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A unifying approach for some nonexpansiveness conditions on modular vector spaces

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Abstract. This paper provides a new, symmetric, nonexpansiveness condition to extend the classical Suzuki mappings. The newly introduced property is proved to be equivalent to condition (E) on Banach spaces, while it leads to an entirely new class of mappings when going to modular vector spaces; anyhow, it still provides an extension for the modular version of condition (C). In connection with the newly defined nonexpansiveness, some necessary and sufficient conditions for the existence of fixed points are stated and proved. They are based on Mann and Ishikawa iteration procedures, convenient uniform convexities and properly selected minimizing sequences.

Keywords: modular vector space, generalized nonexpansive mappings, condition (C), condition (E).

1 Introduction

The idea of looking for new contractive conditions to lead to wider and wider classes of mappings, as well as the effort of extending the metric setting, are two of the main directions in fixed point theory. This paper provides a new contribution related to these directions by defining a new nonexpansiveness property, on Banach spaces initially, and extending it afterwards to modular vector spaces (please see [11, 12] for the definition of modular vector spaces, as well as [1–5, 8–10] and others, for important properties and connections with fixed point theory).

An important step toward analyzing a more general nonexpansiveness condition on Banach spaces was performed by Suzuki in [14]. He defined the so called condition (C), which provided a wider class of mappings than the nonexpansive mappings and stronger than the class of quasinonexpansive mappings. His idea inspired other researchers to introduce even more general properties:

- In [13], the class of Suzuki-type generalized nonexpansive mappings on Banach spaces was changed in connection with an admissible pair of parameters. This new property extends Suzuki's condition, but remains subordinated to quasinonexpansiveness.
- In [7], property (E) was introduced, which extends Suzuki's condition too.
- In [6], it has been defined the so called condition (D_a), which properly contains the nonexpansive mappings. This is stronger than the quasinonexpansiveness property, and it has no inclusion connection with the class of Suzuki mappings; moreover, no direct connection has been proved yet with mappings satisfying condition (E).

In addition, some of the previously listed properties were extended to modular vector spaces, resulting new modular nonexpansiveness conditions:

- Condition (C) in [4] as a modular extension of the Suzuki's nonexpansiveness property;
- The modular Suzuki-type generalized nonexpansive mappings in [3] as a modular extension for the mappings defined in [13], as well as generalization for mappings satisfying condition (ρC);
- Condition (ρE) in [8], i.e. the modular version of condition (E). This condition has not yet been proven to be related to property (ρC) or modular nonexpansiveness (in fact, we will prove later in this paper that condition (ρE) does not extend condition (ρC); moreover, it does not even extend the modular nonexpansiveness).

Obviously, the following two questions appear in connection with the following approaches:

1. *Is it possible to define a nonexpansiveness condition to include all of the above*?
2. *Can this condition be extended to modular vector spaces*?

This paper provides an answer by defining a new property, called condition (CDE), which is proved to be equivalent to condition (E) on Banach spaces and to include the classes of Suzuki-type generalized nonexpansive mappings, as well as the class of mappings satisfying condition (D_a). When it comes to the modular version of the newly defined condition, the generated class of mappings is distinct from the family of mappings satisfying condition (ρE), but it has the interesting property that it still provides an extension for modular Suzuki-type generalized nonexpansive mappings, hence, for modular nonexpansiveness too.

The rest of the paper is organized as follows. Section 2 contains the necessary background regarding modular vector spaces introducing the specific terminology and several important technical elements. Section 3 is a review of several generalized nonexpansiveness conditions on Banach spaces. A new property is introduced, and a comparison analysis with the already existing classes is performed. Section 4 includes the elements of the previous section in modular setting. The newly introduced modular nonexpansiveness condition is proved to be wider than most of the preexisting classes. Nevertheless, one notices that some properties holding true on Banach spaces are lost when going to modular framework. The main outcome gives a necessary and sufficient condition for the existence of fixed points of a generalized nonexpansive mapping in connection with Mann iterative processes. Last, but not least, Section 5 reconsiders this approach via Ishikawa iteration procedures. This issue will require stronger assumptions regarding the uniform convexity and will use minimizing sequences for a mixed ρ-type function. Section 6 contains the conclusions and policy implications.

