $M_\mu(\theta) = \sum_{n=1}^\infty (-\theta)^{n-1} / ((n-1)\Gamma(1 - \mu n))$, $\mu \in (0, 1)$, $\theta \in \mathbb{C}$, and $S_{\alpha,\beta}(t) = I_{0+}^{\beta(1-\alpha)} K_\alpha(t)$. Note that the weight function $M_\mu(\theta)$ satisfies the equality $\int_0^\infty \theta^\tau M_\mu(\theta)\, \mathrm{d}\theta = \Gamma(1 + \tau) / \Gamma(1 + \tau\mu)$ for $\theta \geqslant 0$.

Remark 2. From [7, Remark 2.14]) we have:

(i) $D_{0+}^{\beta(1-\alpha)} S_{\alpha,\beta}(t) = K_\alpha(t)$, $t \in (0, b]$.

(ii) When $\beta = 0$, the fractional equation (2) reduces to the classical Riemann–Liouville fractional equation, which was studied by Zhou et al. [24]. Note that $S_{\alpha,0}(t) = K_\alpha(t) = t^{\alpha-1} P_\alpha(t)$.

(iii) When $\beta = 1$, the fractional equation (2) reduces to the classical Caputo fractional equation, which was studied by Zhou et al. [24]. Note $S_{\alpha,1} = S_\alpha(t)$, where $S_\alpha(t)$ is defined in [24].

Lemma 1. (*See* [7, *Props*. 2.15, 2.16].) *Suppose the semigroup $T(t)$, $t \geqslant 0$, satisfies the condition*

(H_1) *$T(t)$ is continuous for the uniform operator topology for $t > 0$, and there is $M > 1$ such that $\sup_{t\geqslant 0} \|T(t)\| \leqslant M$.*

Then we have

(i) *$P_\alpha(t)$ is continuous for the uniform operator topology for $t > 0$.*
(ii) *For any fixed $t > 0$, $S_{\alpha,\beta}(t)$ and $K_\alpha(t)$ are linear bounded operators, and for any fixed $x \in E$, $\|S_{\alpha,\beta}(t)x\| \leqslant (Mt^{\gamma-1}/\Gamma(\gamma))\|x\|, \gamma = \alpha + \beta - \alpha\beta$, and $\|K_\alpha(t)x\| \leqslant (Mt^{\alpha-1}/\Gamma(\alpha))\|x\|$.*
(iii) *$\{K_\alpha(t),\, t > 0\}$ and $\{S_{\alpha,\beta}(t),\, t > 0\}$ are strongly continuous, which means that for any $x \in E$ and $0 < t_1 < t_2 \leqslant b$, we have $\|K_\alpha(t_1)x - K_\alpha(t_2)x\| \to 0$ and $\|S_{\alpha,\beta}(t_1)x - S_{\alpha,\beta}(t_2)x\| \to 0$ as $t_1 \to t_2$.*

Based on Definition 1 we present the concept of mild solutions of (1).

Definition 2. A function $x \in PC_{1-\gamma}\,[0, b]$ is called a mild solution of problem (1) if there is $f \in S^1_{F(\cdot,x(\cdot))}$ such that

$$x(t) = \begin{cases} S_{\alpha,\beta}(t)(x_0 + g(x)) + \frac{1}{\Gamma(\alpha)} \int_0^t K_\alpha(t-s)(f(s) + Bu(s))\,\mathrm{d}s, \\ \quad t \in (0, t_1], \\ g_i(t, x(t_i^-)), \quad t \in (t_i, s_i],\ i = 1, \dots, m, \\ S_{\alpha,\beta}(t - s_i)g_i(s_i, x(t_i^-)) + \int_{s_i}^t K_\alpha(t-s)(f(s) + B(u(s))\,\mathrm{d}s, \\ \quad t \in (s_i, t_{i+1}],\ i = 1, \dots, m. \end{cases}$$

Definition 3. System (1) is said to be controllable on J if for every $x_0, x_1 \in E$, there exists a control function $u \in L^p(J, X)$ such that a mild solution of (1) satisfies $I_{0^+}^{1-\gamma}x(0) = x_0 + g(x)$ and $x(b) = x_1$.

Lemma 2. (*See* [15].) *Let $C \subset L^1(J, E)$ be a countable set such that there is a $h \in L^1(J, E)$ with $f(t) \leqslant h(t)$, a.e. $t \in J$ and every $f \in C$. Then the function $t \to \chi\{f(t), f \in C\}$ belongs to $L^1(J, E)$, and $\chi\{\int_0^b f(s)\,\mathrm{d}s,\ f \in C\} \leqslant 2\int_0^b \chi\{f(s),\ f \in C\}\,\mathrm{d}s$.*

Lemma 3. (*See* [6].) *Let $\chi_{C(J,E)}$ be the Hausdorff measure of noncompactness on $C(J, E)$. If $W \subseteq C(J, E)$ is bounded, then for every $t \in J$, $\chi(W(t)) \leqslant \chi_{C(J,E)}(W)$, where $W(t) = \{x(t),\ x \in W\}$. Furthermore, if W is equicontinuous on J, then the map $t \to \chi\{x(t),\ x \in W\}$ is continuous on J, and $\chi_{C(J,E)}(W) = \sup_{t\in J} \chi\{x(t),\ x \in W\}$.*

Lemma 4. (*See* [16, *Thm*. 3.1].) *Let D be a closed convex subset of a Banach space X and $N : D \to P_c(D)$. Assume the graph of N is closed, N maps compact sets into* *relatively compact sets and that for some* $x_0 \in U$*, one has*

$$Z \subseteq D, \quad Z = \operatorname{conv}\bigl(\{x_0\} \cup N(Z)\bigr),\ \overline{Z} = \overline{C} \text{ with } C \subseteq Z \text{ countable} \\ \implies \quad Z \text{ is relatively compact.} \tag{3}$$

Then N *has a fixed point.*

3 Controllability results

In this section, we establish some controllability results for (1).

Theorem 1. *Let* $F : J \times E \to P_{\mathrm{ck}}(E)$ *be a multifunction,* X *a Banach space,* $B : X \to E$ *a bounded linear operator,* $g_i : [t_i, s_i] \times E \to E$ $(i = 1, 2, \dots, m)$ *and* p *be a real number such that* $p > 1/\alpha$*. In addition,* (H_1) *holds and we assume the following conditions*:

- (F_1) *For every* $x \in PC_{1-\gamma}(J, E)$*, the multifunction* $t \to F(t, x(t))$ *has a strong measurable selection, and for almost every* $t \in J$*,* $z \to F(t, z)$*, is upper semicontinuous.*
- (F_2) *There exist a function* $\varphi \in L^p(J, \mathbb{R}^+)$ *and a continuous nondecreasing function* $\Omega : [0, \infty) \to (0, \infty)$ *such that for every* $x \in PC_{1-\gamma}(J, E)$*,* $\|F(t, x(t)\| \leqslant \varphi(t)\Omega(\|x\|_{PC_{1-\gamma}(J,E)})$ *for* $t \in J$ *and* $\liminf_{n\to\infty} \|\Omega(n)\|/n = \upsilon < \infty$*.*
- (F_3) *There exists a function* $\varsigma \in L^p(J, \mathbb{R}^+)$ *such that for any bounded subset* $D \subseteq E$ *and any* $k = 0, 1, \dots, m$*,* $\chi(F(t, D)) \leqslant (t - s_k)^{1-\gamma}\varsigma(t)\chi(D)$*, a.e.* $t \in J$*, and*

$$b^{1-\gamma} M \|\varsigma\|_{L^p(J,\mathbb{R}^+)} \left(\frac{2\eta}{\Gamma(\alpha)} + \frac{2\eta N^2}{\Gamma(\alpha)^2} \right) < 1, \tag{4}$$

  where $\eta = b^{\alpha-1/p}((p-1)/(p\alpha-1))^{(p-1)/p}$*, and* χ *is the Hausdorff measure of noncompactness on* E*.*
- (H_g) $g : PC_{1-\gamma}(J, E) \to E$ *is continuous, compact, and* $\liminf_{\|x\|\to\infty} \|g(x)\|/\|x\|_{PC_{1-\gamma}(J,E)} = 0$*.*
- (H_{g_i}) *For every* $i = 1, 2, \dots, m$*,* $g_i : [t_i, s_i] \times E \to E$ *is uniformly continuous on bounded sets, and for any* $t \in J$*,* $g_i(t, \cdot)$ *is compact, and there exists a positive constant* h_i *such that for any* $x \in E$*,* $\|g_i(t, x)\| \leqslant h_i(t_i - s_{i-1})^{1-\gamma}\|x\|$*,* $t \in [t_i, s_i]$*.*
- (HW) *The linear bounded operator* $W : L^p(J, X) \to E$*, which is defined by* $W(u) = \int_{s_m}^b K_\alpha(b - s)B(u(s))\,\mathrm{d}s$*, has an invertible* $W^{-1} : E \to L^p(J, X)/\mathrm{Ker}(W)$*, and there exists a positive constant* N *such that* $\|W^{-1}\| \leqslant N$ *and* $\|B\| \leqslant N$*.*

Then system (1) *is controllable on* J*, provided that*

$$\frac{M\eta\upsilon b^{1-\gamma}}{\Gamma(\alpha)}\|\varphi\|_{L^p(J,\mathbb{R}^+)} + \frac{Mb^{1-\gamma}N^2}{\Gamma(\alpha)}\eta\left[\frac{M(b-s_m)^{\gamma-1}}{\Gamma(\gamma)}h \right. \\ \left. + \frac{\upsilon M}{\Gamma(\alpha)}\eta\|\varphi\|_{L^p(J,\mathbb{R}^+)}\right] + h + \frac{hM}{\Gamma(\gamma)} < 1, \tag{5}$$

Proof. Note that W is well defined. In fact, since $p > 1/\alpha$, the functions $s \to (b-s)^{\alpha-1}$ belongs to $L^{p/(p-1)}([0,b],\mathbb{R}^+)$. Then by the Hölder's inequality, for any $u \in L^p(J,X)$, we have

$$\|W(u)\| \leqslant \frac{MN}{\Gamma(\alpha)} \int_{s_m}^{b} (b-s)^{\alpha-1}\|u(s)\|\,\mathrm{d}s \\ \leqslant \frac{MN}{\Gamma(\alpha)}\|u\|_{L^p(J,X)}\left(\frac{p-1}{\alpha p-1}\right)^{(p-1)/p} b^{\alpha-1/p}.$$

Now, in view of (F_1) for every $x \in PC_{1-\gamma}(J,E)$, the multifunction $t \to F(t,x(t))$ has a measurable selection f, and by (F_2),

$$\|f(t)\| \leqslant \varphi(t)\Omega(\|x\|_{PC_{1-\gamma}(J,E)}), \tag{6}$$

so $f \in S^p_{F(\cdot,x(\cdot))} = \{z \in L^p(J,E)\colon z(t) \in F(t,x(t))$ a.e. for $t \in \bigcup_{i=0}^{i=m}(s_i,t_{i+1}]\}$. Next, $\|S_{\alpha,\beta}(b-s_m)g_m(s_m,x(t_m^-))\| \leqslant (M(b-s_m)^{\gamma-1}/\Gamma(\gamma))h_m(t_m-s_{m-1})^{1-\gamma} \times \|x(t_m^-)\|$. Thus, for any $x \in PC_{1-\gamma}(J,E)$ and any $f \in S^p_{F(\cdot,x(\cdot))}$, we can define, using (HW), the control function $u_{x,f} \in L^p(J,X)$ by

$$u_{x,f} = W^{-1}\left[x_1 - S_{\alpha,\beta}(b-s_m)g_m\big(s_m,x\big(t_m^-\big)\big) - \int_{s_m}^{b} K_\alpha(b-s)f(s)\,\mathrm{d}s\right]. \tag{7}$$

Therefore, we can define a multifunction $R : PC_{1-\gamma}(J,E) \to 2^{PC_{1-\gamma}(J,E)}$ as follows. For any $x \in PC_{1-\gamma}(J,E)$, a function $y \in R(x)$ if and only if

$$y(t) = \begin{cases} S_{\alpha,\beta}(t)(x_0 + g(x)) + \int_0^t K_\alpha(t-s)(f(s) + B(u_{x,f}(s)))\,\mathrm{d}s, \\ \quad t \in (0,t_1], \\ g_i(t,x(t_i^-)), \quad t \in (t_i,s_i],\ i = 1,\dots,m, \\ S_{\alpha,\beta}(t-s_i)g_i(s_i,x(t_i^-)) + \int_{s_i}^t K_\alpha(t-s)(f(s) + B(u_{x,f}(s)))\,\mathrm{d}s, \\ \quad t \in (s_i,t_{i+1}],\ i = 1,\dots,m, \end{cases}$$

where $f \in S^p_{F(\cdot,x(\cdot))}$.

