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Global exponential synchronization of quaternion-valued memristive neural networks with time delays*

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Abstract. This paper extends the memristive neural networks (MNNs) to quaternion field, a new class of neural networks named quaternion-valued memristive neural networks (QVMNNs) is then established, and the problem of drive-response global synchronization of this type of networks is investigated in this paper. Two cases are taken into consideration: one is with the conventional differential inclusion assumption, the other without. Criteria for the global synchronization of these two cases are achieved respectively by appropriately choosing the Lyapunov functional and applying some inequality techniques. Finally, corresponding simulation examples are presented to demonstrate the correctness of the proposed results derived in this paper.

Keywords: memristor, exponential synchronization, drive-response systems, quaternion, time delays.

1 Introduction

The memristor is considered to be the fourth fundamental circuit element, except resistor, capacitor, and inductor. The concept of memristor is predicted by Chua in 1971 [10]. However, the first practical appliance was invented by HP company until 2008 [33]. Except for the properties of resistors, memristor also possesses lots of other superiority like low power, high density, and good scalability. Thus, widely possible applications of memristor have appeared in different areas [11, 27, 34]. Compared with conventional resistors, memristors has a special characteristic that it can remember its recent value between the period that the voltage is turned off and the next time it turned on. Due to this wonderful character, memristor can be applied to act as the synapses in neurons to better modify human brain. Furthermore, it has the potential to improve the application of associative memory and data processing [4, 28].

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In the past few decades, the study of memristors has become an important topic among various research areas. By introducing memristor into the connection weights of conventional neural networks, the memristive neural networks (MNNs) is then constructed. It is worth noting that MNNs are a particular type of switching nonlinear system dependent on the state value. Recently, the investigation of its dynamical behavior has become a hot topic, and numerous interesting results have been presented [2, 5, 6, 25, 40, 45]. However, all the above-mentioned problems are mainly discussed in the real- or complex-valued fields, the relevant study in quaternion field are relatively few till now.

Quaternion is a special case of Clifford algebra invented by British mathematician Hamilton in 1843 [1]. Compared with real number and complex number, some basic operation rules like commutativity of multiplication are not suitable for quaternion yet. Due to this reason, the quaternion has not receive much research attention for the last few decades. In recent years, the study of quaternion-valued systems has draw much interests due to its broadly potential applications in various fields like attitude control [36], computer graphics [49], image processing [41], and prediction of 3D wind processing [29].

With the introduction of quaternion value into NNs, the quaternion-valued neural networks (QVNNs) was realized. As the extension of complex-valued memristive neural networks (CVNNs), the states, connection weights, and activation functions of QVNNs are all in quaternion field. Compared with RVNNs and CVNNs, the QVNNs possess the superiority of low dimension and high efficiency in handling the multidimensional information. For example, in image compression [31], it takes three real- or complex-valued neurons to transmit one color signal. However, for QVNNs, it only takes one quaternion neuron to transmit one color via 3 channels i, j, k, thus bring about a significant decrease in the scale of system and to an improvement in computation speed. Moreover, some optimization and estimation problem solved by RVNNs or CVNNs [26, 32] can be dealt with the QVNNs with better performance. Recently, as the rapid development of QVNNs, some interesting results were reported [7–9, 12, 23, 30, 37, 38, 42, 43, 48], such as global stability [7], multi-stability [43], robust stability [38], μstability [8], passivity analysis [42], state estimation [9]. For instance, the global stability problem of QVNNs is considered in [23] with the technique of quaternion-valued LMI, criterion of global μstability are derived for the considered QVNNs. The multi-stability problem for delayed QVNNs were studied in [43], some essential dynamical characteristics of the delayed QVNNs are investigated by using the decomposition of the state space. [37] addresses the problem of stability for continuous version and discrete version QVNNs with linear threshold neurons. Via the plural decomposition method of quaternion, some criteria are established. To the best of our knowledge, the study of dynamical behavior of QVNNs mainly concentrate on the stability issue, the investigation of synchronization problem of QVNNs is still quite few, let alone the memristive QVNNs. Recently, some researchers introduced the memristive connection weights into CVNNs to construct a new model, which bring about some interesting results [39, 46]. However, as far as we considered, the works combining the memristor with QVNNs is still very few [24], which is a new and challenging topic. This gives the motivation for our current study.

An unavoidable phenomenon in various practical systems is time delay, which is generated by infinite switching ratio of amplifiers or data transmission. Unfortunately, it may cause oscillation, instability, and other poor performance [3, 22, 39, 46–48]. The influence of time delay on the dynamic systems has become a fundamental problem in the research of natural science and engineering technology [13–21, 35, 44]. Thus, it is meaningful to take time delays into the study of dynamical behavior of QVMNNs.

Considering the above-mentioned discussion, the main objective of this work is to investigate the global exponential synchronization of QVMNNs in drive-response scheme. The main contributions of this thesis are presented as follows.

(i) In this work, the model of QVMNNs is established, which is a challenging model with characteristics of both MNNs and QVNNs. Thus, our research is the generalization and improvement for previous literature, more complicated dynamical behavior of nonlinear system are coped with in this work.

(ii) It is the first time that synchronization problems for QVMNNs are investigated. Two cases about the differential inclusion are considered here, several synchronization criteria for QVMNNs are derived with different feedback controllers, which are easy to verify.

(iii) In many previous literature, QVNNs are usually decomposed into four RVNNs or two CVNNs for analysis. Different from that, our work directly investigate QVNNs as an entirety without any decomposition, thus the property of quaternion is fully utilized and the computation complexity can be reduced effectively.

The frame of this paper is arranged as follows. In Section 2, the model is formulated, and some useful preliminaries are introduced. Main results are presented in Section 3. In Section 4, two simulation examples are proposed to verity the correctness of our theorem results. Finally, conclusions are achieved in Section 5.

Notations. In this paper, let $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Q}$ stand for the real field, complex field, and quaternion field, respectively. $C^{(1)}([-\tau, 0], \mathbb{R}^n)$ denotes the family of continuous functions from $[-\tau, 0]$ to $\mathbb{R}^n$. $\mathrm{co}\{F_1, F_2\}$ denotes closure of the convex hull of $\mathbb{Q}$ produced by quaternion values F_1, F_2.

2 Preliminaries and model formulation

The quaternion is a set of supercomplex number composed of one real part and three imaginary parts. A quaternion $q \in \mathbb{Q}$ can be described in the form

$$q = q^R + q^I i + q^J j + q^K k,$$

where q^R, q^I, q^J, $q^K \in \mathbb{R}$, the imaginary parts i, j, k obey the Hamilton rule:

$$i^2 = j^2 = k^2 = -1, \qquad ij = -ji = k, \qquad jk = -kj = i, \qquad ki = -ik = j.$$

Remark 1. Different from real number and complex number, the commutativity does not hold for quaternion multiplication, i.e., for any $x, y \in \mathbb{Q}$, it can not be guaranteed that $xy = yx$. Due to this reason, some good properties in real field and complex field does not hold for quaternion field. Thus, previous method dealing with RVNNs or CVNNs can not be directly applied to QVNNs, which lead to the need to develop new techniques and theories to cope with QVNNs.

The conjugate of q is denoted by $\bar{q} = q^R - q^I i - q^J j - q^K k$. The modulus of q is defined as

$$|q| = \sqrt{\bar{q}q} = \sqrt{\left(q^R\right)^2 + \left(q^I\right)^2 + \left(q^J\right)^2 + \left(q^K\right)^2}.$$

For two quaternions $h = h^R + h^I i + h^J j + h^K k$ and $q = q^R + q^I i + q^J j + q^K k$, the addition between them is defined as

$$h + q = h^R + q^R + \left(h^I + q^I\right)i + \left(h^J + q^J\right)j + \left(h^K + q^K\right)k.$$

By Hamilton rule, the product between them is defined as

$$\begin{aligned} hq = {}& \left(h^R q^R - h^I q^I - h^J q^J - h^K q^K\right) + \left(h^R q^I + h^I q^R + h^J q^K - h^K q^J\right)i \\ & + \left(h^R q^J + h^J q^R + h^K q^I - h^I q^K\right)j + \left(h^R q^K + h^K q^R + h^I q^J - h^J q^I\right)k. \end{aligned}$$