2 Preliminaries

We start with a short description of basic elements regarding modular vector spaces.

Definition 1. [See [11, 12].) Let X be a real vector space. A function $\rho : X \to [0, \infty]$ satisfying

(a) $\rho(x) = 0 \Leftrightarrow x = 0$;
(b) $\rho(-x) = \rho(x)$ for all $x \in X$;
(c) $\rho(\alpha x + (1-\alpha)y) \leqslant \alpha\rho(x) + (1-\alpha)\rho(y)$ for all $\alpha \in [0, 1]$, for all $x, y \in X$

is called convex modular.

The set

$$X_\rho = \Big\{x \in X\colon \lim_{\alpha\to 0} \rho(\alpha x) = 0\Big\}$$

is called a modular vector space.

Similarly to normed spaces, concepts as modular convergent sequences, modular Cauchy sequences, modular completeness, modular boundedness or modular closeness of sets can be defined in connection with a given convex modular.

Definition 2. (See [1,4,8,11,12].) Let ρ be a convex modular defined on a vector space X.

(a) A sequence $\{x_n\} \subset X_\rho$ is called ρ-convergent to the point $x \in X_\rho$ if and only if $\lim_{n\to\infty} \rho(x_n - x) = 0$. Note that the ρ-limit is unique if it exists.
(b) We say that a sequence $\{x_n\} \subset X_\rho$ is ρ-Cauchy if $\lim_{n,m\to\infty} \rho(x_n - x_m) = 0$.
(c) We say that the modular space X_ρ is ρ-complete if and only if any ρ-Cauchy sequence in X_ρ is ρ-convergent.
(d) A set $C \subset X_\rho$ is called ρ-closed if for any sequence $\{x_n\} \subset C$, which ρ-converges to x, it follows $x \in C$.
(e) A set $C \subset X_\rho$ is called ρ-bounded if $\operatorname{diam}_\rho(C) = \sup\{\rho(x-y)\colon x, y \in C\} < \infty$.
(f) ρ is said to satisfy the Fatou property if $\rho(x) \leqslant \liminf_{n\to\infty} \rho(x_n)$ whenever $\{x_n\}$ ρ-converges to x for any x and $\{x_n\}$ in X_ρ.
(g) The modular ρ is said to satisfy the Δ_2-condition if there exists $K \geqslant 0$ such that $\rho(2x) \leqslant K\rho(x)$ for all $x \in X_\rho$. The smallest constant K with this property is usually denoted by $\omega(2)$ and, divided by two and rewritten $\mu = \omega(2)/2$, is called the modular factor.

Let us point out next an important property of the modular factor, which follows directly from the Δ_2-condition and the convexity.

Remark 1. For all $x, y \in X_\rho$, the following inequality holds true:

$$\rho(x+y) \leqslant \mu\big[\rho(x) + \rho(y)\big].$$

The next elements were initially stated in the setting of modular function spaces (see [9]). Nevertheless, they can be easily reconsidered for modular vector spaces too (see [1]).

Definition 3. (See [1], cf. [9].) Let $r, \epsilon > 0$. Consider the set

$$D_1(r,\epsilon) = \big\{(x,y)\colon\ x, y \in X_\rho,\ \rho(x) \leqslant r,\ \rho(y) \leqslant r,\ \rho(x-y) \geqslant \epsilon r\big\}$$

and

$$\delta_1(r,\epsilon) = \begin{cases} \inf\{1 - \frac{1}{r}\rho(\frac{x+y}{2})\colon\ x, y \in D_1(r,\epsilon)\} & \text{if } D_1(r,\epsilon) \neq \emptyset; \\ 1 & \text{if } D_1(r,\epsilon) = \emptyset. \end{cases}$$

The convex modular ρ satisfies property (UUC1) if for every $s \geqslant 0$ and $\epsilon > 0$, there exists $\eta_1(s,\epsilon) > 0$ such that $\delta_1(r,\epsilon) > \eta_1(s,\epsilon) > 0$ for $r > s$.