Let us show that using the control function defined by (7), any fixed point for R is a mild solution for (1) and satisfies $x(0) = x_0$ and $x(b) = x_1$. In fact, if x is a fixed point for R, then from (7) we have

$$x(b) = S_{\alpha,\beta}(b-s_i)g_m\big(s_m,x\big(t_m^-\big)\big) \\ + \int_{s_m}^{b} K_\alpha(b-s)f(s)\,\mathrm{d}s + \int_{s_m}^{b} K_\alpha(b-s)B\big(u_{x,f}(s)\big)\,\mathrm{d}s$$

$$\begin{aligned}
&= S_{\alpha,\beta}(b-s_i)g_m\big(s_m, x\big(t_m^-\big)\big) + \int\limits_{s_m}^{b} K_\alpha(b-s)f(s)\,\mathrm{d}s + W(u_{x,f}) \\
&= S_{\alpha,\beta}(b-s_i)g_m\big(s_m, x\big(t_m^-\big)\big) + \int\limits_{s_m}^{b} K_\alpha(b-s)f(s)\,\mathrm{d}s \\
&\quad + x_1 - S_{\alpha,\beta}(b-s_i)g_m\big(s_m, x\big(t_m^-\big)\big) - \int\limits_{s_m}^{b} K_\alpha(b-s)f(s)\,\mathrm{d}s \\
&= x_1.
\end{aligned}$$

We now prove using Lemma 4 that R has a fixed point. The proof will be given in several steps. It is easy to see that the values of R are convex.

Step 1. In this step, we claim that there is a natural number n such that $R(B_n) \subseteq B_n$, where $B_n = \{x \in PC_{1-\gamma}(J,E)\colon \|x\|_{PC_{1-\gamma}(J,E)} \leqslant n\}$. Suppose the contrary. Then for any $n \in \mathbb{N}$, there are $x_n, y_n \in PC_{1-\gamma}(J,E)$ with $y_n \in R(x_n)$, $\|x_n\|_{PC_{1-\gamma}(J,E)} \leqslant n$ and $\|y_n\|_{PC_{1-\gamma}(J,E)} > n$. Then there is a $f_n \in S^p_{F(\cdot,x(\cdot))}$, $n \geqslant 1$, such that

$$y_n(t) = \begin{cases} S_{\alpha,\beta}(t)(x_0 + g(x_n)) + \int_0^t K_\alpha(t-s)(f_n(s) + B(u_{x_n,f_n}(s)))\,\mathrm{d}s, \\ \quad t \in (0, t_1], \\ g_i(t, x_n(t_i^-)), \quad t \in (t_i, s_i],\ i = 1, \dots, m, \\ S_{\alpha,\beta}(t-s_i)g_i(s_i, x_n(t_i^-)) + \int_{s_i}^t K_\alpha(t-s)(f_n(s) + B(u_{x_n,f_n}(s)))\,\mathrm{d}s, \\ \quad t \in (s_i, t_{i+1}],\ i = 1, \dots, m. \end{cases} \tag{8}$$

Then, if $t \in [0, t_1]$, using Hölder's inequality, we have

$$\begin{aligned}
&\sup_{t\in[0,t_1]} t^{1-\gamma}\big\|y_n(t)\big\| \\
&\quad \leqslant \sup_{t\in[0,t_1]} t^{1-\gamma}\big\|S_{\alpha,\beta}(t)\big(x_0 + g(x)\big)\big\| \\
&\qquad + \sup_{t\in[0,t_1]} \frac{Mt^{1-\gamma}\Omega(\|x_n\|_{PC_{1-\gamma}(J,E)})}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1}\varphi(s)\,\mathrm{d}s \\
&\qquad + \sup_{t\in[0,t_1]} \frac{MNt^{1-\gamma}}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1}\big\|u_{x_n,f_n}(s)\big\|\,\mathrm{d}s \\
&\quad \leqslant \frac{M}{\Gamma(\gamma)}\big[\|x_0\| + \big\|g(x_n)\big\|\big] + \frac{Mb^{1-\gamma}}{\Gamma(\alpha)}\Omega(n)\|\varphi\|_{L^p(J,\mathbb{R}^+)}\eta \\
&\qquad + \frac{MNb^{1-\gamma}}{\Gamma(\alpha)}\|u_{x_n,f_n}\|_{L^p(J,\mathbb{R}^+)}\eta.
\end{aligned} \tag{9}$$

From (9) and Remark 1(ii) we get

$$\begin{aligned}
&\|u_{x_n,f_n}\|_{L^p(J,X)} \\
&\quad \leqslant \|W^{-1}\| \biggl[\|x_1\| + \bigl\|S_{\alpha,\beta}(b-s_m)g_m\bigl(s_m, x\bigl(t_m^-\bigr)\bigr)\bigr\| + \int_{s_m}^{b} K_\alpha(b-s)f(s)\,\mathrm{d}s \biggr] \\
&\quad \leqslant N \biggl[\|x_1\| + \frac{M(b-s_m)^{\gamma-1}}{\Gamma(\gamma)} h_m(t_m - s_{m-1})^{1-\gamma} \bigl\|x\bigl(t_m^-\bigr)\bigr\| \\
&\qquad + \frac{M\Omega(\|x_n\|_{PC_{1-\gamma}(J,E)})}{\Gamma(\alpha)} \|\varphi\|_{L^p(J,\mathbb{R}^+)}\eta \biggr] \\
&\quad = N \biggl[\|x_1\| + \frac{M(b-s_m)^{\gamma-1}}{\Gamma(\gamma)} h\|x_n\|_{PC_{1-\gamma}(J,E)} + \frac{M\Omega(n)}{\Gamma(\alpha)}\eta\|\varphi\|_{L^p(J,\mathbb{R}^+)} \biggr] \\
&\quad \leqslant N \biggl[\|x_1\| + \frac{M(b-s_m)^{\gamma-1}}{\Gamma(\gamma)} hn + \frac{M\Omega(n)}{\Gamma(\alpha)}\eta\|\varphi\|_{L^p(J,\mathbb{R}^+)} \biggr]. \qquad (10)
\end{aligned}$$

It follows from (9) and (10) that

$$\begin{aligned}
&\sup_{t\in[0,t_1]} t^{1-\gamma}\bigl\|y_n(t)\bigr\| \\
&\quad \leqslant \frac{M}{\Gamma(\gamma)}\bigl[\|x_0\| + \bigl\|g(x_n)\bigr\|\bigr] + \eta\frac{Mb^{1-\gamma}}{\Gamma(\alpha)}\Omega(n)\|\varphi\|_{L^p(J,\mathbb{R}^+)} \\
&\qquad + \frac{Mb^{1-\gamma}N^2}{\Gamma(\alpha)}\eta\biggl[\|x_1\| + \frac{M(b-s_m)^{\gamma-1}}{\Gamma(\gamma)}hn + \frac{M\Omega(n)}{\Gamma(\alpha)}\eta\|\varphi\|_{L^p(J,\mathbb{R}^+)}\biggr]. \qquad (11)
\end{aligned}$$

If $t \in (t_i, s_i]$, $i = 1, 2, \dots, m$, then from Remark 1(ii),

$$\sup_{t\in[t_i,s_i]} \bigl\|y_n(t)\bigr\| \leqslant h(t_i - s_{i-1})^{1-\gamma}\bigl\|x_n\bigl(t_i^-\bigr)\bigr\| \leqslant h\|x_n\|_{PC_{1-\gamma}(J,E)} \leqslant hn. \qquad (12)$$

Similarly, we get for $t \in (s_i, t_{i+1}]$, $i = 1, 2, \dots, m$,

$$\begin{aligned}
&\sup_{t\in[s_i,t_{i+1}]} (t-s_i)^{1-\gamma}\bigl\|y_n(t)\bigr\| \\
&\quad \leqslant \sup_{t\in[s_i,t_{i+1}]} \frac{M\|g_i(s_i, x_n(t_i^-))\|}{\Gamma(\gamma)} + \frac{Mb^{1-\gamma}}{\Gamma(\alpha)}\Omega(n)\|\varphi\|_{L^p(J,\mathbb{R}^+)}\eta \\
&\qquad + \frac{Mb^{1-\gamma}N^2}{\Gamma(\alpha)}\eta\biggl[\|x_1\| + \frac{M(b-s_m)^{\gamma-1}}{\Gamma(\gamma)}hn + \frac{M\Omega(n)}{\Gamma(\alpha)}\eta\|\varphi\|_{L^p(J,\mathbb{R}^+)}\biggr] \\
&\quad \leqslant \frac{Mhn}{\Gamma(\gamma)} + \frac{Mb^{1-\gamma}}{\Gamma(\alpha)}\Omega(n)\|\varphi\|_{L^p(J,\mathbb{R}^+)}\eta \\
&\qquad + \frac{Mb^{1-\gamma}N^2}{\Gamma(\alpha)}\eta\biggl[\|x_1\| + \frac{M(b-s_m)^{\gamma-1}}{\Gamma(\gamma)}hn + \frac{M\Omega(n)}{\Gamma(\alpha)}\|\varphi\|_{L^p(J,\mathbb{R}^+)}\eta\biggr]. \qquad (13)
\end{aligned}$$

From (11), (12) and (13) we have

$$\begin{aligned} n &< \|y_n\|_{PC_{1-\gamma}(J,E)} \\ &\leqslant \frac{M}{\Gamma(\gamma)}\big[\|x_0\| + \big\|g(x_n)\big\|\big] + \eta\frac{Mb^{1-\gamma}}{\Gamma(\alpha)}\Omega(n)\|\varphi\|_{L^p(J,\mathbb{R}^+)} \\ &\quad + \frac{Mb^{1-\gamma}N^2}{\Gamma(\alpha)}\eta\biggl[\|x_1\| + \frac{M(b-s_m)^{\gamma-1}}{\Gamma(\gamma)}hn + \frac{M\Omega(n)}{\Gamma(\alpha)}\eta\|\varphi\|_{L^p(J,\mathbb{R}^+)}\biggr] \\ &\quad + hn + \frac{Mhn}{\Gamma(\gamma)}. \end{aligned}$$

Divide both sides by n and pass to the limit as $n \to \infty$, and we obtain

$$\begin{aligned} 1 &\leqslant \frac{M\eta\upsilon b^{1-\gamma}}{\Gamma(\alpha)}\|\varphi\|_{L^p(J,\mathbb{R}^+)} \\ &\quad + \frac{Mb^{1-\gamma}N^2}{\Gamma(\alpha)}\eta\biggl[\frac{M(b-s_m)^{\gamma-1}}{\Gamma(\gamma)}h + \frac{\upsilon M}{\Gamma(\alpha)}\eta\|\varphi\|_{L^p(J,\mathbb{R}^+)}\biggr] + h + \frac{hM}{\Gamma(\gamma)}, \end{aligned}$$

which contradicts (5).

