On the Denseness of One Set of Series

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Abstract

In the paper the denseness of the set of all convergent series related to a class of multiplicative functions is obtaned.

Keywords: function of exponential type, multiplicative function, space of analytic functions.

Let $s = \sigma + it$ is a complex variable, and let, for $\sigma > 1$,

$$Z(s) = \sum_{m=1}^{\infty} \frac{g(m)}{m^s}.$$

Here $g(m)$ is a multiplicative function ($g(m,n) = g(m)g(n)$ for $(m,n) = 1$), $|g(m)| \le 1$. It is known [1] that for some class of functions $g(m)$ the function $Z(s)$ is universal, i.e. any analytic function can be approximated uniformly on compact sets by translations of $Z(s)$. To prove this, usually we need a denseness of a certain set of convergent series. The aim of this note is to obtain the denseness of a set of convergent series related to a new class of multiplicative functions $g(m)$.

We say that a multiplicative function $g(m)$ belongs to the class $\mathcal{M}_{\eta,\theta}(C)$ if the following conditions are satisfied:

1^0. $|g(m)| \le 1$ for all naturals m;

2^0. There exists a constant θ such that, for $x \to \infty$,

$$\sum_{p \le x} |g(p)|^2 = \frac{\theta x}{\log x}(1 + o(1));$$

3^0. There exist constants η and C, $0 < C < 1$, $1/2 \le \eta < 1$, such that

$$\sum_{\alpha=1}^{\infty} \frac{|g(p^\alpha)|}{p^{\eta\alpha}} \le C$$

for all primes p.

Note that in [1] a class of multiplicative functions with a condition

$$\inf_p |g(p)| > 0$$

was considered. In the class $\mathcal{M}_{\eta,\theta}$ the latter condition is replaced by the asymptotical condition 2^0.

By $\mathbb{C}$ and $\mathcal{P}$ denote the complex plane and the set of all prime numbers, respectevely. Let, for $N > 0$, $D_N = \{s \in \mathbb{C} : \eta < \sigma < 1, |t| < N\}$, and let $H(D_N)$ denote the space of analytic on D_N functions equipped with the topology of uniform converegence on compacta. Moreover, let γ stand for the unit circle on the complex plane, and, for $a_p \in \gamma$, $s \in D_N$,

$$f_p(s) = f_p(s, a_p) = \log\left(1 + \sum_{\alpha=1}^{\infty} \frac{g(p^\alpha)a_p^\alpha}{p^{\alpha s}}\right),$$

where, for $|z| < 1$,

$$\log(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots.$$

In virtue of condition 3^0 of the class $\mathcal{M}_{\eta,\theta}(C)$ we have that the definition of $f_p(s)$ is correct.

Theorem. *Suppose that a multiplicative function* $g(m) \in \mu_{\eta,\theta}(C)$. *Then the set of all convergent series*

$$\sum_p f_p(s, a_p)$$

is dense in $H(D_N)$.

For the proof of the theorem we will apply some properties of functions of exponential type. Let $0 < \theta_0 \le \pi$. We recall, that a function $f(s)$ analytic in the closed angular region $|\arg s| \le \theta_0$ is said to be of exponential type if

$$\limsup_{r \to \infty} \frac{\log |f(re^{i\theta})|}{r} < \infty$$

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uniformly in θ, $|\theta| \leq \theta_0$.

Denote by $\mathcal{B}(\mathbb{C})$ the class of Borel sets of the complex plane. Let μ be a complex measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact support contained in D_N, and

$$\rho(z) = \int_{\mathbb{C}} e^{-sz} \mathrm{d}\mu(s), \quad z \in \mathbb{C}.$$

Lemma 1. *Suppose that*

$$\sum_p |g(p)||\rho(\log p)| < \infty.$$

Then

$$\int_{\mathbb{C}} s^r \mathrm{d}\mu(s) = 0, \quad r = 0, 1, 2, \cdots.$$

Proof of the lemma leans on a version of the Bernstein theorem, see Theorem 6.4.12 [1].

Lemma 2. *Let $f(s)$ be an entire function of exponentional type, and let $\{\lambda_m\}$ be a sequence of complex numbers such that*
1^0. $\limsup\limits_{y \to \infty} \frac{\log |f(\pm iy)|}{y} \leq \alpha$;

2^0. $|\lambda_m - \lambda_n| \geq \delta|m - n|$;

3^0. $\lim\limits_{m \to \infty} \frac{\lambda_m}{m} = \beta$;

4^0. $\alpha\beta < \pi$.