Lemma 1. (*Cf.* [9].) *Suppose that ρ is endowed with the* (UUC1) *property, and let $\{\alpha_n\}$ be a sequence bounded away from* 0 *and* 1 (*i.e.* $0 < a \leqslant \alpha_n \leqslant b < 1$ *for all* $n \in \mathbb{N}$). *If there exists $r > 0$ such that* $\limsup_{n\to\infty} \rho(x_n) \leqslant r$, $\limsup_{n\to\infty} \rho(y_n) \leqslant r$ *and* $\limsup_{n\to\infty} \rho(\alpha_n x_n + (1-\alpha_n) y_n) = r$, *then* $\lim_{n\to\infty} \rho(x_n - y_n) = 0$.

Definition 4. (See [1].) Let $\{x_n\}$ be a sequence in X_ρ, and let $S \subset X_\rho$ be a nonempty subset.

(a) The function $\tau : S \to [0,\infty]$, $\tau(x) = \limsup_{n\to\infty} \rho(x - x_n)$, is called a ρ-type function.
(b) The value $r(S) = \inf\{\tau(x)\colon\ x \in S\}$ is called the asymptotic radius of $\{x_n\}$ relative to S.
(c) A sequence $\{c_n\}$ in S is called a minimizing sequence of τ if $\lim_{n\to\infty} \tau(c_n) = r(S)$.

Lemma 2. (*See* [1].) *Let X_ρ be a ρ-complete modular space. Assume that ρ satisfies the Fatou property. Let S be a nonempty ρ-closed convex subset of X_ρ, and let $\{x_n\}$ be a sequence in X_ρ with finite asymptotic radius relative to S* (*i.e.* $r(S) = \inf\{\tau(x)\colon x \in S\} < \infty$). *If ρ satisfies* (UUC1) *property, then all the minimizing sequences of τ are ρ-convergent to the same limit.*

In addition to the modular related elements listed above, the following general outcome will provide an important tool for our further analysis.

Lemma 3. *Let $\{a_n\}$ and $\{b_n\}$ be two bounded real sequences. Then*

(i) $\liminf_{n\to\infty} \min\{a_n, b_n\} = \min\{\liminf_{n\to\infty} a_n, \limsup_{n\to\infty} b_n\}$ *and* $\limsup_{n\to\infty} \min\{a_n, b_n\} \leqslant \min\{\limsup_{n\to\infty} a_n, \limsup_{n\to\infty} b_n\}$.
(ii) *Let* $c_n = \alpha_n a_n + (1-\alpha_n) b_n$ *with* $\alpha_n \in [0,1]$ *convergent to a real number* $\alpha \in [0,1]$. *Then* $\limsup_{n\to\infty} c_n \leqslant \alpha \limsup_{n\to\infty} a_n + (1-\alpha) \limsup_{n\to\infty} b_n$.

3 Classes of generalized nonexpansive mappings on Banach spaces

Let us start by recalling several important properties, which were meant to extend the class of nonexpansive mappings on Banach spaces.

B.1 Mappings with condition (C) (Suzuki nonexpansive mappings)

Definition 5. (See [14].) Let $S \subset X$ be a nonempty subset of a Banach space $(X, \|\cdot\|)$. A mapping $T : S \to S$ is said to satisfy condition (C) (or to be a Suzuki nonexpansive mapping) if $\|Tx - Ty\| \leqslant \|x - y\|$ whenever $\|x - Tx\|/2 \leqslant \|x - y\|$.

The definition above makes clear the fact that each nonexpansive mapping is also a Suzuki mapping, while each mapping satisfying condition (C) is quasinonexpansive.

The following lemma refers to an essential property of mappings under condition (C).