Thus we deduce that there is a natural number n_0 such that $R(B_{n_0}) \subseteq B_{n_0}$.

Step 2. Let $K = \{z \in PC_{1-\gamma}(J,E),\ z \in R(B_{n_0})\}$. We claim that the subsets $K_{|\overline{J_k}}$ $(k = 0,1,\dots,m)$ and $K_{|\overline{T_i}}$ $(i = 1,2,\dots,m)$ are equicontinuous, where $K_{|\overline{J_k}} = \{z\colon \overline{J_k} \to E,\ z(t) = (t-s_k)^{1-\gamma}y(t),\ t \in J_k,\ z(s_k) = \lim_{t\to s_k}(t-s_k)^{1-\gamma}z(t),\ y \in R(x),\ x \in B_{n_0}\}$ and $K_{|\overline{T_i}} = \{y^* \in C(\overline{T_i},E)\colon y^*(t) = y(t),\ t \in [t_i,s_i],\ y^*(t_i) = y(t_i^+),\ y \in R(x),\ x \in B_{n_0}\}$.

Case 1. Let $z \in K_{|\overline{J_0}}$. Then there is a $x \in B_{n_0}$ and a $f \in S^p_{F(\cdot,x(\cdot))}$ such that for $t \in (0,t_1]$,

$$z(t) = t^{1-\gamma}\Biggl[S_{\alpha,\beta}(t)\big(x_0 + g(x)\big) + \int_0^t K_\alpha(t-s)\big(f(s) + B\big(u_{x,f}(s)\big)\big)\,\mathrm{d}s\Biggr],$$

and $z(0) = \lim_{t\to 0+} t^{1-\gamma}y(t)$. It follows for $t = 0$, $\delta \in (0,t_1]$ that

$$\lim_{\delta\to 0+} z(\delta) = \lim_{\delta\to 0+}\delta^{1-\gamma}y(\delta) = \lim_{t\to 0+}t^{1-\gamma}y(t) = z(0).$$

Let t, $t+\delta$ be two points in $(0,t_1]$. Then

$$\begin{aligned} &\big\|z(t+\delta) - z(t)\big\| \\ &\quad\leqslant \big\|(t+\delta)^{1-\gamma}S_{\alpha,\beta}(t+\delta)\big(x_0 + g(x)\big) - t^{1-\gamma}S_{\alpha,\beta}(t)\big(x_0 + g(x)\big)\big\| \end{aligned}$$

$$
\begin{aligned}
&+ \left\| (t+\delta)^{1-\gamma} \int\limits_0^{t+\delta} K_\alpha(t+\delta-s)\bigl(f(s)+B\bigl(u_{x,f}(s)\bigr)\bigr)\,\mathrm{d}s \right. \\
&\left. - t^{1-\gamma} \int\limits_0^{t} K_\alpha(t-s)\bigl(f(s)+B\bigl(u_{x,f}(s)\bigr)\bigr)\,\mathrm{d}s \right\| \\
&\leqslant \sum_{i=1}^{i=8} I_i,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= (t+\delta)^{1-\gamma}\bigl\|S_{\alpha,\beta}(t+\delta)\bigl(x_0+g(x)\bigr) - S_{\alpha,\beta}(t)\bigl(x_0+g(x)\bigr)\bigr\|, \\
I_2 &= \bigl|(t+\delta)^{1-\gamma} - t^{1-\gamma}\bigr|\bigl\|S_{\alpha,\beta}(t)(x_0+g(x))\bigr\|, \\
I_3 &= \left\|(t+\delta)^{1-\gamma} \int\limits_t^{t+\delta} K_\alpha(t+\delta-s)f(s)\,\mathrm{d}s\right\|, \\
I_4 &= \left\| \int\limits_0^t \bigl[(t+\delta)^{1-\gamma}K_\alpha(t+\delta-s)f(s) - t^{1-\gamma}(t-s)^{\alpha-1}P_\alpha(t+\delta-s)f(s)\bigr]\,\mathrm{d}s\right\|, \\
I_5 &= \left\| \int\limits_0^t \bigl[t^{1-\gamma}(t-s)^{\alpha-1}P_\alpha(t+\delta-s) - t^{1-\gamma}K_\alpha(t-s)\bigr]f(s)\,\mathrm{d}s\right\|, \\
I_6 &= \left\|(t+\delta)^{1-\gamma} \int\limits_t^{t+\delta} K_\alpha(t+\delta-s)B\bigl(u_{x,f}(s)\bigr)\,\mathrm{d}s\right\|, \\
I_7 &= \left\| \int\limits_0^t \bigl[(t+\delta)^{1-\gamma}K_\alpha(t+\delta-s) - t^{1-\gamma}(t-s)^{\alpha-1}P_\alpha(t+\delta-s)\bigr] \right. \\
&\qquad \left. \times B\bigl(u_{x,f}(s)\bigr)\,\mathrm{d}s\right\|, \\
I_8 &= \left\| \int\limits_0^t \bigl[t^{1-\gamma}(t-s)^{\alpha-1}P_\alpha(t+\delta-s) - t^{1-\gamma}K_\alpha(t-s)\bigr]B\bigl(u_{x,f}(s)\bigr)\,\mathrm{d}s\right\|.
\end{aligned}
$$

In view of Lemma 1, it follows that

$$
\begin{aligned}
\lim_{\delta\to 0} I_1 &= \lim_{\delta\to 0}(t+\delta)^{1-\gamma}\bigl\|S_{\alpha,\beta}(t+\delta)\bigl(x_0+g(x)\bigr) - S_{\alpha,\beta}(t)\bigl(x_0+g(x)\bigr)\bigr\| \\
&= 0
\end{aligned}
$$

and

$$\lim_{\delta\to 0} I_2 = \lim_{\delta\to 0} \left|(t+\delta)^{1-\gamma} - t^{1-\gamma}\right| \left\|S_{\alpha,\beta}(t)\big(x_0 + g(x)\big)\right\|$$
$$\leqslant \frac{Mt^{\gamma-1}}{\Gamma(\gamma)} \left\|\big(x_0 + g(x)\big)\right\| \lim_{\delta\to 0} \left|(t+\delta)^{1-\gamma} - t^{1-\gamma}\right| = 0.$$

From Lemma 1 and (F_2) we get

$$\lim_{\delta\to 0} I_3 = \lim_{\delta\to 0} \left\|(t+\delta)^{1-\gamma} \int_t^{t+\delta} K_\alpha(t+\delta-s) f(s)\,\mathrm{d}s\right\|$$
$$\leqslant \frac{M\Omega(n_0)}{\Gamma(\alpha)} \lim_{\delta\to 0} (t+\delta)^{1-\gamma} \int_t^{t+\delta} (t+\delta-s)^{\alpha-1} \varphi(s)\,\mathrm{d}s = 0.$$

Similarly,

$$\lim_{\delta\to 0} I_4 \leqslant \lim_{\delta\to 0} \left\| \int_0^t \big[(t+\delta)^{1-\gamma} K_\alpha(t+\delta-s) f(s) - t^{1-\gamma}(t-s)^{\alpha-1} P_\alpha(t+\delta-s) f(s)\big]\,\mathrm{d}s \right\|$$
$$= \lim_{\delta\to 0} \left\| \int_0^t \big[(t+\delta)^{1-\gamma}(t+\delta-s)^{\alpha-1} P_\alpha(t+\delta-s) f(s) - t^{1-\gamma}(t-s)^{\alpha-1} P_\alpha(t+\delta-s) f(s)\big]\,\mathrm{d}s \right\|$$
$$\leqslant \frac{M\Omega(n_0)}{\Gamma(\alpha)} \lim_{\delta\to 0} \int_0^t \left|(t+\delta)^{1-\gamma}(t+\delta-s)^{\alpha-1} - t^{1-\gamma}(t-s)^{\alpha-1}\right| \varphi(s)\,\mathrm{d}s.$$

Since $\varphi \in L^p(J,\mathbb{R}^+)$, $\int_0^t [(t+\delta)^{1-\gamma}(t+\delta-s)^{\alpha-1} - t^{1-\gamma}(t-s)^{\alpha-1}]\varphi(s)\,\mathrm{d}s$ exists, and from Lebesgue dominated convergence theorem we see that $\lim_{\delta\to 0} I_4 = 0$.

For I_5, note that

$$\lim_{\delta\to 0} I_5 = \lim_{\delta\to 0} \left\| \int_0^t \big[t^{1-\gamma}(t-s)^{\alpha-1} P_\alpha(t+\delta-s) - t^{1-\gamma} K_\alpha(t-s)\big] f(s)\,\mathrm{d}s \right\|$$
$$= \lim_{\delta\to 0} \left\| \int_0^t \big[t^{1-\gamma}(t-s)^{\alpha-1} P_\alpha(t+\delta-s) - t^{1-\gamma}(t-s)^{\alpha-1} P_\alpha(t-s)\big] f(s)\,\mathrm{d}s \right\|.$$