With the introduction of memristive connection weights into QVNNs, the model of quaternion-valued memristive neural networks is introduced as follows:

$$\begin{aligned} \frac{\mathrm{d}x_p(t)}{\mathrm{d}t} = {}& -d_p x_p(t) + \sum_{q=1}^{n} a_{pq}\left(x_p(t)\right) f_q\left(x_q(t)\right) \\ & - \sum_{q=1}^{n} b_{pq}\left(x_p(t)\right) f_q\left(x_q\left(t - \tau(t)\right)\right), \quad t \geqslant 0, \end{aligned} \tag{1}$$

where $p = 1, 2, \dots, n$; $x(t) = (x_1(t), \dots, x_n(t))^{\mathrm{T}} \in \mathbb{Q}^n$. $x_p(t) \in \mathbb{Q}$ denotes the state vector of the pth neuron at time t. $d_p > 0$ is the self-feedback coefficient; $f(x(\cdot)) = (f_1(x_1(\cdot)), \dots, f_n(x_n(\cdot)))^{\mathrm{T}}$ denotes the activation function. $a_{pq}(x_p(t))$ and $b_{pq}(x_p(t))$ stand for the quaternion-valued memristive connection weights. $\tau(t)$ is the time delay satisfying $\dot{\tau}(t) \leqslant \mu < 1$ and $0 \leqslant \tau(t) < \tau$, where μ and τ are positive constants. The initial condition of system (1) is taken as $x(s) = \phi(s)$, $-\tau \leqslant s \leqslant 0$, where $\phi(s) = (\phi_1(s), \dots, \phi_n(s))^{\mathrm{T}} \in C^{(1)}([-\tau, 0], \mathbb{Q}^n)$. The memristor connection weights can be defined by

$$a_{pq}\left(x_p(t)\right) = \frac{W_{apq}(x_q(t))\,\mathrm{sgn}_{pq}}{C_p} \qquad b_{pq}\left(x_p(t)\right) = \frac{W_{bpq}(x_q(t))\,\mathrm{sgn}_{pq}}{C_p}$$

$$\mathrm{sgn}_{pq} = \begin{cases} 1, & p = q, \\ -1, & p \neq q, \end{cases}$$

where $W_{apq}(x_q(t))$ and $W_{bpq}(x_q(t))$ stand for the memductances of memristors M_{apq} and M_{bpq}, respectively. C_p is the capacitor. M_{apq} denotes the memristor between the activation function $f_p(x_p(t))$ and $x_p(t)$. Similarly, M_{bpq} represents the memristor between $x_p(t)$ and $f_p(x_p(t - \tau(t)))$. Based on the current-voltage characteristics and nature of memristor, the memristive weights can be defined as the state-dependent switching case:

$$a_{pq}^R\big(x_p^R(t)\big) = \begin{cases} \hat{a}_{pq}^R, & |x_p^R(t)| \leqslant T_p, \\ \check{a}_{pq}^R, & |x_p^R(t)| > T_p, \end{cases} \qquad a_{pq}^I\big(x_p^I(t)\big) = \begin{cases} \hat{a}_{pq}^I, & |x_p^I(t)| \leqslant T_p, \\ \check{a}_{pq}^I, & |x_p^I(t)| > T_p, \end{cases}$$

$$a_{pq}^J\big(x_p^J(t)\big) = \begin{cases} \hat{a}_{pq}^J, & |x_p^J(t)| \leqslant T_p, \\ \check{a}_{pq}^J, & |x_p^J(t)| > T_p, \end{cases} \qquad a_{pq}^K\big(x_p^K(t)\big) = \begin{cases} \hat{a}_{pq}^K, & |x_p^K(t)| \leqslant T_p, \\ \check{a}_{pq}^K, & |x_p^K(t)| > T_p, \end{cases}$$

$$b_{pq}^R\big(x_p^R(t)\big) = \begin{cases} \hat{b}_{pq}^R, & |x_p^R(t)| \leqslant T_p, \\ \check{b}_{pq}^R, & |x_p^R(t)| > T_p, \end{cases} \qquad b_{pq}^I\big(x_p^I(t)\big) = \begin{cases} \hat{b}_{pq}^I, & |x_p^I(t)| \leqslant T_p, \\ \check{b}_{pq}^I, & |x_p^I(t)| > T_p, \end{cases}$$

$$b_{pq}^J\big(x_p^J(t)\big) = \begin{cases} \hat{b}_{pq}^J, & |x_p^J(t)| \leqslant T_p, \\ \check{b}_{pq}^J, & |x_p^J(t)| > T_p, \end{cases} \qquad b_{pq}^K\big(x_p^K(t)\big) = \begin{cases} \hat{b}_{pq}^K, & |x_p^K(t)| \leqslant T_p, \\ \check{b}_{pq}^K, & |x_p^K(t)| > T_p, \end{cases}$$

where the switching jump $T_p > 0$, $\hat{a}_{pq}^l$, $\check{a}_{pq}^l$, $\hat{b}_{pq}^l$, $\check{b}_{pq}^l$ are known constants with respect to memristances.

Remark 2. For the QVMNNs model proposed in [24], the switching rules are dependent on the modulus of quaternion-valued state, so there are 2^n possible switching conditions. Different from that, in our work, each imaginary part of the quaternion connection weights is determined by corresponding part of state value. Thus, there are 2^{4n} possible conditions for the connection weights, which arouse more complex dynamical behaviors.

Definition 1. For $\hat{a}_{pq}, \check{a}_{pq}, \hat{b}_{pq}, \check{b}_{pq} \in \mathbb{Q}$, the following definition is given:

$$\begin{aligned} \max\{\hat{a}_{pq}, \check{a}_{pq}\} &= \max\{\hat{a}_{pq}^R, \check{a}_{pq}^R\} \\ &\quad + i\max\{\hat{a}_{pq}^I, \check{a}_{pq}^I\} + j\max\{\hat{a}_{pq}^J, \check{a}_{pq}^J\} + k\max\{\hat{a}_{pq}^K, \check{a}_{pq}^K\}, \\ \min\{\hat{a}_{pq}, \check{a}_{pq}\} &= \min\{\hat{a}_{pq}^R, \check{a}_{pq}^R\} \\ &\quad + i\min\{\hat{a}_{pq}^I, \check{a}_{pq}^I\} + j\min\{\hat{a}_{pq}^J, \check{a}_{pq}^J\} + k\min\{\hat{a}_{pq}^K, \check{a}_{pq}^K\}, \\ \max\{\hat{b}_{pq}, \check{b}_{pq}\} &= \max\{\hat{b}_{pq}^R, \check{b}_{pq}^R\} \\ &\quad + i\max\{\hat{b}_{pq}^I, \check{b}_{pq}^I\} + j\max\{\hat{a}_{pq}^J, \check{a}_{pq}^J\} + k\max\{\hat{a}_{pq}^K, \check{a}_{pq}^K\}, \\ \min\{\hat{b}_{pq}, \check{b}_{pq}\} &= \min\{\hat{b}_{pq}^R, \check{b}_{pq}^R\} \\ &\quad + i\min\{\hat{b}_{pq}^I, \check{b}_{pq}^I\} + j\min\{\hat{a}_{pq}^J, \check{a}_{pq}^J\} + k\min\{\hat{a}_{pq}^K, \check{a}_{pq}^K\}, \end{aligned}$$

$$\begin{aligned} \acute{a}_{pq} &= \max\{|\hat{a}_{pq}^R|, |\check{a}_{pq}^R|\} \\ &\quad + i\max\{|\hat{a}_{pq}^I|, |\check{a}_{pq}^I|\} + j\max\{|\hat{a}_{pq}^J|, |\check{a}_{pq}^J|\} + k\max\{|\hat{a}_{pq}^K|, |\check{a}_{pq}^K|\}, \\ \acute{b}_{pq} &= \max\{|\hat{b}_{pq}^R|, |\check{b}_{pq}^R|\} \\ &\quad + i\max\{|\hat{b}_{pq}^I|, |\check{b}_{pq}^I|\} + j\max\{|\hat{b}_{pq}^J|, |\check{b}_{pq}^J|\} + k\max\{|\hat{b}_{pq}^K|, |\check{b}_{pq}^K|\}, \end{aligned}$$

$$a_{pq}^+ = \max\{\hat{a}_{pq}, \check{a}_{pq}\}, \qquad a_{pq}^- = \min\{\hat{a}_{pq}, \check{a}_{pq}\},$$
$$b_{pq}^+ = \max\{\hat{b}_{pq}, \check{b}_{pq}\}, \qquad b_{pq}^- = \min\{\hat{b}_{pq}, \check{b}_{pq}\}.$$

Since the connection weights $a_{pq}(x_p(t))$ and $b_{pq}(x_p(t))$ are discontinuous, solutions of (1) are in Filippovs sense. Based on the theory of set valued map and the principal of differential inclusion [8], it yields from (1) that

$$\frac{\mathrm{d}x_p(t)}{\mathrm{d}t} \in -d_p x_p(t) + \sum_{q=1}^{n} \mathrm{co}\{a_{pq}^-, a_{pq}^+\} f_q(x_q(t)) + \sum_{q=1}^{n} \mathrm{co}\{b_{pq}^-, b_{pq}^+\} f_q(x_q(t-\tau(t))),$$

or equivalently, there exists $a'_{pq} \in \mathrm{co}\{a_{pq}^-, a_{pq}^+\}$, $b'_{pq} \in \mathrm{co}\{b_{pq}^-, b_{pq}^+\}$ such that

$$\frac{\mathrm{d}x_p(t)}{\mathrm{d}t} = -d_p x_p(t) + \sum_{q=1}^{n} a'_{pq} f_q(x_q(t)) + \sum_{q=1}^{n} b'_{pq} f_q(x_q(t-\tau(t))).$$

