The Numerical Simulation in Ballistics

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Abstract

In this paper we examine the movement of the solid body thrown with some angle to the horizon (for example the shot mine). Such movement is described by non-linear system of equations. This system is being approximated by linear system, in segments. The experiment results have been approximated and the dependence of air resistance coefficient from mean value of velocity along the trajectory was found. From the point of view of mathematics the incorrect problem must be solved because the initial conditions of system corresponding to fixed values of solutions (the coordinates of target points) must been estimated. In the case of linear system it is possible to examine the influence of non-large increment of initial conditions to the final result. In this work the probability of destruction of some fixed target and the mean square deviation of shooting regression. On the other hand has been estimated the possibility of destruction of group target using the method of Monte-Carlo.

Keywords: mathematical model, Monte-Carlo method, ballistics, symbolic and numerical simulation.

1 Introduction and Main System of Equations

We examine the problems of holding under control the trajectory of mine. The movement of the solid is described by non-linear system of equations because the air force resistance is expressed by the polynomial of the third power. In our case the velocity of mine flies not exceed that of sound so the member of the third degree of polynomial may be given up. Having in

mind that coefficient of air resistance is the function of mean velocity we can linearize the non-linear system of equations. The starting angle of the mine and the angle of the correction must be found, the influence of the side wind must be taken into account. We suppose that initial speed of a mine is known and the meteorological conditions are taken into consideration. The movement of mine may be described by the following system of the equations [1]:

$$\begin{cases} m\frac{d^2x}{dt^2} &= -k\frac{dx}{dt} + F_1 cos(\phi), \\ m\frac{d^2y}{dt^2} &= -k\frac{dy}{dt} + F_2 sin(\phi), \\ m\frac{d^2z}{dt^2} &= -k\frac{dz}{dt} - mg, \end{cases} \tag{1}$$

with the following initial conditions

$$x(0) = 0, y(0) = 0, z(0) = 0, \tag{2}$$

$$m\frac{dx}{dt} = v_0 cos(\alpha_1) cos(\alpha_2), m\frac{dy}{dt} = v_0 cos(\alpha_1) sin(\alpha_2), m\frac{dz}{dt} = v_0 sin(\alpha_1),$$

where t denotes time of flying of mine, k is the coefficient of aerodynamic resistance, m is the mass of a missile, g - the acceleration due to the force of gravity, $x(t)$ and $y(t)$ (x_t, y_t) denote the horizontal coordinates of the moving body at time moment t $(x - axis$ is turn to the target), $z(t)(z_t)$ - vertical coordinate, F_1 - the front force of wind, F_2 - the side force of wind, ϕ - is the angle between the directions of a wind and $x - axis$, v_0 - initial velocity.

We suppose that the direction of the initial velocity vector coincides with the direction of the trench-mortar tube. When aiming, the position of the trench-mortar tube is determined by two angles: α_1 - angle between the initial velocity vector and xy - plane and α_2 - angle of correction, that is the angle of compensation of influence of side winter.

Using the MAPLE (a symbolic computation package) [2, 3, 4], we get the analytic solution of system of equations (1) with the initial conditions (2)

$$x = \{[m(1 - e^{-\frac{kt}{m}})(F_1 cos(\phi) + kv_0 cos(\alpha_1) cos(\alpha_2))]/k - tF_1 cos(\phi)\}/k,$$
$$y = \{[m(e^{-\frac{kt}{m}} - 1)(F_2 sin(\phi) + kv_0 sin(\alpha_1) sin(\alpha_2))]/k + tF_2 sin(\phi)\}/k,$$
$$z = \{[m(1 - e^{-\frac{kt}{m}})(mg + kv_0 sin(\alpha_1))]/k - tmg\}/k. \tag{3}$$

It is rather difficult to choose the aerodynamic resistance coefficient k, which may be established by experiment [5]. It depends on the speed of mine and on meteorological conditions: pressure of atmosphere, air temperature and air humidity in every point of trajectory. These parameters change during the movement because the mine achieves greatest attitude. We propose to choose it by coordinating the calculus and the results of the experience, which that'is generalized in the following tables [6]. We choose the coefficient of air resistance k correspondent to different distances from the target in accordance with the data of tables [6] and calculus and we construct the approximation polynomial. Having performed this calculus we have got the dependence of air resistance coefficient from mean value of speed during the flight. The results of the calculus are presented in fig.1. (correspondent dependence of mean velocity of mine moving along the trajectory)
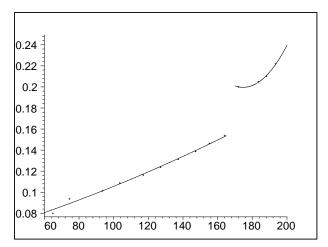

Fig.1 Dependence of the coefficient of resistance k from the mean value of mine's velocity

In the table 1, the distance to the target, the initial speed of mine and corresponding mean speed of mine during the trajectories are presented.