Lemma 4. *(See [14].) If $S \subset X$ is a nonempty subset of a Banach space $(X, \|\cdot\|)$ and $T : S \to S$ is a Suzuki nonexpansive mapping, then*

(i) *for each $x \in S$, one has $\|Tx - T^2x\| \leqslant \|x - Tx\|$;*
(ii) *for any $x, y \in S$, either $\|x - Tx\|/2 \leqslant \|x - y\|$ or $\|Tx - T^2x\|/2 \leqslant \|Tx - y\|$.*
(iii) *the inequality $\|x - Ty\| \leqslant 3\|Tx - x\| + \|x - y\|$ holds for all $x, y \in S$.*

B.2 (α, β)-Suzuki-type generalized nonexpansive mappings

Definition 6. (See [13].) Let $(X, \|\cdot\|)$ be a Banach space and $S \subset X$ a nonempty subset. Let $\alpha > 0$ and $\beta \geqslant 0$. A mapping $T : S \to S$ is called (α, β)-Suzuki-type generalized nonexpansive if, for all $x, y \in S$,

$$\frac{1}{2}\|x - Tx\| \leqslant \|x - y\|$$
$$\implies \quad \alpha\|Tx - Ty\| + (1 - \alpha)\|x - Ty\| \leqslant \beta\|Tx - y\| + (1 - \beta)\|x - y\|.$$

We notice that the class of Suzuki nonexpansive mappings can be retrieved when considering $\alpha = 1$ and $\beta = 0$. Also, each (α, β)-Suzuki-type generalized nonexpansive mapping is quasinonexpansive.

The following lemma recalls some basic properties of the (α, β)-Suzuki mappings as they were stated and proved in [13] for properly chosen parameters (we shall refer to such convenient parameters as admissible parameters).

Lemma 5. (*See* [13].) *Let $(X, \|\cdot\|)$ be a Banach space, and let $S \subset X$ a nonempty subset. Assume that $\alpha \geqslant 1$, $\beta \geqslant 0$ and $\alpha - \beta \leqslant 1$, and let $T : S \to S$ be an (α, β)-Suzuki-type mapping. Then the following inequalities hold*:

(i) $\|Tx - T^2x\| \leqslant \|x - Tx\|$ *for all $x \in S$*;
(ii) $\|x - Tx\|/2 \leqslant \|x - y\|$ *or* $\|Tx - T^2x\|/2 \leqslant \|Tx - y\|$ *for all $x, y \in S$*;
(iii) $\|x - Ty\| \leqslant (3\alpha + \beta)\|Tx - x\| + \|x - y\|$ *for all $x, y \in S$.*

B.3 Mappings satisfying condition ($\mathbf{D}_a$)

Definition 7. (See [6].) Let $S \subset X$ be a nonempty subset of a Banach space $(X, \|\cdot\|)$. For $a \in (1/2, 1)$, we say that a mapping $T: S \to S$ satisfies condition (D_a) if

$$\|Tx - Ty\| \leqslant \|x - y\| \quad \forall \alpha \in [a, 1],\ x \in S,\ y \in S(T, x, \alpha),$$

where

$$S(T, x, \alpha) = \bigl\{(1 - \alpha)p + \alpha Tq\colon\ p, q \in S,\ \|Tp - p\|,\ \|Tq - q\| \leqslant \|Tx - x\|\bigr\}.$$

B.4 Mappings satisfying condition (D)

If we reduce the admissible set of pairs involved in Definition 7 by fixing $\alpha = 1$, we could define a more general class of mappings. We shall further refer to the resulting condition by calling it condition (D). Its definition is stated below.

Definition 8. Let $S \subset X$ be a nonempty subset of a Banach space $(X, \|\cdot\|)$. We say that a mapping $T: S \to S$ satisfies condition (D) if

$$\|Tx - Ty\| \leqslant \|x - y\| \quad \forall x \in S,\ y \in S(T, x),$$

where

$$S(T, x) = \bigl\{Tp\colon\ p \in S,\ \|Tp - p\| \leqslant \|Tx - x\|\bigr\}.$$

We notice that the notion of mappings satisfying condition (D) is more general than the notion of mappings satisfying condition (D_a) for any $a \in (1/2, 1)$.

The following result is an immediate consequence of the definition and requires no special clarifications.

Proposition 1. *Each nonexpansive mapping satisfies condition (D). If T satisfies condition (D), then T is quasinonexpansive.*

In addition, the following list provides basic properties of mappings satisfying condition (D).