Consider system (1) as the drive system, then choose the response system as below:

$$\begin{aligned}\frac{\mathrm{d}x_p^*(t)}{\mathrm{d}t} &= -d_p x_p^*(t) + \sum_{q=1}^{n} a_{pq}(x_p^*(t)) f_q(x_q^*(t)) \\ &\quad + \sum_{q=1}^{n} b_{pq}(x_p^*(t)) f_q(x_q^*(t-\tau(t))) + u_p(t),\end{aligned} \tag{2}$$

or equivalently, there exists $a''_{pq} \in \mathrm{co}\{a_{pq}^-, a_{pq}^+\}$, $b''_{pq} \in \mathrm{co}\{b_{pq}^-, b_{pq}^+\}$ such that

$$\frac{\mathrm{d}x_p^*(t)}{\mathrm{d}t} = -d_p x_p^*(t) + \sum_{q=1}^{n} a''_{pq} f_q(x_q^*(t)) + \sum_{q=1}^{n} b''_{pq} f_q(x_q^*(t-\tau(t))) + u_p(t),$$

where $u_p(t)$ is the controller to be designed later to realize the synchronization objective. Choose the initial state of system (2) as $x^*(s) = \varphi(s) = (\varphi_1(s), \dots, \varphi_n(s))^{\mathrm{T}} \in C^{(1)}([-\tau, 0], \mathbb{Q}^n)$, $-\tau \leqslant s \leqslant 0$. Then let $e(t) = (e_1(t), \dots, e_n(t))$ be the synchronization error, where $e_p(t) = x_p(t) - x_p^*(t)$. Thus, the following error system is achieved

$$\begin{aligned}\frac{\mathrm{d}e_p(t)}{\mathrm{d}t} &= -d_p e_p(t) + \sum_{q=1}^{n} a'_{pq} f_q(x_q(t)) - \sum_{q=1}^{n} a''_{pq} f_q(x_q^*(t)) \\ &\quad + \sum_{q=1}^{n} b'_{pq} f_q(x_q(t-\tau(t))) - \sum_{q=1}^{n} b''_{pq} f_q(x_q^*(t-\tau(t))) - u_p(t)\end{aligned}$$

with initial condition $\psi(s) = \phi(s) - \varphi(s)$, $-\tau \leqslant s \leqslant 0$.

Definition 2. Under a properly designed control input, for any initial conditions ϕ and φ, if there exists constants $M \geqslant 1$ and $\epsilon > 0$ such that

$$\sum_{p=1}^{n} \bar{e}_p(t) e_p(t) \leqslant M \|\psi(s)\| \mathrm{e}^{-\epsilon t},$$

where $\|\psi(s)\| = \sum_{p=1}^{n} \sup_{-\tau \leqslant s \leqslant 0} \{\bar{\psi}_p(s) \psi_p(s)\}$, then the global exponential synchronization is achieved between the drive system (1) and response system (2).

Lemma 1. *For all $x, y \in \mathbb{Q}$, the following properties hold*:

(i) $\overline{x+y} = \bar{x} + \bar{y}$;

(ii) $\overline{xy} = \bar{y}\bar{x}$.

Lemma 2. *For any $x, y \in \mathbb{Q}, \epsilon \in \mathbb{R}^+$, the following inequality holds*:

$$xy + \bar{y}\bar{x} \leqslant \epsilon\bar{x}x + \frac{1}{\epsilon}y\bar{y}.$$

Assumption 1. For all $x, y \in \mathbb{Q}$, there exists positive constants l_p, $p = 1, \ldots, n$, such that

$$\bigl(\overline{f_p(x) - f_p(y)}\bigr)\bigl(f_p(x) - f_p(y)\bigr) \leqslant l_p^2(\overline{x-y})(x-y).$$

3 Main results

In this part, we focus on the drive-response global exponential synchronization of memristive QVNNs. The synchronization problem are considered in two cases.

Case 1. Based on the theory of differential inclusion, the following assumption is introduced.

Assumption 2. (See [25].) Suppose that the following conditions hold for (1) and (2):

$$\begin{aligned}
&\mathrm{co}\bigl\{a_{pq}^-, a_{pq}^+\bigr\} f_q\bigl(x_q(t)\bigr) - \mathrm{co}\bigl\{a_{pq}^-, a_{pq}^+\bigr\} f_q\bigl(x_q^*(t)\bigr)\\
&\quad\subseteq \mathrm{co}\bigl\{a_{pq}^-, a_{pq}^+\bigr\}\bigl[f_q\bigl(x_q(t)\bigr) - f_q\bigl(x_q^*(t)\bigr)\bigr],\\
&\mathrm{co}\bigl\{b_{pq}^-, b_{pq}^+\bigr\} f_q\bigl(x_q(t)\bigr) - \mathrm{co}\bigl\{b_{pq}^-, b_{pq}^+\bigr\} f_q\bigl(x_q^*(t)\bigr)\\
&\quad\subseteq \mathrm{co}\bigl\{b_{pq}^-, b_{pq}^+\bigr\}\bigl[f_q\bigl(x_q(t)\bigr) - f_q\bigl(x_q^*(t)\bigr)\bigr],
\end{aligned}$$

where $x(t) = (x_1(t), \ldots, x_n(t))^{\mathrm{T}}$, $x^*(t) = (x_1^*(t), \ldots, x_n^*(t))^{\mathrm{T}}$ are two solutions of systems (1) and (2) with initial values $\phi(s) = (\phi_1(s), \ldots, \phi_n(s))^{\mathrm{T}}$, $\varphi(s) = (\varphi_1(s), \ldots, \varphi_n(s))^{\mathrm{T}}$, respectively.

Remark 3. Assumption 2 is a conventional assumption frequently used in many previous literature on MNNs, which solves the effect of parameter mismatch of memristive connection weights, thus effectively simplify the analysis process of synchronization problem of memristive neural networks.

Now, the state feed-back controller is designed as

$$u_p(t) = k_p\bigl(x_p(t) - x_p^*(t)\bigr), \quad k_p > 0. \tag{3}$$

Thus, the following error system is achieved:

$$\begin{aligned}
\frac{\mathrm{d}e_p(t)}{\mathrm{d}t} \in{}& -d_p e_p(t) + \sum_{q=1}^{n} \mathrm{co}\bigl\{a_{pq}^-, a_{pq}^+\bigr\} F_q\bigl(e_q(t)\bigr)\\
&+ \sum_{q=1}^{n} \mathrm{co}\bigl\{b_{pq}^-, b_{pq}^+\bigr\} F_q\bigl(e_q\bigl(t - \tau(t)\bigr)\bigr) - k_p e_p(t)
\end{aligned}$$

with the initial value $\psi(s) = \phi(s) - \varphi(s)$, $-\tau \leqslant s \leqslant 0$, where $F_q(e_q(t)) = f_q(x_q(t)) - f_q(x_q^*(t))$, $F_q(e_q(t-\tau(t))) = f_q(x_q(t-\tau(t))) - f_q(x_q^*(t-\tau(t)))$. Or equivalently, there exists $\tilde{a}_{pq} \in \mathrm{co}\{a_{pq}^-, a_{pq}^+\}$, $\tilde{b}_{pq} \in \mathrm{co}\{b_{pq}^-, b_{pq}^+\}$ such that

$$\frac{\mathrm{d}e_p(t)}{\mathrm{d}t} = -d_p e_p(t) + \sum_{q=1}^{n} \tilde{a}_{pq} F_q\big(e_q(t)\big) + \sum_{q=1}^{n} \tilde{b}_{pq} F_q\big(e_q\big(t-\tau(t)\big)\big) - k_p e_p(t). \quad (4)$$

Based on above discussion, the following criteria for global synchronization of master-slave QVMNNs can be derived.