Comparing the data in fig.1. and table 1 we state that dependence of resistance coefficient from the mean value of speed remains linear whole initial velocity of mine does not exceed one's of sound. Probably the linear dependence is possible where the force of resistance is determined by air friction (viscosity), but not by the difference of pressures. Because the mean value of speed is linearly connected with the distance to the target, provide in fig. 2 the corresponding data.

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Table 1.

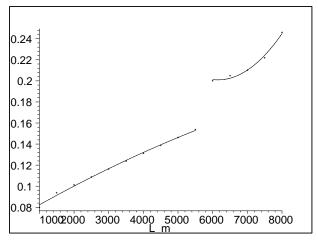| Distance to the target (m)	Initial speed (m/s)	Mine speed value (m/s)
1000	124	74
2000	163	103
3000	197	127
4000	228	147
4900	256	164
5800	280	179
5800	352	220

Fig. 2 Dependence of the coefficient of resistance k from distance to the target

The dependence of resistance coefficient from the distance to target, which is more convenient for practical using. If k is known, it is possible to calculate the trajectory of mine's flying. Inserting the numerical system (1) coefficients values and parameters of initials conditions in the expression of solutions (3), we transform this solutions in the functions of angles α_1, α_2 and of time t_n. It can be solved using the numerical methods the system of nonlinear equations:

$$\begin{cases} x(\alpha_1, \alpha_2, t) = L, \\ y(\alpha_1, \alpha_2, t) = 0, \\ z(\alpha_1, \alpha_2, t) = 0, \end{cases} \tag{4}$$

where L - the distance to the target.

Having solved the system (4), using the method of simple iteration, can be found the angle of aiming α_1, the angle of correction α_2 and the time of flying t_0 ($40^0 \leq \alpha_1 \leq 85^0, -8^0 \leq \alpha_2 \leq 8^0, 10 \leq t_n \leq 80sec$). The example of such calculus is presented in the fig.3.
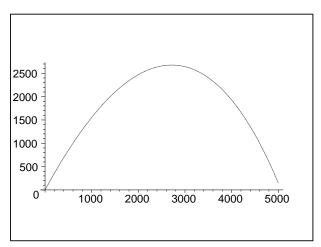

Fig.3 The trajectories of mine flying

In this case distance to the target is $5558m$, the angle of throwing $68^0 22'$, the altitude of trajectory (the maximal altitude of the trajectory) $2333m$, the length of flying $t_n - 44s$. In addition, using the analytical solution we can find duration all necessary parameters for practical application: the instantaneous speed, the altitude of trajectory, the angle of a fall and etc. application without difficulty.

From the point of view of mathematics this problem is incorrect, because we must choose the initial conditions (the angle of starting) when the solution is known. It is possible to examine the different interesting cases. For example, in the case when the target is moving in the mountainous country, the trench mortar and target may be in different altitudes with respect to sea-level. Inserting in the right side of the system (4) the corresponding non-zero parameters, we get the following solutions

$$\begin{cases} x(\alpha_1, \alpha_2, t) = L \pm v_1 cos(\theta)t, \\ \quad y(\alpha_1, \alpha_2, t) = \pm v_1 sin(\theta)t, \\ \quad\quad z(\alpha_1, \alpha_2, t) = \Delta h, \end{cases} \quad (4a)$$

where L - the distance to the target, v_1 - the speed of the target(par example the speed of the tank is $v_1 = 10m/s$), θ - the angle between the vector