To find this limit, let $\epsilon > 0$ be enough small. We have

$$\begin{aligned}
\lim_{\delta\to 0} I_5 &\leqslant \Omega(n_0)t^{1-\gamma}\lim_{\delta\to 0}\int_0^{t-\epsilon}(t-s)^{\alpha-1}\varphi(s)\sup_{s\in[0,t-\epsilon]}\big\|P_\alpha(t+\delta-s)-P_\alpha(t-s)\big\|\,\mathrm{d}s \\
&\quad+\lim_{\delta\to 0}\int_{t-\epsilon}^{t} t^{1-\gamma}(t-s)^{\alpha-1}\big\|P_\alpha(t+\delta-s)f(s)-P_\alpha(t-s)f(s)\,\mathrm{d}s\big\| \\
&\leqslant \Omega(n_0)t^{1-\gamma}\lim_{\delta\to 0}\int_0^{t-\epsilon}(t-s)^{\alpha-1}\varphi(s)\sup_{s\in[0,t-\epsilon]}\big\|P_\alpha(t+\delta-s)-P_\alpha(t-s)\big\|\,\mathrm{d}s \\
&\quad+\frac{2M\Omega(n_0)}{\Gamma(\alpha)}\lim_{\delta\to 0}\int_{t-\epsilon}^{t} t^{1-\gamma}(t-s)^{\alpha-1}\varphi(s)\,\mathrm{d}s \\
&\leqslant \Omega(n_0)t^{1-\gamma}\lim_{\delta\to 0}\int_0^{t-\epsilon}(t-s)^{\alpha-1}\varphi(s)\sup_{s\in[0,t-\epsilon]}\big\|P_\alpha(t+\delta-s)-P_\alpha(t-s)\big\|\,\mathrm{d}s \\
&\quad+\frac{2M\Omega(n_0)}{\Gamma(\alpha)}\lim_{\delta\to 0}\Bigg[\int_0^{t} t^{1-\gamma}(t-s)^{\alpha-1}\varphi(s)\,\mathrm{d}s \\
&\quad-\int_0^{t-\epsilon}(t-\epsilon)^{1-\gamma}(t-\epsilon-s)^{\alpha-1}\varphi(s)\,\mathrm{d}s\Bigg] \\
&\quad+\frac{2M\Omega(n_0)}{\Gamma(\alpha)}\lim_{\delta\to 0}\Bigg[\int_0^{t-\epsilon}(t-\epsilon)^{1-\gamma}(t-\epsilon-s)^{\alpha-1}\varphi(s)\,\mathrm{d}s \\
&\quad-\int_0^{t-\epsilon} t^{1-\gamma}(t-s)^{\alpha-1}\varphi(s)\,\mathrm{d}s\Bigg].
\end{aligned}$$

From Lemma 1, $\lim_{\delta\to 0}\sup_{s\in[0,t-\epsilon]}\|P_\alpha(t+\delta-s)-P_\alpha(t-s)\| = 0$, and since $\varphi \in L^p(J,\mathbb{R}^+)$, then from the Lebesgue dominated convergence theorem we see that $I_5 \to 0$ as $\delta \to 0$ and $\epsilon \to 0$.

Next, it follows from (10) that

$$\begin{aligned}
\lim_{\delta\to 0} I_6 &= \lim_{\delta\to 0}(t+\delta)^{1-\gamma}\Bigg\|\int_t^{t+\delta}K_\alpha(t+\delta-s)B\big(u_{x,f}(s)\big)\,\mathrm{d}s\Bigg\| \\
&\leqslant \lim_{\delta\to 0}\frac{M(t+\delta)^{1-\gamma}N}{\Gamma(\alpha)}\int_t^{t+\delta}(t+\delta-s)^{\alpha-1}\big\|u_{x,f}(s)\big\|\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \lim_{\delta\to 0}\frac{M(t+\delta)^{1-\gamma}N}{\Gamma(\alpha)}\|u_{x,f}\|_{L^p(J,X)}\\
&\quad\times\left(\int\limits_t^{t+\delta}(t+\delta-s)^{(\alpha-1)p/(p-1)}\,\mathrm{d}s\right)^{(p-1)/p}\\
&\leqslant N\left[\|x_1\|+\frac{M(b-s_m)^{\gamma-1}}{\Gamma(\gamma)}hn+\frac{M\Omega(n)}{\Gamma(\alpha)}\eta\|\varphi\|_{L^p(J,\mathbb{R}+)}\right]\\
&\quad\times\lim_{\delta\to 0}\frac{M(t+\delta)^{1-\gamma}N}{\Gamma(\alpha)}\left(\int\limits_t^{t+\delta}(t+\delta-s)^{(\alpha-1)p/(p-1)}\,\mathrm{d}s\right)^{(p-1)/p}\\
&=0.
\end{aligned}
$$

For I_7, note that

$$
\begin{aligned}
&\big\|\big[(t+\delta)^{1-\gamma}(t+\delta-s)^{\alpha-1}-t^{1-\gamma}(t-s)^{\alpha-1}\big]f(s)\big\|\\
&\quad\leqslant \Omega(n_0)\big[(t+\delta)^{1-\gamma}(t-s)^{\alpha-1}+t^{1-\gamma}(t-s)^{\alpha-1}\big]\varphi(s),\quad \text{a.e. } s\in[0,t].
\end{aligned}
$$

Since $\varphi\in L^p(J,\mathbb{R}+)$ and $\int_0^t[(t+\delta)^{1-\gamma}(t+\delta-s)^{\alpha-1}-t^{1-\gamma}(t-s)^{\alpha-1}]\varphi(s)\,\mathrm{d}s$ exists, then from the Lebesgue dominated convergence theorem we see that

$$
\begin{aligned}
\lim_{\delta\to 0}I_7 &\leqslant \lim_{\delta\to 0}\left\|\int\limits_0^t\big[(t+\delta)^{1-\gamma}K_\alpha(t+\delta-s)-t^{1-\gamma}(t-s)^{\alpha-1}P_\alpha(t+\delta-s)\big]\right.\\
&\qquad\left.\times B\big(u_{x,f}(s)\big)\,\mathrm{d}s\right\|\\
&=\lim_{\delta\to 0}\int\limits_0^t\big|(t+\delta)^{1-\gamma}(t+\delta-s)^{\alpha-1}-t^{1-\gamma}(t-s)^{\alpha-1}\big|\\
&\qquad\times\big\|P_\alpha(t+\delta-s)B\big(u_{x,f}(s)\big)\big\|\,\mathrm{d}s\\
&\leqslant\frac{NM\Omega(n_0)}{\Gamma(\alpha)}\lim_{\delta\to 0}\int\limits_0^t\big|(t+\delta)^{1-\gamma}(t+\delta-s)^{\alpha-1}-t^{1-\gamma}(t-s)^{\alpha-1}\big|\\
&\qquad\times\big\|u_{x,f}(s)\big\|\,\mathrm{d}s\\
&\leqslant\frac{NM\Omega(n_0)}{\Gamma(\alpha)}\|u_{x,f}\|_{L^p(J,X)}\\
&\quad\times\lim_{\delta\to 0}\left(\int\limits_0^t\big|(t+\delta)^{1-\gamma}(t+\delta-s)^{\alpha-1}-t^{1-\gamma}(t-s)^{\alpha-1}\big|^{p/(p-1)}\,\mathrm{d}s\right)^{(p-1)/p}
\end{aligned}
$$

$$\leqslant \frac{NM\Omega(n_0)}{\Gamma(\alpha)}\left[\|x_1\| + \frac{M(b-s_m)^{\gamma-1}}{\Gamma(\gamma)}h_m n_0 + \frac{M\Omega(n_0)}{\Gamma(\alpha)}\eta\|\varphi\|_{L^p(J,\mathbb{R}^+)}\right]$$
$$\times \lim_{\delta\to 0}\left(\int_0^t \left|(t+\delta)^{1-\gamma}(t+\delta-s)^{\alpha-1} - t^{1-\gamma}(t-s)^{\alpha-1}\right|^{p/(p-1)}\,\mathrm{d}s\right)^{(p-1)/p}$$
$$= 0.$$

Next,

$$\lim_{\delta\to 0} I_8 = \lim_{\delta\to 0}\left\|\int_0^t \left[t^{1-\gamma}(t-s)^{\alpha-1}P_\alpha(t+\delta-s) - t^{1-\gamma}K_\alpha(t-s)\right] \times B\big(u_{x,f}(s)\big)\,\mathrm{d}s\right\|$$
$$= \lim_{\delta\to 0}\left\|\int_0^t \left[t^{1-\gamma}(t-s)^{\alpha-1}P_\alpha(t+\delta-s) - t^{1-\gamma}(t-s)^{\alpha-1}P_\alpha(t-s)\right] \times B\big(u_{x,f}(s)\big)\,\mathrm{d}s\right\|.$$

To find this limit, let $\epsilon > 0$ be enough small. We have

$$\lim_{\delta\to 0} I_8 \leqslant N\Omega(n_0)t^{1-\gamma}\lim_{\delta\to 0}\int_0^{t-\epsilon}(t-s)^{\alpha-1}\|u_{x,f}(s)\| \times \sup_{s\in[0,t-\epsilon]}\|P_\alpha(t+\delta-s) - P_\alpha(t-s)\|\,\mathrm{d}s$$
$$+ \lim_{\delta\to 0}\int_{t-\epsilon}^t t^{1-\gamma}(t-s)^{\alpha-1}\|P_\alpha(t+\delta-s)B\big(u_{x,f}(s)\big) \times -P_\alpha(t-s)B\big(u_{x,f}(s)\big)\|\,\mathrm{d}s$$
$$\leqslant N\Omega(n_0)t^{1-\gamma}\lim_{\delta\to 0}\int_0^{t-\epsilon}(t-s)^{\alpha-1}\|u_{x,f}(s)\| \times \sup_{s\in[0,t-\epsilon]}\|P_\alpha(t+\delta-s)f(s) - P_\alpha(t-s)\|\,\mathrm{d}s$$
$$+ \frac{2MN\Omega(n_0)}{\Gamma(\alpha)}\lim_{\delta\to 0}\int_{t-\epsilon}^t t^{1-\gamma}(t-s)^{\alpha-1}\|u_{x,f}(s)\|\,\mathrm{d}s$$

$$
\begin{aligned}
&\leqslant N\varOmega(n_0)t^{1-\gamma}\lim_{\delta\to 0}\int_0^{t-\epsilon}(t-s)^{\alpha-1}\big\|u_{x,f}(s)\big\| \\
&\qquad\times\sup_{s\in[0,t-\epsilon]}\big\|P_\alpha(t+\delta-s)f(s)-P_\alpha(t-s)\big\|\,\mathrm{d}s \\
&\quad+\frac{2NM\varOmega(n_0)}{\Gamma(\alpha)}\lim_{\delta\to 0}\Bigg[\int_0^t t^{1-\gamma}(t-s)^{\alpha-1}\big\|u_{x,f}(s)\big\|\,\mathrm{d}s \\
&\quad-\int_0^{t-\epsilon}(t-\epsilon)^{1-\gamma}(t-\epsilon-s)^{\alpha-1}\big\|u_{x,f}(s)\big\|\,\mathrm{d}s\Bigg] \\
&\quad+\frac{2MN\varOmega(n_0)}{\Gamma(\alpha)}\lim_{\delta\to 0}\Bigg[\int_0^{t-\epsilon}(t-\epsilon)^{1-\gamma}(t-\epsilon-s)^{\alpha-1}\big\|u_{x,f}(s)\big\|\,\mathrm{d}s \\
&\quad-\int_0^{t-\epsilon}t^{1-\gamma}(t-s)^{\alpha-1}\big\|u_{x,f}(s)\big\|\,\mathrm{d}s\Bigg].
\end{aligned}
$$

From Lemma 1, $\lim_{\delta\to 0}\sup_{s\in[0,t-\epsilon]}\|P_\alpha(t+\delta-s)-P_\alpha(t-s)\| = 0$, and since $u_{x,f}\in L^p(J,X)$, then from the Lebesgue dominated convergence theorem we see that $I_8\to 0$ as $\delta\to 0$ and $\epsilon\to 0$.

Case 2. Let $y\in K_{|T_i}, i=1,2,\dots,m$. Then $y(t)=g_i(t,x(t_i^-))$, $t\in(t_i,s_i]$, $i=1,\dots,m$. Let $i\in\{1,2,\dots,m\}$ be fixed and $t,t+\delta\in(t_i,s_i]$. Since $\|x\|_{PC_{1-\gamma}(J,E)}\leqslant n_0$, it follows from the uniform continuity of g_i on bounded sets that $\lim_{\delta\to 0}\|y(t+\delta)-y(t)\|=\lim_{\delta\to 0}\|g_i(t+\delta,x(t_i^-))-g_i(t,x(t_i^-))\|=0$, independent of x.