Theorem 1. *Under Assumptions 1 and 2, if there exists positive constants k_p, α, ϵ_1, ϵ_2 such that the following condition holds*

$$\alpha - 2d_p - 2k_p + \epsilon_1 \sum_{q=1}^{n} \acute{a}_{pq} \bar{\acute{a}}_{pq} + \epsilon_2 \sum_{q=1}^{n} \acute{b}_{pq} \bar{\acute{b}}_{pq} + \frac{1}{\epsilon_1} n l_p^2 + \frac{n l_p^2}{\epsilon_2(1-\mu)} \mathrm{e}^{\alpha\tau} \leqslant 0 \quad (5)$$

for $p = 1, \dots, n$, then the drive system (1) *and response system* (2) *can reach global exponential synchronization with convergence rate α under control input* (3)*, i.e., $\sum_{p=1}^{n} \bar{e}_p(t) \times e_p(t) \leqslant M\|\psi\|\mathrm{e}^{-\alpha t}$, $M > 1$.*

Proof. Considering the Lyapunov functional as

$$V(t) = \mathrm{e}^{\alpha t} \sum_{p=1}^{n} \bar{e}_p(t) e_p(t) + \sum_{p=1}^{n} \frac{\gamma}{1-\mu} \int_{t-\tau(t)}^{t} \mathrm{e}^{\alpha(s+\tau)} \bar{F}_p\big(e_p(s)\big) F_p\big(e_p(s)\big)\, \mathrm{d}s,$$

where $\alpha > 0$ is the adjustable number, which act as the convergence rate, $\gamma > 0$ is the known constant to be determined later in the proof.

Calculating the derivative of $V(t)$ along the trajectory (4) yields

$$\begin{aligned}
\frac{\mathrm{d}V(t)}{\mathrm{d}t} \leqslant{}& \alpha \mathrm{e}^{\alpha t} \sum_{p=1}^{n} \bar{e}_p(t) e_p(t) + \mathrm{e}^{\alpha t} \sum_{p=1}^{n} \big\{ \dot{\bar{e}}_p(t) e_p(t) + \bar{e}_p(t) \dot{e}_p(t) \big\} \\
&+ \sum_{p=1}^{n} \frac{\gamma \mathrm{e}^{\alpha(t+\tau)}}{1-\mu} \bar{F}_p\big(e_p(t)\big) F_p\big(e_p(t)\big) \\
&- \sum_{p=1}^{n} \gamma \mathrm{e}^{\alpha t} \bar{F}_p\big(e_p\big(t-\tau(t)\big)\big) F_p\big(e_p\big(t-\tau(t)\big)\big) \\
\leqslant{}& \alpha \mathrm{e}^{\alpha t} \sum_{p=1}^{n} \bar{e}_p(t) e_p(t) + \mathrm{e}^{\alpha t} \sum_{p=1}^{n} \Bigg[-d_p \bar{e}_p(t) + \sum_{q=1}^{n} \bar{F}_q\big(e_q(t)\big) \bar{\tilde{a}}_{pq} \\
&+ \sum_{q=1}^{n} \bar{F}_q\big(e_q\big(t-\tau(t)\big)\big) \bar{\tilde{b}}_{pq} - k_p \bar{e}_p(t) \Bigg] e_p(t)
\end{aligned}$$

$$+\,\mathrm{e}^{\alpha t}\sum_{p=1}^{n}\bar{e}_p(t)\Biggl[-d_p e_p(t)+\sum_{q=1}^{n}\tilde{a}_{pq}F_q\bigl(e_q(t)\bigr)$$
$$+\sum_{q=1}^{n}\tilde{b}_{pq}F_q\bigl(e_q\bigl(t-\tau(t)\bigr)\bigr)-k_p e_p(t)\Biggr]+\sum_{p=1}^{n}\frac{\gamma \mathrm{e}^{\alpha(t+\tau)}}{1-\mu}\bar{F}_p\bigl(e_p(t)\bigr)F_p\bigl(e_p(t)\bigr)$$
$$-\sum_{p=1}^{n}\gamma \mathrm{e}^{\alpha t}\bar{F}_p\bigl(e_p\bigl(t-\tau(t)\bigr)\bigr)F_p\bigl(e_p\bigl(t-\tau(t)\bigr)\bigr).$$

Immediately, one has

$$V(t)\leqslant \mathrm{e}^{\alpha t}\sum_{p=1}^{n}(\alpha-2d_p-2k_p)\bar{e}_p(t)e_p(t)+\mathrm{e}^{\alpha t}\sum_{p=1}^{n}\sum_{q=1}^{n}\bar{F}_q\bigl(e_q(t)\bigr)\bar{\tilde{a}}_{pq}e_p(t)+\mathrm{e}^{\alpha t}$$
$$\sum_{p=1}^{n}\sum_{q=1}^{n}\bar{e}_p(t)\tilde{a}_{pq}F_q\bigl(e_q(t)\bigr)+\mathrm{e}^{\alpha t}\sum_{p=1}^{n}\sum_{q=1}^{n}\bar{F}_q\bigl(e_q\bigl(t-\tau(t)\bigr)\bigr)\bar{\tilde{b}}_{pq}e_p(t)$$
$$+\,\mathrm{e}^{\alpha t}\sum_{p=1}^{n}\sum_{q=1}^{n}\bar{e}_p(t)\tilde{b}_{pq}F_q\bigl(e_q\bigl(t-\tau(t)\bigr)\bigr)+\sum_{p=1}^{n}\frac{\gamma \mathrm{e}^{\alpha(t+\tau)}}{1-\mu}\bar{F}_p\bigl(e_p(t)\bigr)F_p\bigl(e_p(t)\bigr)$$
$$-\sum_{p=1}^{n}\gamma \mathrm{e}^{\alpha t}\bar{F}_p\bigl(e_p\bigl(t-\tau(t)\bigr)\bigr)F_p\bigl(e_p\bigl(t-\tau(t)\bigr)\bigr).$$

According to Lemma 2 and Definition 1, there exists constants $\epsilon_1, \epsilon_2 > 0$ such that

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t}\leqslant \mathrm{e}^{\alpha t}\sum_{p=1}^{n}\Biggl\{(\alpha-2d_p-2k_p)\bar{e}_p(t)e_p(t)+\epsilon_1\sum_{q=1}^{n}\bar{e}_p(t)\acute{a}_{pq}\bar{\acute{a}}_{pq}e_p(t)$$
$$+\frac{1}{\epsilon_1}\sum_{q=1}^{n}\bar{F}_q\bigl(e_q(t)\bigr)F_q\bigl(e_q(t)\bigr)+\epsilon_2\sum_{q=1}^{n}\bar{e}_p(t)\acute{b}_{pq}\bar{\acute{b}}_{pq}e_p(t)$$
$$+\frac{1}{\epsilon_2}\sum_{q=1}^{n}\bar{F}_q\bigl(e_q\bigl(t-\tau(t)\bigr)\bigr)F_q\bigl(e_q\bigl(t-\tau(t)\bigr)\Biggr\}$$
$$+\,\mathrm{e}^{\alpha t}\sum_{p=1}^{n}\Biggl\{\frac{\gamma \mathrm{e}^{\alpha\tau}}{1-\mu}\bar{F}_p\bigl(e_p(t)\bigr)F_p\bigl(e_p(t)\bigr)$$
$$-\gamma\bar{F}_p\bigl(e_p\bigl(t-\tau(t)\bigr)\bigr)F_p\bigl(e_p\bigl(t-\tau(t)\bigr)\bigr)\Biggr\}$$
$$=\mathrm{e}^{\alpha t}\sum_{p=1}^{n}\Biggl(\alpha-2d_p-2k_p+\epsilon_1\sum_{q=1}^{n}\acute{a}_{pq}\bar{\acute{a}}_{pq}$$
$$+\epsilon_2\sum_{q=1}^{n}\acute{b}_{pq}\bar{\acute{b}}_{pq}+\frac{1}{\epsilon_1}nl_p^2+\frac{\gamma l_p^2}{1-\mu}\mathrm{e}^{\alpha\tau}\Biggr)\bar{e}_p(t)e_p(t)$$

$$+ \mathrm{e}^{\alpha t} \sum_{p=1}^{n} \biggl\{ \frac{1}{\epsilon_2} n \bar{F}_p\bigl(e_p\bigl(t-\tau(t)\bigr)\bigr) F_p\bigl(e_p\bigl(t-\tau(t)\bigr)\bigr) - \gamma \bar{F}_p\bigl(e_p\bigl(t-\tau(t)\bigr)\bigr) F_p\bigl(e_p\bigl(t-\tau(t)\bigr)\bigr) \biggr\}.$$

Choosing $\gamma = n/\epsilon_2$, it follows that

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \leqslant \mathrm{e}^{\alpha t} \sum_{p=1}^{n} \Biggl(\alpha - 2d_p - 2k_p + \epsilon_1 \sum_{q=1}^{n} \acute{a}_{pq} \bar{\acute{a}}_{pq} + \epsilon_2 \sum_{q=1}^{n} \acute{b}_{pq} \bar{\acute{b}}_{pq} + \frac{1}{\epsilon_1} n l_p^2 + \frac{n l_p^2}{\epsilon_2 (1-\mu)} \mathrm{e}^{\alpha \tau} \Biggr) \bar{e}_p(t) e_p(t).$$

Due to (5), we have $\mathrm{d}V(t)/\mathrm{d}t \leqslant 0$. Hence,

$$V(t) \leqslant V(0) \leqslant \biggl(1 + \frac{\gamma}{1-\mu} \tau l_p^2 \mathrm{e}^{\alpha \tau} \biggr) \|\psi\|.$$

Let $M = 1 + (\gamma/(1-\mu))\tau l_p^2 \mathrm{e}^{\alpha\tau}$, which implies that $\sum_{p=1}^{n} \bar{e}_p(t) e_p(t) \leqslant M \|\psi\| \mathrm{e}^{-\alpha t}$. Therefore, according to Definition 2, the drive system (1) and response system (2) can reach exponential synchronization under control input (3). □

Remark 4. Different from the technique adopted in [7, 26, 43], where the QVNNs are decomposed into four RVNNs or two equivalent CVNNs. In this work, we directly discuss the QVMNNs as an entirety without any decomposition, the advantage of our results is that it can be applied to the case where activation functions cannot be expressed explicitly by real-imaginary parts. To solve the difficulty caused by the non-commutativity of quaternion multiplication, the property $x\bar{x} = \bar{x}x = |x|^2 \in \mathbb{R}$ and Lemma 2 are fully utilized. Thus, some interesting and satisfying results are obtained in this work.