of speed of target (tank) and the principal direction of firing, Δh - the difference between sea levels of target and trench mortar (if the trench mortar is below the target, we insert the negative number). When the altitude of target is chosen, the speed of moving and direction of moving, one get three-dimensional solutions (because the third coordinate y is not equal to zero). The corresponding mine moving trajectory is represented in fig 4. The parameters in this case is: distance to the target $L = 3525m$, it's speed $v_1 = 10m/s$, the angle between the vector of speed and the direction of shooting $\theta = 45^0$, the difference of altitudes is equal $\Delta h = 500m$.
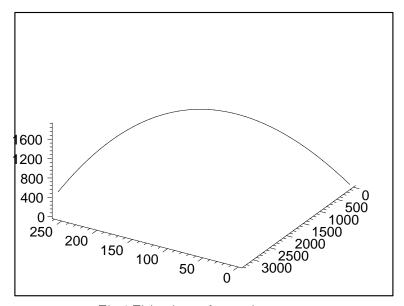

Fig.4 Firing in to the moving target

We take into account the more important possibility. We may insert the time $t + t_0$ in the system (4a)(where t_0 is the time necessary for prepare the shooting). Having solved the system (4a) we shall know the time t_0 that is the moment of time after which we must shoot hitting the moving target.

We can get the random errors of interesting variables by using the generating program. For example, we can indicate the initial speed of mine v_0, the coefficient of resistance k and the influence of wind in the following manner

$$v_0 + random[normal]\Delta v_0,, k + random[normal]\Delta k,$$
$$F_1 + random[normal]\Delta F_1, F_2 + random[normal]\Delta F_2, \tag{5}$$

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here random [normal] - MAPLE command generating standard normal distribution $N(0; 1)$ and $\Delta v_0, \Delta k, \Delta F_1, \Delta F_2$ - the maximal random errors of indicated variables discused in specific situation.

2   The Establishment of the Trajectory of Flying

Actually it is impossible to avoid the most random errors. For example, the gusts of wind, may occur the changeable air humidity, air temperature and density are possible. Some errors may is appended due to the perfection of technology of mines and arms production, for example the variation of initial speed. By generating the random initial conditions (5) and solving system (4) and (4a), we can find the angles $\alpha_1 and \alpha_2$. For these angles $\alpha_1 and \alpha_2$ we find the scatter of points of the fall of mines caused by random errors. In this way we get the dispersion Ellipse of points hitting the target, corresponding to the chosen dispersion of our parameters. We present the example of such calculus in fig 5.
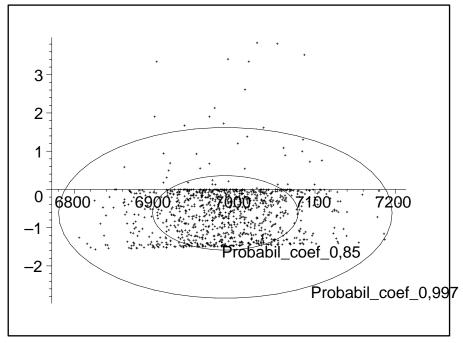
Fig.5 The ellipse of dispersion of points hitting

The coordinates of the hitting points are calculated with $0, 5\%$ maximal error of initial velocity and 1% maximal error of coefficient of air resistance.

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The distance to the target $L_x = 7000m$. The ellipse of dispersion of the hitting points with the confidence level equals to 0,85 and 0,997. The standard quadratic variance $\sigma_x = 63, 1, \sigma_y = 0, 61$. If the distance to the target increase, the initial speed and the length of trajectory increases, too. The results of those calculations are generalized in table 2:

Table 2.

$L_x(m)$	$\sigma_x(m)$
1000	9,9
2000	19,4
3000	28,7
4000	37,7
5000	46,1
6000	54
7000	63
8000	67,5

Approximating the points of table 2 by quadratic polynomial (using method of least squares) we get such expression of the function:

$$f_\sigma = -0,9227910^-7(L_x)^2 + 0,009568L_x + 0,60359(6)$$
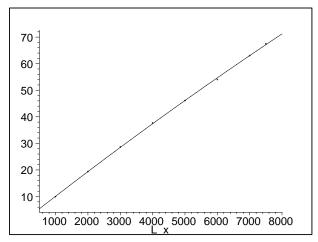
These results are represented in fig. 6.

Fig.6 The dependence of standard quadratic variation $\sigma_x(m)$ of distance to the target $L_x(m)$

In this picture the points correspond to the results of table 2 and the line - the graph of function f_σ.