Lemma 6. *Suppose $T: S \to S$ satisfies condition* (D). *Then*

(i) $\|T^2x - Tx\| \leqslant \|Tx - x\|$ *for all* $x \in S$;
(ii) *for all $x, y \in S$, at least one of the inequalities $\|T^2x - Ty\| \leqslant \|Tx - y\|$ or $\|T^2y - Tx\| \leqslant \|Ty - x\|$ is satisfied*;
(iii) $\|x - Ty\| \leqslant 3\|Tx - x\| + \|x - y\|$ *whenever* $\|Tx - x\| \leqslant \|Ty - y\|$;
(iv) $\|Tx - Ty\| \leqslant 2\min\{\|Tx - x\|, \|Ty - y\|\} + \|x - y\|$ *for all* $x, y \in S$.

Proof. (i) Since $\|Tx - x\| \leqslant \|Tx - x\|$, it follows that $Tx \in S(T, x)$, therefore, according to the definition, $\|T^2x - Tx\| \leqslant \|Tx - x\|$.

(ii) If $\|Tx - x\| \leqslant \|Ty - y\|$, it follows that $Tx \in S(T, y)$ and $\|T^2x - Ty\| \leqslant \|Tx - y\|$. Similarly, when $\|Ty - y\| < \|Tx - x\|$, one finds $\|T^2y - Tx\| \leqslant \|Ty - x\|$.

(iii) Assume that $\|Tx - x\| \leqslant \|Ty - y\|$. Then, using the subadditivity of the norm and the inequalities stated at (i) and (ii), we obtain

$$\begin{aligned}\|x - Ty\| &\leqslant \|x - Tx\| + \big\|Tx - T^2x\big\| + \big\|T^2x - Ty\big\| \\ &\leqslant \|x - Tx\| + \|x - Tx\| + \|Tx - y\| \\ &\leqslant 3\|Tx - x\| + \|x - y\|.\end{aligned}$$

(iv) Suppose $\|Tx - x\| \leqslant \|Ty - y\|$. Then

$$\begin{aligned}\|Tx - Ty\| &\leqslant \big\|Tx - T^2x\big\| + \big\|T^2x - Ty\big\| \\ &\leqslant \|Tx - x\| + \|Tx - y\| \\ &\leqslant 2\|Tx - x\| + \|x - y\| \\ &= 2\min\big\{\|Tx - x\|, \|Ty - y\|\big\} + \|x - y\|.\end{aligned}$$

By switching the order of x and y we find that the inequality is satisfied when $\|Ty - y\| < \|Tx - x\|$ too. Therefore, the inequality holds for every pair (x, y). □

B.5 Mappings satisfying condition (E)

Definition 9. (See [7].) Let S be a nonempty subset of a Banach space $(X, \|\cdot\|)$. For $\lambda \geqslant 1$, we say that a mapping $T : S \to S$ satisfies condition (E_λ) if

$$\|x - Ty\| \leqslant \lambda\|x - Tx\| + \|x - y\| \quad \forall x, y \in S.$$

The mapping T is said to satisfy condition (E) whenever T satisfies (E_λ) for some $\lambda \geqslant 1$.

Proposition 2. (*See* [7].) *Let $T : S \to S$ be a mapping, which satisfies condition* (E) *on S. If T has fixed points, then T is quasinonexpansive. The converse is not true.*

Moreover, Lemma 4(iii) and Lemma 5(iii) ensure us that each Suzuki nonexpansive mapping, as well as each (α, β)-Suzuki-type nonexpansive mapping with convenient pair of parameters, satisfies also condition (E). In particular, each nonexpansive mapping satisfies condition (E). Nevertheless, we notice from Lemma 6(iii) that this property is not necessarily satisfied by mappings endowed with property (D) in general.

B.6 A new class of nonexpansive mappings on Banach spaces

To summarize, we notice that all the conditions listed above are wider than nonexpansiveness and, simultaneously, remain stronger than quasinonexpansiveness. The (α, β)-Suzuki-type generalized nonexpansive mappings with $\alpha \geqslant 1$, $\beta \geqslant 0$ and $\alpha - \beta \leqslant 1$ satisfy condition (E) for $\mu = 3\alpha + \beta$. In particular, the Suzuki nonexpansive mappings satisfy condition (E). However, at this moment, we cannot state for sure that mappings satisfying condition (D) have also the property (E). These lead us to the problem of finding an even larger class of mappings that would include all of the above.