Remark 5. For the first time, the memistive connection weights are brought into QVNNs. As the extension of memristive RVNNs and CVNNs, the weight connections $a_{pq}^R(x_p(t))$, $a_{pq}^I(x_p(t))$, $a_{pq}^J(x_p(t))$, and $a_{pq}^K(x_p(t))$ are decided by the corresponding imaginary unit of state vector $x_p(t)$. Thus, the character of both MNNs and QVNNs are combined in this new model, which lead to more complex dynamical behavior in nonlinear systems. Hence, our work serves as the supplement for the previous results and enrich the theory of QVNNs.

Theorem 1 is based on the Assumption 2, which is a conventional assumption adopted in many previous literature. Though it can simplify the designing process of the feedback controller effectively. However, this assumption is rather conservative and not suitable for many practical situations. Due to this reason, in the following, we focus on the case that Assumption 2 is not valid.

Case 2. In this condition, the error system is achieved as

$$\frac{\mathrm{d}e_p(t)}{\mathrm{d}t} = -d_p e_p(t) + \sum_{q=1}^{n} a'_{pq} F_q\big(e_q(t)\big) + \sum_{q=1}^{n} (a'_{pq} - a''_{pq}) f_q\big(x_q^*(t)\big) + \sum_{q=1}^{n} b'_{pq} F_q\big(e_q\big(t-\tau(t)\big)\big) + \sum_{q=1}^{n} (b'_{pq} - b''_{pq}) f_q\big(x_q^*\big(t-\tau(t)\big)\big) - u_p(t), \tag{6}$$

where a'_{pq}, $a''_{pq} \in \mathrm{co}\{a^-_{pq}, a^+_{pq}\}$, b'_{pq}, $b''_{pq} \in \mathrm{co}\{b^-_{pq}, b^+_{pq}\}$. For convenience of the proof, the following assumption is given.

Assumption 3. There exists positive constants $M_p > 0$ such that

$$\bar{f}_p(\cdot) f_p(\cdot) \leqslant M_p, \quad p = 1, 2, \ldots, n.$$

The state feedback control law is designed as

$$u_p(t) = k_p e_p(t) + \lambda_p \frac{e_p(t)}{\bar{e}_p(t) e_p(t)}, \tag{7}$$

where k_p, λ_p are positive parameters to be designed later.

Theorem 2. *Under Assumptions 1 and 3, if there exists positive constants k_p, λ_p, α, ϵ_1, ϵ_2, ϵ_3, ϵ_4 such that*

$$-2k_p + \alpha - 2d_p + \epsilon_1 \sum_{q=1}^{n} \acute{a}_{pq} \bar{\acute{a}}_{pq} + \epsilon_2 \sum_{q=1}^{n} \acute{b}_{pq} \bar{\acute{b}}_{pq} + \frac{n l_p^2}{\epsilon_1} + \frac{n l_p^2}{\epsilon_2 (1-\mu)} \mathrm{e}^{\alpha\tau} + \epsilon_3 \sum_{q=1}^{n} (\hat{a}_{pq} - \check{a}_{pq})(\bar{\hat{a}}_{pq} - \bar{\check{a}}_{pq}) + \epsilon_4 \sum_{q=1}^{n} (\hat{b}_{pq} - \check{b}_{pq})(\bar{\hat{b}}_{pq} - \bar{\check{b}}_{pq}) \leqslant 0, \tag{8}$$

$$-2\lambda_p + \left(\frac{1}{\epsilon_3} + \frac{1}{\epsilon_4}\right) n M_p \leqslant 0,$$

then global exponential synchronization can be reached between drive system (1) *and response system* (2) *under control input* (7).

Proof. Constructing the Lyapunov functional as

$$V(t) = \mathrm{e}^{\alpha t} \sum_{p=1}^{n} \bar{e}_p(t) e_p(t) + \sum_{p=1}^{n} \frac{\gamma}{1-\mu} \int_{t-\tau(t)}^{t} \mathrm{e}^{\alpha(s+\tau)} \bar{F}_p\big(e_p(s)\big) F_p\big(e_p(s)\big)\, \mathrm{d}s.$$

Computing the derivative of $V(t)$ along the trajectory (6), we have

$$\begin{aligned}
\frac{\mathrm{d}V(t)}{\mathrm{d}t} &= \alpha \mathrm{e}^{\alpha t}\sum_{p=1}^{n}\bar{e}_p(t)e_p(t) + \mathrm{e}^{\alpha t}\sum_{p=1}^{n}\dot{\bar{e}}_p(t)e_p(t) + \mathrm{e}^{\alpha t}\sum_{p=1}^{n}\bar{e}_p(t)\dot{e}_p(t) \\
&\quad + \sum_{p=1}^{n}\frac{\gamma}{1-\mu}\mathrm{e}^{\alpha(t+\tau)}\bar{F}_p\big(e_p(t)\big)F_p\big(e_p(t)\big) \\
&\quad - \sum_{p=1}^{n}\gamma \mathrm{e}^{\alpha t}\bar{F}_p\big(e_p\big(t-\tau(t)\big)\big)F_p\big(e_p\big(t-\tau(t)\big)\big) \\
&\leqslant \alpha \mathrm{e}^{\alpha t}\sum_{p=1}^{n}\bar{e}_p(t)e_p(t) + \mathrm{e}^{\alpha t}\sum_{p=1}^{n}\Bigg[-d_p\bar{e}_p(t) + \sum_{q=1}^{n}\bar{F}_q\big(e_q(t)\big)\bar{a}'_{pq} \\
&\quad + \sum_{q=1}^{n}\bar{f}_q\big(x_q^*(t)\big)(\bar{a}'_{pq}-\bar{a}''_{pq}) + \sum_{q=1}^{n}\bar{F}_q\big(e_q\big(t-\tau(t)\big)\big)\bar{b}'_{pq} \\
&\quad + \sum_{q=1}^{n}\bar{f}_q(x_q^*\big(t-\tau(t)\big)(\bar{b}'_{pq}-\bar{b}''_{pq}) - \bar{u}_p(t)\Bigg]e_p(t) \\
&\quad + \mathrm{e}^{\alpha t}\sum_{p=1}^{n}\bar{e}_p(t)\Bigg[-d_pe_p(t) + \sum_{q=1}^{n}a'_{pq}F_q\big(e_q(t)\big) + \sum_{q=1}^{n}(a'_{pq}-a''_{pq})f_q\big(x_q^*(t)\big) \\
&\quad + \sum_{q=1}^{n}b'_{pq}F_q\big(e_q\big(t-\tau(t)\big)\big) + \sum_{q=1}^{n}(b'_{pq}-b''_{pq})f_q\big(x_q^*\big(t-\tau(t)\big)\big) - u_p(t)\Bigg] \\
&\quad + \gamma \mathrm{e}^{\alpha t}\sum_{p=1}^{n}\bigg\{\frac{\mathrm{e}^{\alpha\tau}}{1-\mu}\bar{F}_p\big(e_p(t)\big)F_p\big(e_p(t)\big) \\
&\quad - \bar{F}_p\big(e_p\big(t-\tau(t)\big)\big)F_p\big(e_p\big(t-\tau(t)\big)\big)\bigg\}. \qquad (9)
\end{aligned}$$