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Repeatedly generating the random variables using the same program one can get the same numbers. It is possible to repeat the numerical experience and to the examine the influence of same parameters by changing their values. The numerical experience can help to estimate the possibilities of certain fire - arm and to estimate properly the meteorological factors to the quality of shooting. Let us examine the influence of initial conditions to the final results, so we analyze the change of shooting result depending on initial conditions. We shall discus random errors tending to affect the moving of mines and shall estimate its maximal values. First of all the "wearing of gun tube" is of grand importance to the results of shooting and it determines the random error of initial speed Δv_0. This variable is measured before shooting. The increment of mine mass from standard (it is indicated in the passport) and the temperature of charge also influences or differ from the initial speed. The oscillation of air resistance, atmosphere pressure and air humidity in function of different attitudes of mines determine the random error of coefficient of resistance k. The rush (of a wind) determines the random errors of values F_1, F_2. We can insert the maximal random errors corresponding to specific conditions in the system of equations (1) and in the initial conditions (2) in every case. In the case of changing the mine mass, the resistance coefficient of , the wind force, the initial speed, that influence the coefficients of equations system, one must change the angle of laying α_1, the correction of direction of α_2 and the time of flight t_n. For example, we get the different mean square variance for different aims shooting at the target. It's evident that necessity of careful coordination of results presented by the program with shooting results in the polygon, and the calculations may minimize the costs of similar experience. The calculations need insignificant machine time expenditure. For example, in the case of processors cycling density of $400MHz$ the modeling calculus of 1000 shooting with random initial conditions takes about 15 minutes.

3 The Choose of Optimal Numbers of Mines

Let us analyze two problems. The first - how many mines must be shooted to the armed target of size $8 \times 4m^2$ (for example the tank) and situated at some distance for hitting with the confidence probability 0,9. The second - how many volleys must be shooted to a group target $200 \times 300m$ situated at some distance for hitting more 95% of the target by the shell and splitters.

In the first case using the methodology described above we calculate the random coordinates of the points scatter of the fall in the cases of $0, 5\%$ of

maximal error of initial speed and of 1% of maximal error of air resistance coefficient. For example, one shoot (imitation by numerical experience) sixteen times to the target, situated at the $4000m$ distance. One finds the number of hits (that is the number of mines fall sufficiently close to the target). One shoots 50 times $(N = 50)$, each time contains sixteen series and one gets the following hits result:

[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 5, 5]

The mean value of hits n_{vid} and mean square variance σ_n are correspondingly equal:

$$n_{vid} = 1, 24, \sigma_n = 0, 21$$

that is the armored target $8 \times 4m^2$ will be hit.

By increasing the number of repeated series N we get the necessary accuracy n_{vid} and σ_n that is we find such N of repetitions of experience, that some fixed number of digits of n_{vid} and σ_n would not be changed. 'We present the results of numerical experience with the probability of confidence P = 0,9 for different distances to the target L_x and the numbers of shooting n_x, in the table 3.

Table 3.

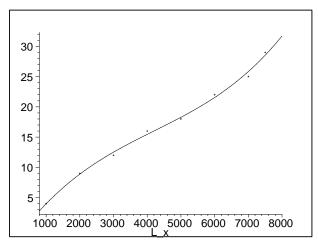| $L_x(m)$	n_x	n_{vid}	σ_n
1000	4	1,48	0,18
2000	9	1,26	0,2
3000	12	1,12	0,19
4000	16	1,24	0,21
5000	18	1,28	0,17
6000	22	1,5	0,21
7000	25	1,16	0,23
8000	29	1,26	0,24

Fig.7 The necessary number of mines for hurt the target

The relation between L_x and n_x may be approximated by the polynomial. In this case it will be possible for any distance to find the necessary number of mines for hitting the target with the probability of confidence 0,9. This relation is presented in the fig. 7. In this picture one the number of shooting is indicated that the target would be hit with the probability of confidence $P = 0,9$. The coordinates of points are the results of numerical experiments .