Then
$$\limsup_{m \to \infty} \frac{\log |f(\lambda_m)|}{|\lambda_m|} = \limsup_{r \to \infty} \frac{\log |f(r)|}{r}.$$

Proof of Lemma 1. We take $f(s) = \rho(s)$ in Lemma 2. By the definition of $\rho(s)$ we have that

$$|\rho(\pm iy)| \leq e^{Ny} \int_{\mathbb{C}} |\mathrm{d}\mu(s)|$$

for $y > 0$, therefore

$$\limsup_{y \to \infty} \frac{\log |\rho(\pm iy)|}{y} \leq N,$$

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and the condition 1^0 of Lemma 2 is valid with $\alpha = N$. Let a fixed positive number β satisfy

$$\beta < \frac{\pi}{N},$$

and

$$A = \{ m \in \mathbb{N} : \exists r \in \left((m - \tfrac{1}{4})\beta, (m + \tfrac{1}{4})\beta \right] \quad \text{with} \quad |\rho(r)| \le e^{-r} \}.$$

We fix a number μ, $0 < \mu < \theta$, and denote

$$\mathcal{P}_\mu = \{ p \in \mathcal{P} : |g(p)| > \mu \}.$$

Then the condition of the lemma yields

$$\sum_{p \in \mathcal{P}_\mu} |\rho(\log p)| < \infty. \tag{1}$$

However,

$$\sum_{p \in \mathcal{P}_\mu} |\rho(\log p)| \ge \sum_{m \notin A} {\sum_m}' |\rho(\log p)| \ge \sum_{m \notin A} {\sum_m}' \tfrac{1}{p},$$

where ${\sum_m}'$ denotes the sum over all primes $p \in \mathcal{P}_\mu$ satisfying the inequalities $(m - \tfrac{1}{4})\beta < \log p \le (m + \tfrac{1}{4})\beta$. Therefore, in view of (1)

$$\sum_{m \notin A} \sum_{\substack{p \in \mathcal{P}_\mu \\ a < p \le b}} \frac{1}{p} < \infty, \tag{2}$$

where $a = \exp\{ (m - \tfrac{1}{4})\beta \}$, $b = \exp\{ (m + \tfrac{1}{4})\beta \}$.

Denote

$$\pi_\mu(x) = \sum_{\substack{p \le x \\ p \in \mathcal{P}_\mu}} 1, \quad \pi(x) = \sum_{p \le 1} 1.$$

Then the condition 1^0 of the class $\mathcal{M}_{\beta,\theta}(C)$ gives, for $a \le u \le b$,

$$\sum_{a < p \le u} |g(p)|^2 \le \sum_{\substack{p \in \mathcal{P}_\mu \\ a < p \le u}} 1 + \mu^2 \sum_{\substack{p \notin \mathcal{P}_\mu \\ a < p \le u}} 1 =$$

$$(\pi_\mu(u) - \pi_\mu(a)) + \mu^2 ((\pi(u) - \pi_\mu(u)) - (\pi(a) - \pi_\mu(a))) =$$
$$(1 - \mu^2)(\pi_\mu(u) - \pi_\mu(a)) + \mu^2 (\pi(u) - \pi(a)).$$

Hence, using condition 2^0 of the class $\mathcal{M}_{\eta,\theta}(C)$, we find

$$\theta(\pi(u) - \pi(a))(1 + o(1)) \leq (1 - \mu^2)(\pi_\mu(u) - \pi_\mu(a)) + \mu^2(\pi(u) - \pi(a)).$$

Therefore

$$\pi_\mu(u) - \pi_\mu(a) \geq \frac{\theta - \mu^2}{1 - \mu^2}(\pi(u) - \pi(a))(1 + o(1))$$

for $u \geq a(1 + \delta)$, $m \to \infty$. This and the partial summation yield

$$\sum_{\substack{p \in \mathcal{P}_\mu \\ a < p \leq b}} \frac{1}{p} = \frac{1}{b}\left(\sum_{\substack{p \in \mathcal{P}_\mu \\ a < p \leq b}} 1\right) + \int_a^b \left(\sum_{\substack{p \in \mathcal{P}_\mu \\ a < p \leq u}} 1\right)\frac{\mathrm{d}u}{u^2} =$$

$$= \frac{1}{b}(\pi_\mu(b) - \pi_\mu(a)) + \int_a^b (\pi_\mu(u) - \pi_\mu(a))\frac{\mathrm{d}u}{u^2} \geq$$