According to Lemma 2 and Definition 1, there exists constants $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 > 0$ such that the following inequalities hold:

$$\begin{aligned}
&\bar{F}_q\big(e_q(t)\big)\bar{a}'_{pq}e_p(t) + \bar{e}_p(t)a'_{pq}F_q\big(e_q(t)\big) \\
&\quad \leqslant \epsilon_1\bar{e}_p(t)a'_{pq}\bar{a}'_{pq}e_p(t) + \frac{1}{\epsilon_1}\bar{F}_q\big(e_q(t)\big)F_q\big(e_q(t)\big) \\
&\quad \leqslant \epsilon_1\bar{e}_p(t)\acute{a}_{pq}\bar{\acute{a}}_{pq}e_p(t) + \frac{1}{\epsilon_1}\bar{F}_q\big(e_q(t)\big)F_q\big(e_q(t)\big), \qquad (10)
\end{aligned}$$

$$\begin{aligned}
&\bar{F}_q\big(e_q\big(t-\tau(t)\big)\big)\bar{b}'_{pq}e_p(t) + \bar{e}_p(t)b'_{pq}F_q\big(e_q\big(t-\tau(t)\big)\big) \\
&\quad \leqslant \epsilon_2\bar{e}_p(t)b'_{pq}\bar{b}'_{pq}e_p(t) + \frac{1}{\epsilon_2}\bar{F}_q\big(e_q\big(t-\tau(t)\big)\big)F_q\big(e_q\big(t-\tau(t)\big)\big) \\
&\quad \leqslant \epsilon_2\bar{e}_p(t)\acute{b}_{pq}\bar{\acute{b}}_{pq}e_p(t) + \frac{1}{\epsilon_2}\bar{F}_q\big(e_q\big(t-\tau(t)\big)\big)F_q\big(e_q\big(t-\tau(t)\big)\big), \qquad (11)
\end{aligned}$$

$$\begin{aligned}
&\bar{f}_q\big(x_q^*(t)\big)(\bar{a}'_{pq}-\bar{a}''_{pq})e_p(t)+\bar{e}_p(t)(a'_{pq}-a''_{pq})f_q\big(x_q^*(t)\big)\\
&\quad\leqslant \epsilon_3\bar{e}_p(t)(a'_{pq}-a''_{pq})(\bar{a}'_{pq}-\bar{a}''_{pq})e_p(t)+\frac{1}{\epsilon_3}\bar{f}_q\big(x_q^*(t)\big)f_q\big(x_q^*(t)\big)\\
&\quad\leqslant \epsilon_3\bar{e}_p(t)(\hat{a}_{pq}-\check{a}_{pq})(\bar{\hat{a}}_{pq}-\bar{\check{a}}_{pq})e_p(t)+\frac{1}{\epsilon_3}M_q, \qquad (12)\\
&\bar{f}_q\big(x_q^*\big(t-\tau(t)\big)\big)(\bar{b}'_{pq}-\bar{b}''_{pq})e_p(t)+\bar{e}_p(t)(b'_{pq}-b''_{pq})f_q\big(x_q^*\big(t-\tau(t)\big)\big)\\
&\quad\leqslant \epsilon_4\bar{e}_p(t)(b'_{pq}-b''_{pq})(\bar{b}'_{pq}-\bar{b}''_{pq})e_p(t)+\frac{1}{\epsilon_4}\bar{f}_q\big(x_q^*\big(t-\tau(t)\big)\big)f_q\big(x_q^*\big(t-\tau(t)\big)\big)\\
&\quad\leqslant \epsilon_4\bar{e}_p(t)(\hat{b}_{pq}-\check{b}_{pq})(\bar{\hat{b}}_{pq}-\bar{\check{b}}_{pq})e_p(t)+\frac{1}{\epsilon_4}M_q. \qquad (13)
\end{aligned}$$

Combining (9)–(13), we have

$$\begin{aligned}
\frac{\mathrm{d}V(t)}{\mathrm{d}t}\leqslant{}& \mathrm{e}^{\alpha t}\sum_{p=1}^{n}\Bigg\{(\alpha-2d_p)+\epsilon_1\sum_{q=1}^{n}\acute{a}_{pq}\bar{\acute{a}}_{pq}+\epsilon_2\sum_{q=1}^{n}\acute{b}_{pq}\bar{\acute{b}}_{pq}+\frac{nl_p^2}{\epsilon_1}+\frac{\gamma l_p^2}{1-\mu}\mathrm{e}^{\alpha\tau}\\
&+\epsilon_3\sum_{q=1}^{n}(\hat{a}_{pq}-\check{a}_{pq})(\bar{\hat{a}}_{pq}-\bar{\check{a}}_{pq})+\epsilon_4\sum_{q=1}^{n}(\hat{b}_{pq}-\check{b}_{pq})(\bar{\hat{b}}_{pq}-\bar{\check{b}}_{pq})\Bigg\}\\
&\times\bar{e}_p(t)e_p(t)+\mathrm{e}^{\alpha t}\sum_{p=1}^{n}\left\{\left(\frac{1}{\epsilon_3}+\frac{1}{\epsilon_4}\right)nM_p\right\}\\
&+\mathrm{e}^{\alpha t}\sum_{p=1}^{n}\left(\frac{n}{\epsilon_2}-\gamma\right)\bar{F}_p\big(e_p\big(t-\tau(t)\big)\big)F_p\big(e_p\big(t-\tau(t)\big)\big)\\
&+\sum_{p=1}^{n}\mathrm{e}^{\alpha t}\bar{e}_p(t)\left(-2k_pe_p(t)-2\lambda_p\frac{e_p(t)}{\bar{e}_p(t)e_p(t)}\right).
\end{aligned}$$

Choose $\gamma=n/\epsilon_2$, it follows

$$\begin{aligned}
\frac{\mathrm{d}V(t)}{\mathrm{d}t}\leqslant{}& \mathrm{e}^{\alpha t}\sum_{p=1}^{n}\Bigg\{(\alpha-2d_p)+\epsilon_1\sum_{q=1}^{n}\acute{a}_{pq}\bar{\acute{a}}_{pq}+\epsilon_2\sum_{q=1}^{n}\acute{b}_{pq}\bar{\acute{b}}_{pq}+\frac{nl_p^2}{\epsilon_1}+\frac{nl_p^2}{\epsilon_2(1-\mu)}\mathrm{e}^{\alpha\tau}\\
&+\epsilon_3\sum_{q=1}^{n}(\hat{a}_{pq}-\check{a}_{pq})(\bar{\hat{a}}_{pq}-\bar{\check{a}}_{pq})+\epsilon_4\sum_{j=1}^{n}(\hat{b}_{pq}-\check{b}_{pq})(\bar{\hat{b}}_{pq}-\bar{\check{b}}_{pq})-2k_p\Bigg\}\\
&\times\bar{e}_p(t)e_p(t)+\mathrm{e}^{\alpha t}\sum_{p=1}^{n}\left\{\left(\frac{1}{\epsilon_3}+\frac{1}{\epsilon_4}\right)nM_p-2\lambda_p\right\}.
\end{aligned}$$

Due to condition (8), we have $\mathrm{d}V(t)/\mathrm{d}t\leqslant 0$. With the same discussion in Theorem 1, it follows $\sum_{p=1}^{n}\bar{e}_p(t)e_p(t)\leqslant M\|\psi\|\mathrm{e}^{-\alpha t}$, $M=1+(\gamma/(1-\mu))\tau l_p^2\mathrm{e}^{\alpha\tau}$, which implies that the drive-response system (1) and (2) can reach exponential synchronization with control input (7). □

Remark 6. The synchronization is an extremely important issue to a control system, and it is essential for plenty of applications of neural networks. Recently, the synchronization problem of RVNNs and CVNNs have been considered widely [2, 5, 6, 25, 45]. However, the publications focused on the synchronization problem of QVNNs are still rare, let alone the QVMNNs. Moreover, the QVNNs have better performance compared with CVNNs and RVNNs in dealing with high dimensional data, thus leading to great potential application in practical. Therefore, it is meaningful to investigate the global exponential synchronization of QVMNNs.

Remark 7. The QVMNNs model formulated in our work can be regarded as the extension of traditional RVMNNs and CVMNNs. Thus, the result in this paper can be seen as the generalization of many previous publications on MNNs [6, 40, 45]. Furthermore, the combination of quaternion and memristor reveals more complex dynamical behavior in neural networks, which remains to be challenging topic in future. The quaternion Lyapunov method utilized in this work is efficient and convenient compared with previous publications [7, 26, 32], and our results are easy to be verified by numerical example.

4 Numerical examples

In order to show the effectiveness of our theoretical results, some numerical examples are given in this section.

First, we try to verify the Theorem 1.