We introduce the following definition.

Definition 10. Let S be a nonempty subset of a Banach space $(X, \|\cdot\|)$. We say that a mapping $T : S \to S$ satisfies condition (CDE) if there exists $\lambda \geqslant 1$ such that

$$\|Tx - Ty\| \leqslant \lambda \min\bigl\{\|x - Tx\|, \|y - Ty\|\bigr\} + \|x - y\| \quad \forall x, y, \in S.$$

The statement below describes the connection of the newly defined nonexpansiveness condition with the previously listed classes of mappings.

Proposition 3. *The following statements hold true*:

(i) *Condition* (E) *is equivalent to condition* (CDE): *in particular, Suzuki nonexpansive mappings and* (α, β)*-Suzuki nonexpansive mappings for an admissible set of parameters satisfy condition* (CDE).

(ii) *Each mapping satisfying condition* (D) *satisfies condition* (CDE) *too*; *and, according to* (i)*, it satisfies also condition* (E).

Proof. (i) Suppose that T satisfies condition (E_λ) for some $\lambda \geqslant 1$. Then

$$\begin{aligned}\|Tx - Ty\| &\leqslant \|Tx - x\| + \|x - Ty\| \\ &\leqslant (\lambda + 1)\|Tx - x\| + \|x - y\| \quad \forall x, y \in S.\end{aligned}$$

Changing the order of x and y, one also finds

$$\|Tx - Ty\| \leqslant (\lambda + 1)\|Ty - y\| + \|x - y\| \quad \forall x, y \in S.$$

Combining the two resulted inequalities, one obtains precisely the condition (CDE).

Suppose now that T satisfies condition (CDE) for a given parameter $\lambda \geqslant 1$. Then

$$\begin{aligned}\|x - Ty\| &\leqslant \|x - Tx\| + \|Tx - Ty\| \\ &\leqslant \|x - Tx\| + \lambda \min\bigl\{\|x - Tx\|, \|y - Ty\|\bigr\} + \|x - y\| \\ &\leqslant (\lambda + 1)\|x - Tx\| + \|x - y\|,\end{aligned}$$

which is precisely condition (E).

(ii) It follows directly from Lemma 6(iv). □

4 Extensions to modular vector spaces

In a parallel approach, some of the classes listed above were extended to modular vector spaces as it follows.

M.1 Condition (ρC) or modular Suzuki nonexpansive mappings

Definition 11. (See [4].) Let ρ denote a convex modular satisfying condition Δ_2 on a linear (vector) space X with modular factor μ, and let $S \subset X_\rho$ be a nonempty subset. A mapping $T : S \to S$ is said to satisfy condition (ρC) if

$$\rho(Tx - Ty) \leqslant \rho(x - y) \quad \text{whenever } \frac{1}{2\mu}\rho(x - Tx) \leqslant \rho(x - y).$$

Lemma 7. (*See* [4].) *Let ρ denote a convex modular with Δ_2-property, and let $S \subset X_\rho$ be a nonempty subset. Then*

(i) *for each $x \in S$, one has $\rho(Tx - T^2x) \leqslant \rho(x - Tx)$;*
(ii) *for any $x, y \in S$, either $\rho(x - Tx)/(2\mu) \leqslant \rho(x - y)$ or $\rho(Tx - T^2x)/(2\mu) \leqslant \rho(Tx - y)$.*

M.2 Modular (α, β)-Suzuki mappings

Definition 12. (See [3].) Let ρ be a convex modular satisfying Δ_2-condition on a vector space X, and let $S \subset X_\rho$ be a nonempty subset. Let $\alpha \geqslant 1$ and $\beta \geqslant 0$. A mapping $T : S \to S$ is said to be modular (α, β)-Suzuki mapping if, for all $x, y \in S$,

$$\alpha\rho(Tx - Ty) + \frac{1-\alpha}{\mu}\rho(x - Ty) \leqslant \frac{\beta}{\mu}\rho(Tx - y) + (1-\beta)\rho(x - y)$$

whenever $\rho(x - Tx)/(2\mu) \leqslant \rho(x - y)$.