We shall use the method Monte-Carlo to solve the second problem and we shall estimate the probability of group hit target, so we must find the necessary number of mines that the target would be hit with the chosen confidence (certainty) Let us assume that the battery of trench-mortars receiveds an order-mission to annihilate the group target of 300 m in width and 200 m in depth. If one shoots utilizing the sight three and one angular sight, possible schemes of targets are represented in fig.8.
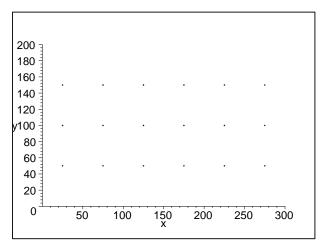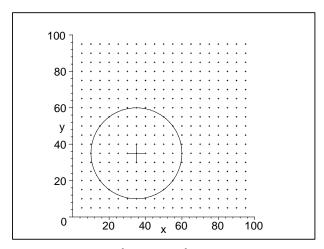
Fig.8 The possible schemes of targets

Fig.9 The fragment of group target.

We form the random variables with the mean values coinciding with the coordinates of points indicated in fig. 8 (18 points) and mean square variances σ_i (for x - axis) and σ_j (for y - axis) ($1 \leq i \leq 6, 1 \leq j \leq 3$). They may be different for distinct mortars. The variables σ_i and σ_j may be found by formula (6).

$$x_i = x_i + \sigma_i \times random[normal] \qquad y_j = y_j + \sigma_j \times random[normal]. (7)$$

Having generated 18 random hits corresponding to the scheme represented by fig.8 we can calculate the results of one realization. For that

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purpose the target is divided in the zones according to the type of target. If the soldiers under fire are without cover they may be hurt by shell-splinters if they are not further than a meter from explosion place.

In the fig. 9 it is represented the fragment of group target. The cross note the hit point. The points in the hit zone are hurt.

The quotient of squares fallen in the zone of hit n_i with all squares n correspond to the hurt part of group target (the ratio between hurt area S_i and whole square S):

$$U_i = \frac{n_i s}{n s} = \frac{S_i}{S}(8)$$

We repeat the numerical experience N times and every time we evaluate the hurt part of target (8). We continue the experience until the mean value of target hurt part U would be evaluated with precision desirable (its value no more change within the limits of given precision when N increase):

$$MU = \frac{1}{N} \sum_{i=1}^{N} U_i (9)$$

Further we determine the variance of this value and find the probability to injure some part U_0 of group target

$$P(U > U_0)$$

and the mean value of mines N_n which missed the target.

The results of the numerical experience are presented in the table 4.

Table 4.

$L_x(m)$	$S_1(1)$	N% (1)	$N_n(1)$	$S_5(2)$	N% (2)	$N_n(2)$	$S_{10}(3)$	N% (3)	$N_n(3)$
1000	1	73	0	5	87	0	10	90	0
2000	1	68	0	5	94	0	10	98	0
3000	1	64	0	5	97	1	10	99	1
4000	1	63	1	5	97	2	10	99,6	4
5000	1	60	1	5	97	4	10	99,7	8
6000	1	56	2	5	97	7	10	99,8	13
7000	1	56	2	5	97	9	10	99,9	17
8000	1	53	3	5	96	13	10	99,8	24

Hitting the target the scheme of fig.8. is the following. The mean quadratic variance σ_i, σ_j may be calculated by the formula (6) (see also formula (7)). After each volleys of eighteen hits one determines the degree of target destruction with given accuracy and calculate the number of mines that have missed the target. If the calculations are repeated of a given

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number of times, one finds mean value and evaluates the percentage $N\%$ injured part of target and average number of missing of the target using the formula (9).

4 Final Remarks

It follows from the data a in table 4 that if the target is dislocated further than $5000m$, one destroys fewer than 60% of group target. One must choose necessary number of volleys to achieve the desirable level of group target destruction, we present in table 4 the results when one destroys the target with five or ten volleys. Using the methodology described above we find the percentage $N\%$ of injured part of a target and average number N_m of missed hits. With these data one can choose the desirable level of target destruction. It is interesting to note that in the case of neigh boring target, no further than $1000m$ or $2000m$, the dispersion of mines is not large and hurt part of target is less than in the case of a large distance to the target. For the sake of ammunition economy, it would be necessary to increase the numbers of target points, indicated in the fig. 8, on the second hand a new formed, scheme and a distance decreases as far as $25m$. This calculation may be fulfilled after evaluating the individual characteristics of fire-arms and it is necessary to choose the target-points in a such manner that the efficiency of its application would be maximal.

5 References

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