Example 1. Consider the memristive QVNNs with 2 neurons as below:

$$\begin{aligned}\frac{\mathrm{d}x_p(t)}{\mathrm{d}t} &= -d_p x_p(t) + \sum_{q=1}^{2} a_{pq}\big(x_p(t)\big) f_q\big(x_q(t)\big) \\ &\quad + \sum_{q=1}^{2} b_{pq}\big(x_p(t)\big) f_q\big(x_q\big(t-\tau(t)\big)\big), \quad p = 1, 2, \end{aligned} \tag{14}$$

where $d_1 = 1$, $d_2 = 2$, and the memristive connection weights are given as below. For convenience, $x_p^l(t)$ is simplified to x_p^l, $l = R, I, J, K$.

$$a_{11}^R\big(x_1^R\big) = \begin{cases} -1, |x_1^R| \leqslant 1, \\ 1, |x_1^R| > 1, \end{cases} \qquad a_{12}^R\big(x_1^R\big) = \begin{cases} 1, |x_1^R| \leqslant 1, \\ -1, |x_1^R| > 1, \end{cases}$$

$$a_{21}^R\big(x_2^R\big) = \begin{cases} -1, |x_2^R| \leqslant 1, \\ 1, |x_2^R| > 1, \end{cases} \qquad a_{22}^R\big(x_2^R\big) = \begin{cases} 1, |x_2^R| \leqslant 1, \\ -1, |x_2^R| > 1, \end{cases}$$

$$a_{11}^I\big(x_1^I\big) = \begin{cases} -1, |x_1^I| \leqslant 1, \\ 1, |x_1^I| > 1, \end{cases} \qquad a_{12}^I\big(x_1^I\big) = \begin{cases} 1, |x_1^I| \leqslant 1, \\ -1, |x_1^I| > 1, \end{cases}$$

$$a_{21}^I\big(x_2^I\big) = \begin{cases} -1, |x_2^I| \leqslant 1, \\ 2, |x_2^I| > 1, \end{cases} \qquad a_{22}^I\big(x_2^I\big) = \begin{cases} 1, |x_2^I| \leqslant 1, \\ -1, |x_2^I| > 1, \end{cases}$$

$$a_{11}^J(x_1^J) = \begin{cases} 2, |x_1^J| \leqslant 1, \\ 1, |x_1^J| > 1, \end{cases} \qquad a_{12}^J(x_1^J) = \begin{cases} 1, |x_1^J| \leqslant 1, \\ -1, |x_1^J| > 1, \end{cases}$$

$$a_{21}^J(x_2^J) = \begin{cases} -1, |x_2^J| \leqslant 1, \\ 2, |x_2^J| > 1, \end{cases} \qquad a_{22}^J(x_2^J) = \begin{cases} 1, |x_2^J| \leqslant 1, \\ -1, |x_2^J| > 1, \end{cases}$$

$$a_{11}^K(x_1^K) = \begin{cases} 2, |x_1^K| \leqslant 1, \\ 1, |x_1^K| > 1, \end{cases} \qquad a_{12}^K(x_1^K) = \begin{cases} 1, |x_1^K| \leqslant 1, \\ -1, |x_1^K| > 1, \end{cases}$$

$$a_{21}^K(x_2^K) = \begin{cases} -1, |x_2^K| \leqslant 1, \\ 1, |x_2^K| > 1, \end{cases} \qquad a_{22}^K(x_2^K) = \begin{cases} 1, |x_2^K| \leqslant 1, \\ -1, |x_2^K| > 1, \end{cases}$$

$$b_{11}^R(x_1^R) = \begin{cases} 2, |x_1^R| \leqslant 1, \\ 1, |x_1^R| > 1, \end{cases} \qquad b_{12}^R(x_1^R) = \begin{cases} 1, |x_1^R| \leqslant 1, \\ -1, |x_1^R| > 1, \end{cases}$$

$$b_{21}^R(x_2^R) = \begin{cases} -1, |x_2^R| \leqslant 1, \\ 2, |x_2^R| > 1, \end{cases} \qquad b_{22}^R(x_2^R) = \begin{cases} 1, |x_2^R| \leqslant 1, \\ -1, |x_2^R| > 1, \end{cases}$$

$$b_{11}^I(x_1^I) = \begin{cases} 2, |x_1^I| \leqslant 1, \\ 1, |x_1^I| > 1, \end{cases} \qquad b_{12}^I(x_1^I) = \begin{cases} 1, |x_1^I| \leqslant 1, \\ -1, |x_1^I| > 1, \end{cases}$$

$$b_{21}^I(x_2^I) = \begin{cases} -1, |x_2^I| \leqslant 1, \\ 2, |x_2^I| > 1, \end{cases} \qquad b_{22}^I(x_2^I) = \begin{cases} 1, |x_2^I| \leqslant 1, \\ -1, |x_2^I| > 1, \end{cases}$$

$$b_{11}^J(x_1^J) = \begin{cases} -1, |x_1^J| \leqslant 1, \\ 1, |x_1^J| > 1, \end{cases} \qquad b_{12}^J(x_1^J) = \begin{cases} 1, |x_1^J| \leqslant 1, \\ -1, |x_1^J| > 1, \end{cases}$$

$$b_{21}^J(x_2^J) = \begin{cases} -1, |x_2^J| \leqslant 1, \\ 1, |x_2^J| > 1, \end{cases} \qquad b_{22}^J(x_2^J) = \begin{cases} 1, |x_2^J| \leqslant 1, \\ -1, |x_2^J| > 1, \end{cases}$$

$$b_{11}^K(x_1^K) = \begin{cases} -1, |x_1^K| \leqslant 1, \\ 1, |x_1^K| > 1, \end{cases} \qquad b_{12}^K(x_1^K) = \begin{cases} 1, |x_1^K| \leqslant 1, \\ -1, |x_1^K| > 1, \end{cases}$$

$$b_{21}^K(x_2^K) = \begin{cases} -1, |x_2^K| \leqslant 1, \\ 1, |x_2^K| > 1, \end{cases} \qquad b_{22}^K(x_2^K) = \begin{cases} 1, |x_2^K| \leqslant 1, \\ -1, |x_2^K| > 1, \end{cases}$$

the response system is given as

$$\begin{aligned} \frac{\mathrm{d}x_p^*(t)}{\mathrm{d}t} &= -d_p x_p^*(t) + \sum_{q=1}^{2} a_{pq}\big(x_p^*(t)\big) f_q\big(x_q^*(t)\big) + \sum_{q=1}^{2} b_{pq}\big(x_p^*(t)\big) f_q\big(x_q^*\big(t-\tau(t)\big)\big) \\ &\quad + k_p\big(x_p(t) - x_p^*(t)\big), \quad p = 1, 2, \end{aligned} \tag{15}$$

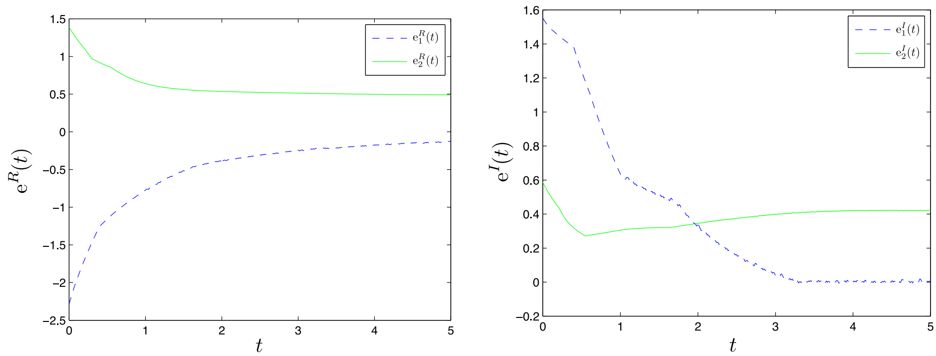

Figure 1. The trajectory of errors $e_1^R(t)$, $e_2^R(t)$, $e_1^I(t)$, $e_2^I(t)$ with no controller.

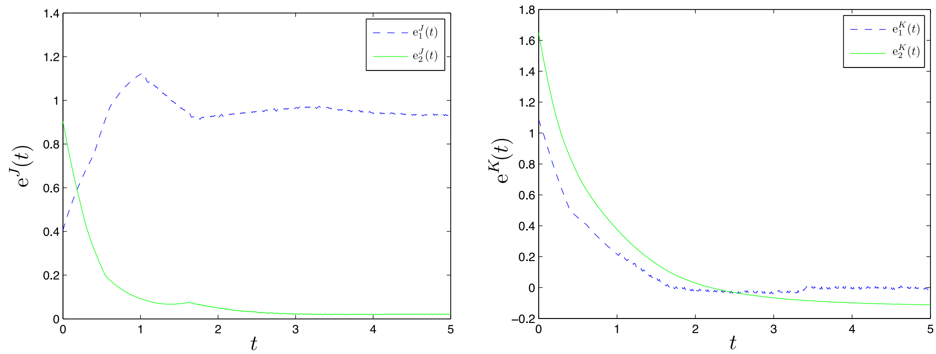

Figure 2. The trajectory of errors $e_1^J(t)$, $e_2^J(t)$, $e_1^K(t)$, $e_2^K(t)$ with no controller.

where the transmission delay is $\tau(t) = 0.2 + 0.5 \sin t$, thus $\tau = 0.7$, $\dot{\tau}(t) \leqslant \mu = 0.5 \leqslant 1$. The activation function is considered as

$$f_i\big(x_i(t)\big) = x_i^R(t) + x_i^I(t)i + x_i^J(t)j + x_i^K(t)k.$$

It is easy to achieve that $l_1 = l_2 = 1$. Firstly, Figs. 1, 2 shows the synchronization errors $e_1^R, e_1^I, e_1^J, e_1^K, e_2^R, e_2^I, e_2^J, e_2^K$ between drive system (14) and response system (15) without control input. Obviously, the drive-response system can not achieve synchronization without external control input. Choose $\alpha = 1$, $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$, $k_1 = k_2 = 21$, it can be checked that the condition of Theorem 1 holds. Choosing 20 initial random conditions in $[-0.3, 0.3]$, Figs. 3, 4 describes the synchronization errors e_1^R, e_1^I, e_1^J, e_1^K, e_2^R, e_2^I, e_2^J, e_2^K between drive system (14) and response system (15) with control input (3), respectively. From the simulation results, the error between drive system (14) and response system (15) tends to be 0 under the designed controller (3), which verify the results of Theorem 1.