$$\geq \frac{1}{b}(\pi_\mu(b) - \pi_\mu(a)) + \int_{a(1+\delta)}^b (\pi_\mu(u) - \pi_\mu(a))\frac{\mathrm{d}u}{u^2} \geq \qquad (3)$$

$$\geq \frac{\theta - \mu^2}{1 - \mu^2}\left(\frac{1}{b}(\pi(b) - \pi(a)) + \int_{a(1+\delta)}^b (\pi(u) - \pi(a))\frac{\mathrm{d}u}{u^2}\right)(1 + o(1)) \geq$$

$$\geq \frac{\theta - \mu^2}{1 - \mu^2}\left(\frac{1}{b}(\pi(b) - \pi(a(1+\delta))) + \int_{a(1+\delta)}^b (\pi(u) - \pi(a(1+\delta)))\frac{\mathrm{d}u}{u^2}\right)(1 + o(1)) \geq$$

$$= \frac{\theta - \mu^2}{1 - \mu^2}\left(\sum_{a(1+\delta) < p \leq b} \frac{1}{p}\right)(1 + o(1))$$

as $m \to \infty$.

It is well known that, for $x > 1$,

$$\sum_{p \leq x} \frac{1}{p} = \log\log x + c_1 + B\exp\{-c_2\sqrt{\log x}\}$$

whith some constants c_1 and $c_2 > 0$. Here B denotes a quantity bounded by a constant. Thus, for $m \to \infty$, in view of the definition of a and b

$$\sum_{a(1+\delta) < p \leq b} \frac{1}{b} = \left(\frac{1}{2} - \frac{\log(1+\delta)}{\beta}\right)\frac{1}{m} + \frac{B}{m^2}.$$

This together with (3) gives

$$\sum_{\substack{p \in \mathcal{P}_\mu \\ a < p \le b}} \frac{1}{p} \ge \frac{\theta - \mu^2}{1 - \mu^2} \left(\frac{1}{2} - \frac{\log(1 + \delta)}{\beta} \right) \frac{1}{m} (1 + o(1)) + \frac{B}{m^2}$$

as $m \to \infty$. Since $0 < \mu < \theta$ and, of course, $\theta < 1$, and $1 + \delta < e^{\beta/2}$, we have that

$$\frac{\theta - \mu^2}{1 - \mu^2} \left(\frac{1}{2} - \frac{\log(1 + \delta)}{\beta} \right) > 0.$$

Consequently, by (2) and (3)

$$\sum_{m \notin A} \frac{1}{m} < \infty. \tag{4}$$

Let

$$A = \{a_m : m = 1, 2, \cdots\}, \quad a_1 < a_2 < \cdots.$$

Then from (4) we deduce that

$$\lim_{m \to \infty} \frac{a_m}{m} = 1.$$

By the definition of the set A there exists a sequence $\{\lambda_m\}$ such that

$$\left(a_m - \frac{1}{4} \right) \beta < \lambda_m \le \left(a_m + \frac{1}{4} \right) \beta$$

and

$$|\rho(\lambda_m)| \le \exp\{-\lambda_m\}.$$

Then we have

$$\lim_{m \to \infty} \frac{\lambda_m}{m} = \beta,$$

and

$$\limsup_{m \to \infty} \frac{|\rho(\lambda_m)|}{\lambda_m} \le -1.$$

Now, applying Lemma 2, we obtain

$$\limsup_{r \to \infty} \frac{|\rho(r)|}{r} \le -1. \tag{5}$$

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If $\rho(z) \not\equiv 0$, then from this and Theorem 6.4.14 of [1] (Let μ be a complex measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact support contained in the half-plane $\sigma > \sigma_0$,

$$f(s) = \int_{\mathbb{C}} e^{sz} \mathrm{d}\mu(z)$$

and $f(s) \not\equiv 0$. Then

$$\limsup_{r \to \infty} \frac{\log |f(r)|}{r} > \sigma_0.)$$

we find that

$$\limsup_{r \to \infty} \frac{\log |f(r)|}{r} > -1,$$

and this contradicts (5). Therefore $\rho(z) \equiv 0$, and hence the lemma follows by differentiation.

We will deduce the assertion of the theorem from the following lemma.

Lemma 3.*Let $\{f_m\}$ be a sequence in $H(D_N)$ which satisfies:*
1^0. *If μ is a complex measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact support contained in D_N such that*

$$\sum_{m=1}^{\infty} \left| \int_{\mathbb{C}} f_m \mathrm{d}\mu(z) \right| < \infty,$$

then

$$\int_{\mathbb{C}} s^r \mathrm{d}\mu(s) = 0$$

for any $r = 0, 1, 2, \cdots$;
2^0. *The series*

$$\sum_{m=1}^{\infty} f_m$$

converges in $H(D_N)$;
3^0. *For any compact $K \subset D_N$*

$$\sum_{m=1}^{\infty} \sup_{s \in K} |f_m(s)|^2 < \infty.$$

Then the set of all convergent series

$$\sum_{m=1}^{\infty} a_m f_m, \quad a_m \in \gamma,$$

is dense in $H(D_N)$.