When $t=t_i$, $i=1,\dots,m$, let $\delta>0$ be such that $t_i+\delta\in(t_i,s_i]$ and $\lambda>0$ such that $t_i<\lambda<t_i+\delta\leqslant s_i$. Then we have $\|y^*(t_i+\delta)-y^*(t_i)\|=\lim_{\lambda\to t_i^+}\|y(t_i+\delta)-y(\lambda)\|=0$.

Case 3. Let $z\in K_{|J_k}$, $k=1,\dots,m$. Then there is a $x\in B_{n_0}$ and a $f\in S^p_{F(\cdot,x(\cdot))}$ such that for $t\in(s_k,t_{k+1}]$,

$$
\begin{aligned}
z(t)=(t-s_k)^{1-\gamma}\Bigg[&S_{\alpha,\beta}(t-s_k)g_k\big(s_k,x\big(t_k^-\big)\big) \\
&+\int_{s_k}^t K_\alpha(t-s)\big(f_n(s)+B\big(u_{x,f}(s)\big)\big)\,\mathrm{d}s\Bigg].
\end{aligned}
$$

Let $k\in\{1,2,\dots,m\}$ be fixed. If $t=s_k$ and $\delta>0$, then

$$
\begin{aligned}
\lim_{\delta\to 0+}z(s_k+\delta)&=\lim_{\delta\to 0+}(s_k+\delta-s_k)^{1-\gamma}y(s_k+\delta) \\
&=\lim_{t\to s_k+}(t-s_k)^{1-\gamma}y(t)=z(s_k).
\end{aligned}
$$

Next, let $t, t+\delta \in (s_i, t_{i+1}]$, $\delta > 0$. Then we have

$$\begin{aligned}\big\|z(t+\delta) - z(t)\big\| &= \big\|(t+\delta-s_k)^{1-\gamma} S_{\alpha,\beta}(t+\delta-s_k) g_k\big(s_k, x\big(t_k^-\big)\big) \\ &\quad - (t-s_k)^{1-\gamma} S_{\alpha,\beta}(t-s_k) g_k\big(s_k, x\big(t_k^-\big)\big)\big\| \\ &\quad + \left\|(t+\delta-s_k)^{1-\gamma}\int_{s_k}^{t} K_\alpha(t+\delta-s)\big(f_n(s) + B\big(u_{x_n,f_n}(s)\big)\big)\,\mathrm{d}s \right. \\ &\quad \left. - (t-s_k)^{1-\gamma}\int_{s_k}^{t+\delta} K_\alpha(t-s)\big(f_n(s) + B\big(u_{x_n,f_n}(s)\big)\big)\,\mathrm{d}s\right\|.\end{aligned}$$

Arguing as in Case 1, we conclude that $\lim_{\delta\to 0}\|z(t+\delta) - z(t)\| = 0$.

Step 3. The graph of the multivalued function $R_{|B_{n_0}} : B_{n_0} \to 2^{B_{n_0}}$ is closed. Consider a sequence $\{x_n\}_{n\geqslant 1}$ in B_{n_0} with $x_n \to x$ in B_{n_0} and let $y_n \in R(x_n)$ with $y_n \to y$ in $PC_{1-\gamma}(J,E)$. We need to show $y \in R(x)$. Recalling the definition of R, for any $n \geqslant 1$, there is a $f_n \in S^p_{F(\cdot,x_n(\cdot))}$ such that (8) holds.

In view of (6), $\|f_n(t)\| \leqslant \varphi(t)\Omega(n_0)$ for every $n \geqslant 1$ and for a.e. $t \in J$. Then $\{f_n,\, n \geqslant 1\}$ is bounded in $L^p(J,E)$. Because $p > 1$, $L^p(J,E)$ is reflexive, and hence, without loss of generality, we can assume that $\{f_n\}$ converges weakly to a function $f \in L^p(J,E)$. From Mazur's lemma, for every natural number j, there is a natural number $k_0(j) > j$ and a sequence of nonnegative real numbers $\lambda_{j,k}$, $k = k_0(j), \dots, j$, such that $\sum_{k=j}^{k_0} \lambda_{j,k} = 1$, and the sequence of convex combinations $z_j = \sum_{k=j}^{k_0} \lambda_{j,k} f_k$, $j \geqslant 1$, converges strongly to f in $L^1(J,E)$ as $j \to \infty$.

Take $\overline{y}_n(t) = \sum_{k=n}^{k_0(n)} \lambda_{n,k} y_k(t)$. Then

$$\overline{y}_n(t) = \begin{cases} S_{\alpha,\beta}(t)(x_0 + g(x_n)) + \int_0^t K_\alpha(t-s)(z_n(s) + B(u_{x_n,z_n}(s)))\,\mathrm{d}s, \\ \quad t \in (0,t_1], \\ g_i(t, x_n(t_i^-)), \quad t \in (t_i, s_i],\ i = 1,\dots,m, \\ S_{\alpha,\beta}(t-s_i) g_i(s_i, x_n(t_i^-)) + \int_{s_i}^t K_\alpha(t-s)(z_n(s) + B(u_{x_n,z_n}(s)))\,\mathrm{d}s, \\ \quad t \in (s_i, t_{i+1}],\ i = 1,\dots,m. \end{cases}$$

From the continuity of W^{-1} and the fact that $z_n(t) \to f(t)$ a.e. it follows from the Lebesgue dominated convergence theorem that $\lim_{n\to\infty} u_{x_n,z_n}(t) = u_{x,f}(t)$, a.e. $t \in J$. Then by the continuity of g and B and by the uniform continuity of g_i on bounded sets it follows from the Lebesgue dominated convergence theorem that $\overline{y}_n(t) \to v(t)$, where

$$v(t) = \begin{cases} S_{\alpha,\beta}(t)(x_0 + g(x)) + \int_0^t K_\alpha(t-s)(f(s) + B(u_{x,f}(s)))\,\mathrm{d}s, \\ \quad t \in (0,t_1], \\ g_i(t, x(t_i^-)), \quad t \in (t_i, s_i],\ i = 1,\dots,m, \\ S_{\alpha,\beta}(t-s_i) g_i(s_i, x(t_i^-)) + \int_{s_i}^t K_\alpha(t-s)(f(s) + B(u_{x,f}(s)))\,\mathrm{d}s, \\ \quad t \in (s_i, t_{i+1}],\ i = 1,\dots,m. \end{cases}$$

Since $y_n \to y$, then $y = v$. For almost everywhere t, $F(t,\cdot)$ is upper semicontinuous with closed convex values, so from [1, Chap. 1, Sect. 4, Thm. 1] it follows that $f(t) \in F(t, x(t))$, a.e. $t \in J$, and hence R is closed.

Step 4. We show (3) holds with $x_0 = 0$.

Let $Z \subseteq B_{n_0}$, $Z = \operatorname{conv}(\{0\} \cup R(Z))$, $\overline{Z} = \overline{C}$ with $C \subseteq Z$ countable. We claim that Z is relatively compact in $PC_{1-\gamma}(J,E)$. Since C is countable and $C \subseteq Z = \operatorname{conv}(\{0\} \cup R(Z))$, we can find a countable set $\mathcal{H} = \{y_n,\, n \geqslant 1\} \subseteq R(Z)$ with $C \subseteq \operatorname{conv}(\{0\} \cup \mathcal{H})$. Now for any $n \geqslant 1$, there exists $x_n \in Z \subseteq B_{n_0}$ with $y_n \in R(x_n)$. Thus there is a $f_n \in S^p_{F(\cdot,x_n(\cdot))}$ such that (8) holds. According to the definition of $\chi_{PC_{1-\gamma}(J,E)}(Z)$, one obtains

$$\begin{aligned}
\chi_{PC_{1-\gamma}(J,E)}(Z) &= \chi_{PC_{1-\gamma}(J,E)}(\overline{Z}) = \chi_{PC_{1-\gamma}(J,E)}(\overline{C}) \\
&= \chi_{PC_{1-\gamma}(J,E)}(C) \leqslant \chi_{PC_{1-\gamma}(J,E)}\bigl(\operatorname{conv}\bigl(\{x_0\} \cup \mathcal{H}\bigr)\bigr) \\
&= \chi_{PC_{1-\gamma}(J,E)}(\mathcal{H}) \\
&= \max\Bigl\{\max_{k=0,1,\dots,m} \chi_{C(\overline{J_k},E)}(\mathcal{H}_{|}\overline{J_k}),\ \max_{i=1,\dots,m} \chi_{C(\overline{T_i},E)}(\mathcal{H}_{|\overline{T_i}})\Bigr\}.
\end{aligned}$$

Since $Z_{|\overline{J_i}}$ and $Z_{|\overline{T_i}}$ are equicontinuous, then from Lemma 3 the last inequality becomes

$$\begin{aligned}
&\chi_{PC_{1-\gamma}(J,E)}(Z) \\
&\quad \leqslant \max\Bigl\{\max_{i=0,1,\dots,m}\max_{t\in\overline{J_k}} \chi\bigl\{y_n^*(t),\, n \geqslant 1\bigr\},\ \max_{i=1,\dots,m}\max_{t\in\overline{T_i}} \chi\bigl\{y_n^*(t),\, n \geqslant 1\bigr\}\Bigr\}, \qquad (14)
\end{aligned}$$

where

$$y_n^*(t) = \begin{cases}
t^{1-\gamma} y(t), \quad t \in (0, t_1], \\
\lim_{t\to 0} t^{1-\gamma} y(t), \quad t = 0, \\
g_i(t, x_n(t_i^-)), \quad t \in (t_i, s_i],\ i = 1,\dots,m, \\
y_n(t_i^+), \quad t = t_i, \\
(t - s_i)^{1-\gamma} y(t), \quad t \in (s_i, t_{i+1}],\ i = 1,\dots,m, \\
\lim_{t\to s_i} (t - s_i)^{\gamma-1} y(t), \quad t = s_i,\ i = 1,\dots,m.
\end{cases}$$

That is,

$$y_n^*(t) = \begin{cases}
t^{1-\gamma} S_{\alpha,\beta}(t)(x_0 + g(x_n)) \\
\qquad + t^{1-\gamma} \int_0^t K_\alpha(t-s)(f_n(s) + B(u_{x_n,f_n}(s)))\,\mathrm{d}s, \quad t \in (0, t_1], \\
\lim_{t\to 0} t^{1-\gamma} y(t), \quad t = 0, \\
g_i(t, x_n(t_i^-)), \quad t \in (t_i, s_i],\ i = 1,\dots,m, \\
g_i(t_i, x_n(t_i^-)), \quad t = t_i, \\
(t - s_i)^{1-\gamma} S_{\alpha,\beta}(t - s_i) g_i(s_i, x_n(t_i^-)) \\
\qquad + (t - s_i)^{1-\gamma} [\int_{s_i}^t K_\alpha(t-s)(f_n(s) + B(u_{x_n,f_n}(s)))\,\mathrm{d}s], \\
\qquad\quad t \in (s_i, t_{i+1}],\ i = 1,\dots,m, \\
\lim_{t\to s_i^+} (t - s_i)^{\gamma-1} y(t), \quad t = s_i, i = 1,\dots,m.
\end{cases}$$