Next, we provide a numerical example to demonstrate the effectiveness of Theorem 2 in this work.

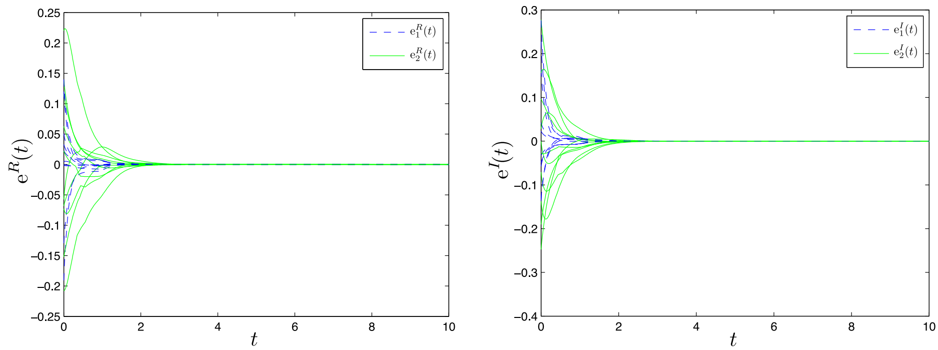

Figure 3. The trajectory of errors $e_1^R(t)$, $e_2^R(t)$, $e_1^I(t)$, $e_2^I(t)$ with controller (3).

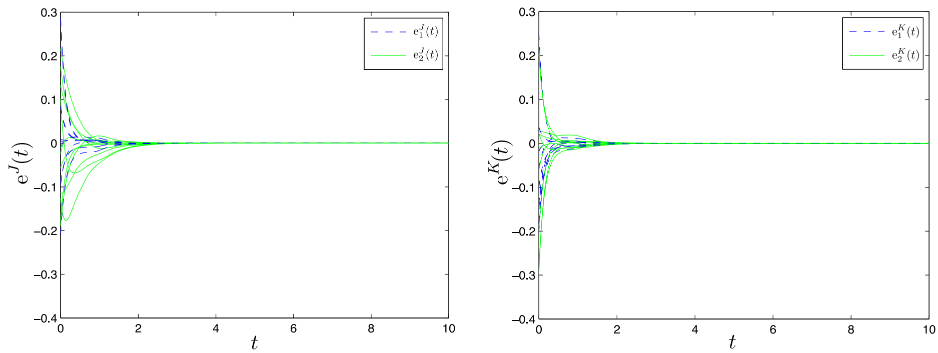

Figure 4. The trajectory of errors $e_1^J(t)$, $e_2^J(t)$, $e_1^K(t)$, $e_2^K(t)$ with controller (3).

Example 2. Consider the QVMNNs (14) as the drive system and the response system with control input (7):

$$\frac{\mathrm{d}x_p^*(t)}{\mathrm{d}t} = -d_i x_p^*(t) + \sum_{q=1}^{2} a_{pq}(x_p^*(t)) f_q\big(x_q^*(t)\big) + \sum_{q=1}^{2} b_{pq}(x_p^*(t)) f_q\big(x_q^*\big(t-\tau(t)\big)\big)$$
$$+ k_p e_p(t) + \lambda_p \frac{e_p(t)}{\bar{e}_p(t) e_p(t)}, \quad p = 1, 2, \tag{16}$$

where the memristive connection weights share the same value of Example 1, choose the activation function as

$$f_p\big(x_p(t)\big) = \frac{1}{1+\mathrm{e}^{x_p^R(t)}} + \frac{1}{1+\mathrm{e}^{x_p^I(t)}} i + \frac{1}{1+\mathrm{e}^{x_p^J(t)}} j + \frac{1}{1+\mathrm{e}^{x_p^K(t)}} k, \quad i = 1, 2.$$

Hence, $l_1 = l_2 = 1$, $M_1 = M_2 = 4$. Choose parameters $\alpha = 1$, $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 1$, control gains $k_1 = 47$, $k_2 = 62$, $\lambda_1 = \lambda_2 = 4$, which means that the condition of Theorem 2 is hold. Figures 5, 6 depicts the synchronization errors e_1^R, e_1^I, e_1^J, e_1^K, e_2^R, e_2^I, e_2^J, e_2^K with 20 initial conditions in $[-0.4, 0.4]$. From the simulation results, we can

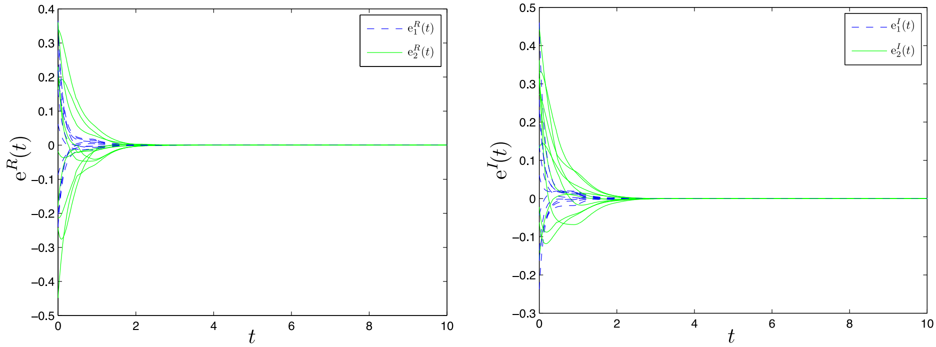

Figure 5. The trajectory of errors $e_1^R(t)$, $e_2^R(t)$, $e_1^I(t)$, $e_2^I(t)$ with controller (7).

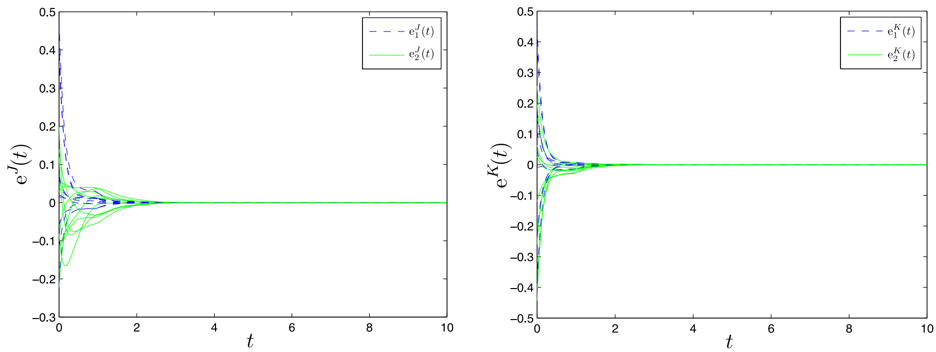

Figure 6. The trajectory of errors $e_1^J(t)$, $e_2^J(t)$, $e_1^K(t)$, $e_2^K(t)$ with controller (7).

see that the drive-response systems (14) and (16) are synchronized under controller (7), which verify the results of Theorem 2.

5 Conclusion

In this paper, we introduced the memristive connection weights into QVNNs to construct the QVMNNs, which is a new class of network model with the character of both MNNs and QVNNs. Then, according to different assumptions of the memristive connection weights, the global synchronization problem of this type of networks are considered in two cases. Applying the theory of set-valued map, differential inclusion and Lyapunov functional technique, several criterias for global exponential synchronization of drive-response memristive QVNNs are obtained. Finally, simulation examples are given to verify the correctness of our theorem.

Our future research will concentrate on two aspects: (i) The dynamics of quaternion-valued memristive neural networks with stochastic case. (ii) Investigating the dynamical behavior of coupled memristive quaternion-valued neural networks with imperfect communication, such as packet dropout and quantization.

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