Proof. The lemma is a special case of Theorem 6.3.10 from [1].

Proof of Theorem. We have that

$$f_p(s; a_p) = \frac{g(p)a_p}{p^s} + r_p(s; a_p) \tag{6}$$

with

$$r_p(s; a_p) = \frac{B}{p^{2\sigma}},$$

uniformly in a_p. Therefore the series

$$\sum_p r_p(s; a_p) \tag{7}$$

converges uniformly on compact subsets of D_N for all $a_p \in \gamma$. Let $p_0 > 0$ and

$$\hat{f}_p = \hat{f}_p(s) = \begin{cases} \frac{g(p)}{p^s} & \text{if} \quad p > p_0, \\ 0 & \text{if} \quad p \le p_0. \end{cases}$$

Then there exists a sequence $\{\hat{a}_p : \hat{a}_p = \pm 1\}$ such that the series

$$\sum_p \hat{a}_p \hat{f}_p$$

converges uniformly on compact subsets of D_N. Now we will prove that the set of all convergent series

$$\sum_p a_p \hat{f}_p, \quad a_p \in \gamma, \tag{8}$$

is dense in $H(D_N)$. For this aim we apply Lemma 3. Clearly, it suffices to show that the series of all convergent series

$$\sum_p a_p g_p, \quad a_p \in \gamma, \tag{9}$$

with $g_p = \hat{a}_p \hat{f}_p$ is dense in $H(D_N)$.

It was mentioned above that the series

$$\sum_p g_p$$

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converges on $H(D_N)$. Moreover, for any compact subset $K \subset D_N$

$$\sum_p \sup_{s \in K} |g_p(s)|^2 < \infty.$$

Therefore it remains to verify the condition 1^0 of Lemma 3.

Let μ be a complex measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact support contained in D_N such that

$$\sum_p \left| \int_{\mathbb{C}} g_p(s) \mathrm{d}\mu(s) \right| < \infty.$$

Hence we have that

$$\sum_p |g(p)| \left| \int_{\mathbb{C}} p^{-s} \mathrm{d}\mu(s) \right| < \infty,$$

or, by notation of Lemma 1,

$$\sum_p |g(p)| |\rho(\log p)| < \infty.$$

In consequence, it follows from Lemma 1 that the condition 1 of Lemma 3 is also satisfied. Thus, Lemma 3 shows the denseness of the set of all convergent series (9), and therefore that of all convergent series (8). This together with uniform convergence of the series (7) in view of (6) proves the lemma. Really, let $x_0(s) \in H(D_N)$, K be a compact subset of D_N, and $\varepsilon > 0$. We choose p_0 for which

$$\sum_{p \geq p_0} \sup_{a_p \in \gamma} \sup_{s \in K} |r_p(s, a_p)| < \frac{\varepsilon}{2}. \tag{10}$$

It follows from the denseness of the set of the series (8) that there exists a sequence $\{\hat{a}_p : \hat{a}_p \in \gamma\}$ such that

$$\sup_{s \in K} \left| x_0(s) - \sum_{p \leq p_0} f_p(s; 1) - \sum_{p > p_0} \hat{a}_p \hat{f}_p(s) \right| < \frac{\varepsilon}{2}. \tag{11}$$

Now let

$$a_p = \begin{cases} 1 & \text{if} \quad p \leq p_0, \\ \hat{a}_p & \text{if} \quad p > p_0. \end{cases}$$

Thus in view of (10) and (11) we find

$$\sup_{s \in K} \left| x_0(s) - \sum_p f_p(s; a_p) \right| \leq$$

$$\sup_{s \in K} \left| x_0(s) - \sum_{p \leq p_0} f_p(s; 1) - \sum_{p > p_0} \hat{a}_p \hat{f}_p(s) \right| +$$

$$\sup_{s \in K} \left| \sum_{p > p_0} \hat{a}_p \hat{f}_p(s) - \sum_{p > p_0} f_p(s; \hat{a}_p) \right| <$$

$$\frac{\varepsilon}{2} + \sup_{s \in K} \left| \sum_{p > p_0} r_p(s; \hat{a}_p) \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

References

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