Then, using the properties of the measure of noncompactness, one has

$$\chi\{y_n^*(t),\, n \geqslant 1\} \leqslant \begin{cases} \chi\{t^{1-\gamma}S_{\alpha,\beta}(t)(x_0+g(x_n)),\, n\geqslant 1\} \\ \quad +t^{1-\gamma}\chi\{\int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)f_n(s)\,\mathrm{d}s,\, n\geqslant 1\} \\ \quad +t^{1-\gamma}\chi\{\int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)B(u_{x_n,f_n}(s)))\,\mathrm{d}s,\, n\geqslant 1\}, \\ \qquad t\in(0,t_1], \\ \chi\{\lim_{t\to 0^+} t^{1-\gamma}y_n(t),\, n\geqslant 1\}, \quad t=0, \\ \chi\{g_i(t,x_n(t_i^-)), n\geqslant 1\}, \quad t\in(t_i,s_i],\ i=1,\dots,m, \\ \chi\{g_i(t_i,x_n(t_i^-)),\, n\geqslant 1\}, \quad t=t_i,\ i=1,\dots,m, \\ \chi\{(t-s_i)^{1-\gamma}S_{\alpha,\beta}(t-s_i)g_i(s_i,x_n(t_i^-)),\, n\geqslant 1\} \\ \quad +(t-s_i)^{1-\gamma}\chi\{\int_{s_m}^t (t-s)^{\alpha-1}f_n(s)\,\mathrm{d}s,\, n\geqslant 1\} \\ \quad +(t-s_i)^{1-\gamma}\,\chi\{\int_{s_m}^t (t-s)^{\alpha-1}B(u_{x_n,f_n}(s)))\,\mathrm{d}s,\, n\geqslant 1\}, \\ \qquad t\in(s_i,t_{i+1}],\ i=1,\dots,m, \\ \chi\{\lim_{t\to s_i}(t-s_i)^{1-\gamma}y_n(t),\, n\geqslant 1\}, \quad t=s_i. \end{cases}$$

From the compactness of g and the continuity of $S_{\alpha,\beta}$ it follows that $\chi\{t^{1-\gamma}S_{\alpha,\beta}(t)\times (x_0+g(x_n)),\, n\geqslant 1\}=0$.

Then, if $t\in J_0$, using Lemma 2, we get that

$$\chi\{y_n^*(t),\, n\geqslant 1\} \leqslant \frac{2t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\chi\{f_n(s),\, n\geqslant 1\}\,\mathrm{d}s + \frac{2t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\chi\{B(u_{x_n,f_n}(s)),\, n\geqslant 1\}\,\mathrm{d}s.$$

Observe that from (F_3) we have

$$\chi\{f_n(s),\, n\geqslant 1\} \leqslant \varsigma(s)s^{1-\gamma}\chi\{x_k(s),\, k\geqslant 1\} \leqslant \varsigma(s)\chi_{PC_{1-\gamma}(J,E)}(Z)$$

for a.e. $s\in J_0$. Thus

$$\int_0^t (t-s)^{\alpha-1}\chi\{f_n(s),\, n\geqslant 1\}\,\mathrm{d}s \leqslant \chi_{PC_{1-\gamma}(J,E)}(Z)\int_0^t (t-s)^{\alpha-1}\varsigma(s)\,\mathrm{d}s \leqslant \chi_{PC_{1-\gamma}(J,E)}(Z)\eta\|\varsigma\|_{L^p(J,\mathbb{R}^+)}. \tag{15}$$

Next, in order to estimate the quantity $\chi\{\int_0^t (t-s)^{\alpha-1}B(u_{x_n,f_n}(s)),\, n\geqslant 1\}$, we consider the operator $\Theta: L^p(\overline{J_0},X)\to E$:

$$\Theta(h)(t) = \int_0^t (t-s)^{\alpha-1}B(h(s))\,\mathrm{d}s;$$

here $h \in L^p(\overline{J_0}, X)$. Now Θ is linear, and for any $h_1, h_2 \in L^p(\overline{J_0}, X)$ and any $t \in J$,

$$\begin{aligned}
&\big\|\Theta(h_1)(t) - \Theta(h_2)(t)\big\| \\
&\quad \leqslant \int_0^t (t-s)^{\alpha-1}\big\|B\big(h_1(s)\big) - B\big(h_2(s)\big)\big\|\,\mathrm{d}s \\
&\quad \leqslant \|B\| \int_0^t (t-s)^{\alpha-1}\|h_1(s) - h_2(s)\|\,\mathrm{d}s \leqslant N\eta\|h_1 - h_2\|_{L^p(J,X)}.
\end{aligned}$$

Thus Θ is linear and continuous (bounded). Moreover, from the linearity and boundedness of W^{-1}, the compactness of g_m, the continuity of $S_{\alpha,\beta}(b-s_m)$, (7) and (15) we have

$$\begin{aligned}
&\chi_{L^p(J,X)}\{u_{x_n,f_n},\, n \geqslant 1\} \\
&\quad \leqslant \chi_{L^p(J,X)}\Bigg(W^{-1}\Bigg\{x_1 - S_{\alpha,\beta}(b-s_m)g_m\big(s_m, x\big(t_m^-\big)\big) \\
&\qquad - \frac{1}{\Gamma(\alpha)}\int_{s_m}^b K_\alpha(b-s)^{\alpha-1}f_n(s)\,\mathrm{d}s,\, n \geqslant 1\Bigg\}\Bigg) \\
&\quad \leqslant N\Bigg[\chi\Bigg\{x_1 - S_{\alpha,\beta}(b-s_m)g_m\big(s_m, x\big(t_m^-\big)\big) \\
&\qquad - \frac{1}{\Gamma(\alpha)}\int_{s_m}^b K_\alpha(b-s)^{\alpha-1}f_n(s)\,\mathrm{d}s,\, n \geqslant 1\Bigg\}\Bigg] \\
&\quad = \frac{N}{\Gamma(\alpha)}\chi\Bigg\{\int_{s_m}^b K_\alpha(b-s)^{\alpha-1}f_n(s)\,\mathrm{d}s,\, n \geqslant 1\Bigg\} \\
&\quad \leqslant \frac{2MN}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\chi\big\{f_n(s),\, n \geqslant 1\big\}\,\mathrm{d}s \\
&\quad \leqslant \frac{2\eta MN}{\Gamma(\alpha)}\chi_{PC_{1-\gamma}(J,E)}(Z)\|\varsigma\|_{L^p(J,\mathbb{R}^+)}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\chi\Bigg\{\int_0^t (t-s)^{\alpha-1}B\big(u_{x_n,f_n}(s)\big),\, n \geqslant 1\Bigg\} \\
&\quad = \chi\big\{\Theta(u_{x_n,f_n}),\, n \geqslant 1\big\} \leqslant \|\Theta\|\chi_{L^p(J,X)}\{u_{x_n,f_n},\, n \geqslant 1\} \\
&\quad \leqslant \frac{2\eta MN^2}{\Gamma(\alpha)}\chi_{PC_{1-\gamma}(J,E)}(Z)\|\varsigma\|_{L^p(J,\mathbb{R}^+)}.
\end{aligned}$$

This inequality with (15) gives

$$\max_{t\in(0,t]}\chi\{y_n^*(t),\, n\geqslant 1\} \leqslant \chi_{PC_{1-\gamma}(J,E)} b^{1-\gamma} M\|\varsigma\|_{L^p(J,\mathbb{R}^+)}\left[\frac{2\eta}{\Gamma(\alpha)}+\frac{2\eta N^2}{\Gamma(\alpha)^2}\right]. \tag{16}$$

Next, from the compactness of g one obtains

$$\begin{aligned}
&\chi\{y_n^*(0),\, n\geqslant 1\}\\
&\quad=\chi\Big\{\lim_{t\to 0^+} t^{1-\gamma}y_n(t),\, n\geqslant 1\Big\}=\chi\Big\{\frac{1}{\Gamma(\gamma)}\big(x_0+g(x_n)\big),\, n\geqslant 1\Big\}\\
&\quad=0.
\end{aligned} \tag{17}$$

Moreover, since $x_n(t_i^-)\to x(t_i^-)$, the set $\{x_n(t_i^-),\, n\geqslant 1\}$ is bounded for every $i=1,2,\dots,m$. Then from the compactness of g_i we get for $i=1,\dots,m$,

$$\chi\{g_i(t, x_n(t_i^-)),\, n\geqslant 1\}=0,\quad t\in(t_i,s_i] \tag{18}$$

and $\chi\{g_i(t_i,x_n(t_i^-)),\, n\geqslant 1\}=0$.

Then for $i=1,2,\dots,m$,

$$\begin{aligned}
\chi\{y_n^*(s_i),\, n\geqslant 1\} &= \chi\Big\{\lim_{t\to s_i^+}(t-s_i)^{\gamma-1}y(t),\, n\geqslant 1\Big\}\\
&= \chi\Big\{\frac{M}{\Gamma(\gamma)}g_i\big(s_i,x_n(t_i^-)\big),\, n\geqslant 1\Big\}=0.
\end{aligned} \tag{19}$$

As above, $\chi\{(t-s_i)^{1-\gamma}S_{\alpha,\beta}(t-s_i)g_i(s_i,x_n(t_i^-)),\, n\geqslant 1\}=0$, $i=1,2,\dots,m$. Then, arguing as above, we see that for any $k=1,2,\dots,m$,

$$\max_{t\in J_k}\chi\{y_n^*(t),\, n\geqslant 1\} \leqslant \chi_{PC_{1-\gamma}(J,E)} b^{1-\gamma} M\|\varsigma\|_{L^p(J,\mathbb{R}^+)}\left[\frac{2\eta}{\Gamma(\alpha)}+\frac{2\eta N^2}{\Gamma(\alpha)^2}\right]. \tag{20}$$

From (4), (14), (16)–(20) we get

$$\begin{aligned}
\chi_{PC_{1-\gamma}(J,E)}(Z) &\leqslant \chi_{PC_{1-\gamma}(J,E)} b^{1-\gamma} M\|\varsigma\|_{L^p(J,\mathbb{R}^+)}\left[\frac{2\eta}{\Gamma(\alpha)}+\frac{2\eta N^2}{\Gamma(\alpha)^2}\right]\\
&< \chi_{PC}(Z).
\end{aligned}$$

Thus $\chi_{PC_{1-\gamma}(J,E)}(Z)=0$, and hence Z is relatively compact.

Step 5. R maps compact sets into relatively compact sets.

Let B be a compact subset of B_{n_0}. Let (y_n), $n\geqslant 1$, be a sequence in $R(B)$. Then there is a sequence (x_n), $n\geqslant 1$, in B, such that $y_n\in R(x_n)$. Thus there is a $f_n\in S^p_{F(\cdot,x_n(\cdot))}$ such that for $t\in J$, (8) holds. We show that $Z=\{y_n,\, n\geqslant 1\}$ is relatively compact in $PC_{1-\gamma}(J,E)$. Since B is compact in $PC_{1-\gamma}(J,E)$, we can assume, without loss of generality, that $x_n\to x$ in B. As in Step 3, we see that there is a subsequence of (y_n), which converges to a function $v\in R(B)$. Then the set $\{y_n,\, n\geqslant 1\}$ is relatively compact in $PC_{1-\gamma}(J,E)$. Thus $R(B)$ is relatively compact.

Now, by applying Lemma 4, there is a $x \in PC_{1-\gamma}(J,E)$ and $f \in S^p_{F(\cdot,x(\cdot))}$ such that

$$x(t) = \begin{cases} S_{\alpha,\beta}(t)(x_0 + g(x)) + \int_0^t K_\alpha(t-s)(f(s) + B(u_{x,f}(s)))\,\mathrm{d}s, \\ \quad t \in (0,t_1] \\ g_i(t, x(t_i^-)), \quad t \in (t_i, s_i],\ i = 1,\dots,m, \\ S_{\alpha,\beta}(t-s_i)g_i(s_i, x(t_i^-)) + \int_{s_i}^t K_\alpha(t-s)(f(s) + B(u_{x,f}(s)))\,\mathrm{d}s, \\ \quad t \in (s_i, t_{i+1}],\ i = 1,\dots,m. \end{cases}$$

This completes the proof. □

Theorem 2. *Assume* (H_1), (F_1), (F_3), (H_g), (H_{g_i}) *and* (HW) *hold and, in addition, suppose*

(F_2^*) *For any natural number* n, *there is a function* $\varphi_n \in L^p(J,\mathbb{R}^+)$ *such that* $\sup_{\|x\|\leqslant n} \|F(t,x)\| \leqslant \varphi_n(t)$, *a.e.* $t \in J$, *and*

$$\liminf_{n\to\infty} \frac{\|\varphi_n\|_{L^p(J,\mathbb{R}^+)}}{n} = 0. \tag{21}$$

Then problem (1) *is controllable, provided that*

$$\frac{M^2 b^{1-\gamma} N^2}{\Gamma(\alpha)} \eta \frac{(b-s_m)^{\gamma-1}}{\Gamma(\gamma)} h + h + \frac{Mh}{\Gamma(\gamma)} < 1. \tag{22}$$

Proof. We have only to prove that there is a natural number n such that $R(B_n) \subseteq B_n$, where $B_n = \{x \in PC_{1-\gamma}(J,E)\colon \|x\|_{PC_{1-\gamma}(J,E)} \leqslant n\}$. Suppose the contrary. Then for any $n \in \mathbb{N}$, there are $x_n, y_n \in PC_{1-\gamma}(J,E)$ with $y_n \in R(x_n)$, $\|x_n\|_{PC_{1-\gamma}(J,E)} \leqslant n$ and $\|y_n\|_{PC_{1-\gamma}(J,E)} > n$. Then there is a $f_n \in S^p_{F(\cdot,x_n(\cdot))}$, $n \geqslant 1$, such that (8) holds. Let $t \in [0,t_1]$. As in Step 1 in Theorem 1, we get

$$\sup_{t\in[0,t_1]} t^{1-\gamma}\|y_n(t)\| \leqslant \frac{M}{\Gamma(\gamma)}\left[\|x_0\| + \|g(x_n)\|\right] + \eta\frac{Mb^{1-\gamma}}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,\mathbb{R}^+)} + \eta\frac{MNb^{1-\gamma}}{\Gamma(\alpha)}\|u_{x_n,f_n}\|_{L^p(J,\mathbb{R}^+)}.$$

Note

$$\begin{aligned} \|u_{x_n,f_n}\|_{L^p(J,\mathbb{R}^+)} &\leqslant \|W^{-1}\|\left[\|x_1\| + \frac{(b-s_m)^{\gamma-1}}{\Gamma(\gamma)}\|g_m(s_m, x(t_m^-))\| + \frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}\varphi_n(s)\,\mathrm{d}s\right] \\ &\leqslant N\left[\|x_1\| + \frac{(b-s_m)^{\gamma-1}}{\Gamma(\gamma)}h_m n + \frac{\eta}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,\mathbb{R}^+)}\right] \\ &= N\left[\|x_1\| + \frac{(b-s_m)^{\gamma-1}}{\Gamma(\gamma)}h_m n + \frac{\eta}{\Gamma(\alpha)}\|\varphi\|_{L^p(J,\mathbb{R}^+)}\right]. \end{aligned} \tag{23}$$

Then

$$\begin{aligned}
&\sup_{t\in[0,t_1]} t^{1-\gamma}\big\|y_n(t)\big\| \\
&\quad\leqslant \frac{1}{\Gamma(\gamma)}\big[\|x_0\| + \big\|g(x_n)\big\|\big] + \eta\frac{b^{1-\gamma}}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,\mathbb{R}^+)} \\
&\qquad + \frac{b^{1-\gamma}N^2}{\Gamma(\alpha)}\eta\biggl[\|x_1\| + \frac{(b-s_m)^{\gamma-1}}{\Gamma(\gamma)}hn + \frac{\eta}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,\mathbb{R}^+)}\biggr].
\end{aligned}$$

If $t\in(t_i,s_i]$, $i=1,2,\dots,m$, then

$$\sup_{t\in[t_i,s_i]}\big\|y_n(t)\big\| = \sup_{t\in[t_i,s_i]}\big\|g_i\big(t,x_n\big(t_i^-\big)\big)\big\| \leqslant hn.$$

Similarly, we get for $t\in(s_i,t_{i+1}]$, $i=1,2,\dots,m$,

$$\begin{aligned}
&\sup_{t\in[s_i,t_{i+1}]}(t-s_i)^{1-\gamma}\big\|y_n(t)\big\| \\
&\quad\leqslant \sup_{t\in[s_i,t_{i+1}]}\frac{M\|g_i(s_i,x_n(t_i^-))\|}{\Gamma(\gamma)} + \frac{Mb^{1-\gamma}}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,\mathbb{R}^+)}\eta \\
&\qquad + \frac{Mb^{1-\gamma}N^2}{\Gamma(\alpha)}\eta\biggl[\|x_1\| + \frac{M(b-s_m)^{\gamma-1}}{\Gamma(\gamma)}hn + \frac{M\eta}{\Gamma(\alpha)}\|\varphi\|_{L^p(J,\mathbb{R}^+)}\biggr] \\
&\quad\leqslant \frac{Mhn}{\Gamma(\gamma)} + \frac{Mb^{1-\gamma}}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,\mathbb{R}^+)}\eta \\
&\qquad + \frac{Mb^{1-\gamma}N^2}{\Gamma(\alpha)}\eta\biggl[\|x_1\| + \frac{M(b-s_m)^{\gamma-1}}{\Gamma(\gamma)}hn + \frac{\eta M}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,\mathbb{R}^+)}\biggr].
\end{aligned}$$

Therefore,

$$\begin{aligned}
n < \|y_n\|_{PC_{1-\gamma}(J,E)} &\leqslant \frac{M}{\Gamma(\gamma)}\big[\|x_0\| + \big\|g(x_n)\big\|\big] + \eta\frac{Mb^{1-\gamma}}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,\mathbb{R}^+)} \\
&\quad + \frac{Mb^{1-\gamma}N^2}{\Gamma(\alpha)}\eta\biggl[\|x_1\| + \frac{M(b-s_m)^{\gamma-1}}{\Gamma(\gamma)}hn + \frac{M\eta}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,\mathbb{R}^+)}\biggr] \\
&\quad + hn + \frac{Mhn}{\Gamma(\gamma)}.
\end{aligned}$$

Divide both sides by n and pass to the limit as $n\to\infty$, and we have from (21) that

$$1 \leqslant \frac{M^2b^{1-\gamma}N^2}{\Gamma(\alpha)}\eta\frac{(b-s_m)^{\gamma-1}}{\Gamma(\gamma)}h + h + \frac{Mh}{\Gamma(\gamma)},$$

which contradicts (22). □

Theorem 3. *Suppose* (H_1), (F_1), (F_2^*), (F_3), (H_{g_i}) *and* (HW) *hold and, in addition, assume*

(H_g^*) $g: PC_{1-\gamma}(J,E)\to E$ *is Lipschitz continuous with Lipschitz constant* k *and compact.*

Then problem (1) *is controllable, provided that*

$$\frac{Mk}{\Gamma(\gamma)} + \frac{M^2 b^{1-\gamma} N^2}{\Gamma(\alpha)} \eta \frac{(b-s_m)^{\gamma-1}}{\Gamma(\gamma)} h + h + \frac{Mh}{\Gamma(\gamma)} < 1. \tag{24}$$

Proof. We have only to prove that there is a natural number n such that $R(B_n) \subseteq B_n$, where $B_n = \{x \in PC_{1-\gamma}(J,E)\colon \|x\|_{PC_{1-\gamma}(J,E)} \leqslant n\}$. Suppose the contrary. Then for any $n \in \mathbb{N}$, there are $x_n, y_n \in PC_{1-\gamma}(J,E)$ with $y_n \in R(x_n)$, $\|x_n\|_{PC_{1-\gamma}(J,E)} \leqslant n$ and $\|y_n\|_{PC_{1-\gamma}(J,E)} > n$. Then there is a $f_n \in S^p_{F(\cdot,x_n(\cdot))}$, $n \geqslant 1$, such that (8) holds. Let $t \in [0,t_1]$. As in Step 1 of Theorem 1, if $t \in [0,t_1]$, we get

$$\begin{aligned}
&\sup_{t\in[0,t_1]} t^{1-\gamma}\|y_n(t)\| \\
&\quad\leqslant \frac{M}{\Gamma(\gamma)} \sup_{t\in[0,t_1]} \bigl[\|x_0\| + \|g(x_n) - g(0)\| + \|g(0)\|\bigr] \\
&\qquad + \sup_{t\in[0,t_1]} \frac{Mt^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\varphi_n(s)\,\mathrm{d}s \\
&\qquad + \sup_{t\in[0,t_1]} \frac{MNt^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\|u_{x_n,f_n}(s)\|\,\mathrm{d}s \\
&\quad\leqslant \frac{M}{\Gamma(\gamma)}\bigl[\|x_0\| + kn + \|g(0)\|\bigr] + \eta\frac{Mb^{1-\gamma}}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,\mathbb{R}^+)} \\
&\qquad + \eta\frac{MNb^{1-\gamma}}{\Gamma(\alpha)}\|u_{x_n,f_n}\|_{L^p(J,\mathbb{R}^+)}.
\end{aligned}$$

Note (23), and we have

$$\begin{aligned}
&\sup_{t\in[0,t_1]} t^{1-\gamma}\|y_n(t)\| \\
&\quad\leqslant \frac{M}{\Gamma(\gamma)}\bigl[\|x_0\| + kn + \|g(0)\|\bigr] + \eta\frac{Mb^{1-\gamma}}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,\mathbb{R}^+)} \\
&\qquad + \frac{Mb^{1-\gamma}N^2}{\Gamma(\alpha)}\eta\biggl[\|x_1\| + \frac{M(b-s_m)^{\gamma-1}}{\Gamma(\gamma)}hn + \frac{M\eta}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,\mathbb{R}^+)}\biggr].
\end{aligned}$$

Therefore,

$$\begin{aligned}
n < \|y_n\|_{PC_{1-\gamma}(J,E)} &\leqslant \frac{M}{\Gamma(\gamma)}\bigl[\|x_0\| + kn + \|g(0)\|\bigr] + \eta\frac{Mb^{1-\gamma}}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,\mathbb{R}^+)} \\
&\quad + \frac{Mb^{1-\gamma}N^2}{\Gamma(\alpha)}\eta\biggl[\|x_1\| + \frac{M(b-s_m)^{\gamma-1}}{\Gamma(\gamma)}hn + \frac{M\eta}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,\mathbb{R}^+)}\biggr] \\
&\quad + hn + \frac{Mhn}{\Gamma(\gamma)}.
\end{aligned}$$

Divide both sides by n and pass to the limit as $n \to \infty$, and we have

$$1 \leqslant \frac{Mk}{\Gamma(\gamma)} + \frac{M^2 b^{1-\gamma} N^2}{\Gamma(\alpha)} \eta \frac{(b-s_m)^{\gamma-1}}{\Gamma(\gamma)} h + h + \frac{Mh}{\Gamma(\gamma)},$$

which contradicts (24). □

4 An example

In this section, we give an example to illustrate our theory.

Take $\alpha = 1/2$, $0 \leqslant \beta \leqslant 1$, $\gamma = \alpha + \beta - \alpha\beta$, $J = [0,1]$ and $E = X = L^2[0,1]$. Now E is a separable Hilbert space. Set $s_0 = 0$, $t_1 = 1/4$, $s_1 = 1/2$ and $t_2 = b = 1$. For any function $x : J \to L^2(J)$ and any $t \in [0,1]$, we let $x(t)(y) := x(t,y)$, $y \in J$. Let $F : J \times E \to P_{\mathrm{ck}}(E)$ be such that $z \in F(t,x) \iff z(y) \in P(t, x(t,y))$, where $P : J \times \mathbb{R} \to P_{\mathrm{ck}}(\mathbb{R})$ is chosen such that (F_1), (F_2^*) and (F_3) are satisfied. Define $g_1 : [t_1, s_1] \times E \to E$ as $g_1(t,x) = t^{1-\gamma} L(x)$, where $L : D(L) = E \to E$ is a compact linear bounded operator. Now (H_{g_1}) holds. Let $B : E \to E$, $B = \gamma I_d$, where I_d is the identity operator and $\gamma > 0$, and let $g : PC_{1-\gamma}(J,E) \to E$, $g(x) = \sum_{i=1}^{2} c_i L(x(t_i))$, where c_i, $i = 1, 2\dots$, are real numbers. Observe that for any $x, y \in PC_{1-\gamma}(J,E)$, $\|g(x) - g(y)\| \leqslant \|L\| \sum_{i=1}^{2} |c_i| (\|x - y\|)$. Moreover, the compactness of L implies that g is compact, and hence (Hg^*) is satisfied.

Now we consider

$$\begin{aligned} &D_{s_i^+}^{1/2,\beta} x(t,y) \in x_{yy}(t,y) + P\big(t, x(t,y)\big) + B\big(u(t)\big), \\ &\quad \text{a.e. } t \in (s_i, t_{i+1}],\ i = 0, 1, \\ &x\big(t_1^+, y\big) = g_1\big(t_1, x\big(t_1^-, y\big)\big), \quad y \in J, \\ &x(t,y) = g_1\big(t, x\big(t_1^-, y\big)\big), \quad t \in (t_1, s_1],\ y \in J, \\ &I_{0^+}^{1-\gamma} x(0,y) = x_0 + g(x)y, \qquad I_{s_i^+}^{1-\gamma} x\big(s_1^+, y\big) = g_1\big(s_1, x\big(t_1^-, y\big)\big), \end{aligned} \tag{25}$$

where $u \in L^2(J, L^2(J))$.

Define $A : D(A) \subseteq L^2[0,1] \to L^2[0,1]$ by $Ax = x_{yy}$, where domain A is given by $D(A) = \{x \in L^2[0,1]\colon x, x_y$ are absolutely continuous, $x_{yy} \in L^2[0,1],\ x(t,0) = x(t,1) = 0\}$. Then A can be written as $Ax = \sum_{n=1}^{\infty} n^2 \langle x, x_n \rangle x_n$, $x \in D(A)$, where $x_n(y) = \sqrt{2} \sin ny$, $n = 1, 2, \dots$, is the orthonormal basis of E. Moreover, for any $x \in L^2[0,1]$, we have $T(t)(x) = \sum_{n=1}^{\infty} \mathrm{e}^{-n^2 t} \langle x, x_n \rangle x_n$. Now A is the infinitesimal generator of the strongly continuous semigroup $\{T(t),\ t \geqslant 0\}$.

Next, the operator $P_{1/2}(\cdot)$ can be written as $P_{1/2}(t) = (1/2) \int_0^\infty \theta \xi_{3/4}(\theta) T(t^{1/2}\theta)\, \mathrm{d}\theta$.

Define $W : L^2(J, L^2(J)) \to L^2(J)$ by $W(u) := \int_{1/2}^{1} (1-s)^{-1/2} T(1-s) u(s)\, \mathrm{d}s$. Now W is linear and bounded. We now show W is surjective. Let $x \in L^2(J)$. Consider the Mittag-Leffler function as follows: $E_{1/2}(-n^2/\sqrt{2}) = \int_0^\infty M_{1/2}(\theta) \mathrm{e}^{-n^2/\sqrt{2})\theta}\, \mathrm{d}\theta$, $n \in \mathbb{N}$.

Note that for any natural number n and any $\theta > 0$, we have $\theta/\sqrt{2} < \theta n^2/\sqrt{2}$, and hence, $\mathrm{e}^{-(n^2/\sqrt{2})\theta} \leqslant \mathrm{e}^{-\theta/\sqrt{2}} < 1$. Thus

$$E_{1/2}\left(-\frac{n^2}{\sqrt{2}}\right) \leqslant E_{1/2}\left(-\frac{1}{\sqrt{2}}\right) < \int_0^\infty M_{1/2}(\theta)\,\mathrm{d}\theta = 1.$$

Then

$$0 < 1 - E_{1/2}\left(-\frac{1}{\sqrt{2}}\right) \leqslant 1 - E_{1/2}\left(-\frac{n^2}{\sqrt{2}}\right) < 1.$$

Define a function $u : J \to L^2(J)$ by

$$u(t) = \sum_{n=1}^\infty \frac{n^2}{\gamma} \frac{\langle x, x_n\rangle x_n}{1 - E_{1/2}(-\frac{n^2}{\sqrt{2}})}, \quad t \in J. \tag{26}$$

Then

$$\begin{aligned}
W(u) &= \gamma \int_{1/2}^1 (1-s)^{-1/2} P_{1/2}(1-s) u(s)\,\mathrm{d}s \\
&= \int_{1/2}^1 (1-s)^{-1/2} P_{1/2}(1-s) \sum_{n=1}^\infty n^2 \frac{\langle x, x_n\rangle x_n}{1-E_{1/2}(-\frac{n^2}{\sqrt{2}})}\,\mathrm{d}s \\
&= \int_{1/2}^1 (1-s)^{-1/2} \left(\frac{1}{2}\int_0^\infty \theta M_{1/2}(\theta) T\big((1-s)^{1/2}\theta\big) \sum_{n=1}^\infty n^2 \frac{\langle x, x_n\rangle x_n}{1-E_{1/2}(-\frac{n^2}{\sqrt{2}})}\,\mathrm{d}\theta\right) \mathrm{d}s \\
&= \int_{1/2}^1 (1-s)^{-1/2} \left(\frac{1}{2}\int_0^\infty \theta M_{1/2}(\theta) \right.\\
&\qquad \left. \times \sum_{m=1}^\infty \mathrm{e}^{-\theta m^2 (1-s)^{1/2}} \left\langle \sum_{n=1}^\infty n^2 \frac{\langle x, x_n\rangle x_n}{1-E_{1/2}(-\frac{n^2}{\sqrt{2}})},\, x_m \right\rangle x_m\,\mathrm{d}\theta\right) \mathrm{d}s \\
&= \int_0^\infty M_{1/2}(\theta) \sum_{m=1}^\infty \frac{1}{1-E_{1/2}(-\frac{m^2}{\sqrt{2}})} \\
&\qquad \times \left(\int_{1/2}^1 \frac{m^2\theta}{2} \mathrm{e}^{-\theta m^2(1-s)^{1/2}} (1-s)^{-1/2}\,\mathrm{d}s\right) \langle x, x_m\rangle x_m\,\mathrm{d}\theta \\
&= \int_0^\infty M_{1/2}(\theta) \sum_{m=1}^\infty \frac{1}{1-E_{1/2}(-\frac{m^2}{\sqrt{2}})} \big[1 - \mathrm{e}^{-m^2\theta/\sqrt{2}}\big]\,\mathrm{d}\theta \langle x, x_m\rangle x_m
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=1}^{\infty} \frac{1}{1-E_{1/2}(-\frac{m^2}{\sqrt{2}})} \int_0^{\infty} M_{1/2}(\theta)\big[1-\mathrm{e}^{-m^2\theta/\sqrt{2}}\big]\,\mathrm{d}\theta \langle x, x_m\rangle x_m \\
&= \sum_{m=1}^{\infty} \frac{1}{1-E_{1/2}(-\frac{m^2}{\sqrt{2}})} \left[\int_0^{\infty} M_{1/2}(\theta)\,\mathrm{d}\theta - \int_0^{\infty} M_{1/2}(\theta)\mathrm{e}^{-m^2\theta/\sqrt{2}}\,\mathrm{d}\theta\right] \langle x, x_m\rangle x_m \\
&= \sum_{m=1}^{\infty} \frac{1}{1-E_{1/2}(-\frac{m^2}{\sqrt{2}})} \left[1 - \int_0^{\infty} M_{1/2}(\theta)\mathrm{e}^{-m^2\theta/\sqrt{2}}\,\mathrm{d}\theta\right] \langle x, x_m\rangle x_m \\
&= \sum_{m=1}^{\infty} \langle x, x_m\rangle x_m = x.
\end{aligned}
$$

From the above computations W is surjective, where $W^{-1}x = u$ and u is given by (26).

Note that W^{-1} is linear, and for $x \in D(A)$,

$$
\|x\| = \big\|A(x)\big\| := \sqrt{\sum_{n=1}^{\infty} n^4 \langle x, x_n\rangle^2}.
$$

Then

$$
\begin{aligned}
&\big\|W^{-1}(x)(t)\big\| \\
&\quad = \sqrt{\sum_{n=1}^{\infty} \frac{n^4}{\gamma^2} \frac{\langle x, x_n\rangle^2}{[1-E_{1/2}(-\frac{n^2}{\sqrt{2}})]^2}} \leqslant \frac{1}{\gamma[1-E_{1/2}(\frac{-1}{\sqrt{2}})]} \sqrt{\sum_{n=1}^{\infty} n^4 \langle x, x_n\rangle^2} \\
&\quad = \frac{1}{\gamma[1-E_{1/2}(\frac{-1}{\sqrt{2}})]} \|x\|.
\end{aligned}
$$

Observe that $W^{-1}(x)$ is independent of $t \in [0,1]$. Consequently, we obtain $\|W^{-1}\| \leqslant 1/(\gamma[1-E_{1/2}(-1/\sqrt{2})])$. Then from Theorem 3 system (25) is controllable.

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