Same as for Banach spaces, the modular Suzuki nonexpansive mappings are modular (α, β)-Suzuki for $\alpha = 1$ and $\beta = 0$. Moreover, when the modular is precisely the norm of a Banach space, then the modular factor is $\mu = 1$, and the definition above describes the (α, β)-Suzuki-type nonexpansiveness condition.

Lemma 8. (*See* [3].) *Let ρ denote a convex modular on X with Δ_2-property, and let μ denote the corresponding modular factor. Consider also a nonempty subset $S \subset X_\rho$. Assume that $(\alpha, \beta) \in \mathcal{A}$, where*

$$\mathcal{A} = \bigl\{(\alpha, \beta)\colon\ \alpha \geqslant 1,\ \beta \geqslant 0,\ \alpha - \beta \leqslant 1,\ (\mu - 1)(\alpha - 1) < 1\bigr\},$$

is a set of admissible parameters, and let $T : S \to S$ be a modular-(α, β)-Suzuki mapping. Then

(i) $\rho(Tx - T^2x) \leqslant \rho(x - Tx)$ *for all* $x \in S$;
(ii) $\rho(x - Tx)/(2\nu) \leqslant \rho(x - y)$ *or* $\rho(Tx - T^2x)/(2\mu) \leqslant \rho(Tx - y)$ *for all* $x, y \in S$;
(iii) *there exists* $\varphi = \varphi(\alpha, \beta, \mu)$ *such that* $\rho(x - Ty) \leqslant \varphi(\alpha, \beta, \mu)\rho(x - Tx) + \mu\rho(x - y)$ *for all* $x, y \in S$.

M.3 Condition (ρE)

Definition 13. (See [8].) Let S be a nonempty subset of a modular vector space X_ρ. For $\lambda \geqslant 1$, we say that a mapping $T : S \to S$ satisfies condition ($\rho\mathrm{E}_\lambda$) if

$$\rho(x - Ty) \leqslant \lambda\rho(x - Tx) + \rho(x - y) \quad \forall x, y \in S.$$

The mapping T is said to satisfy condition (ρE) whenever T satisfies ($\rho\mathrm{E}_\lambda$) for some $\lambda \geqslant 1$.

These definitions inspire us to perform an extension of the newly introduced condition (CDE) from Banach spaces to modular spaces.

M.4 Condition (ρCDE)

Definition 14. Let S be a nonempty subset of a modular vector space X_ρ. For $\lambda \geqslant 1$, we say that a mapping $T : S \to S$ satisfies condition (ρCDE) if

$$\rho(Tx - Ty) \leqslant \lambda \min\bigl\{\rho(x - Tx), \rho(y - Ty)\bigr\} + \rho(x - y) \quad \forall x, y \in S.$$

Remark 2. Unfortunately, apart from Banach spaces, we do not have enough arguments to state that modular (α, β)-Suzuki mappings with $(\alpha, \beta) \in \mathcal{A}$ satisfy also condition (ρE). Lemma 8(iii) gives a description for modular Suzuki mappings of high similitude with the definition of condition (ρE), but the coefficient of $\rho(x - y)$ in the right side is different from 1. We cannot even state for sure that modular Suzuki mappings are satisfying condition (ρE).

However, we shall prove that (α, β)-Suzuki mappings satisfy condition (ρCDE).

Proposition 4. *Let ρ denote a convex modular on X with Δ_2-property, and let $S \subset X_\rho$ be a nonempty subset. Let $T : S \to S$ be a modular (α, β)-Suzuki mapping for a given admissible pair of parameters. Then T satisfies condition* (ρCDE).

Proof. Let $x, y \in S$. According to Lemma 8(ii), we have to consider two distinct cases:

Case 1. Suppose that $\rho(x - Tx)/(2\mu) \leqslant \rho(x - y)$. Then, since T is (α, β)-Suzuki mapping, it follows

$$\begin{aligned}
\alpha\rho(Tx - Tp) &\leqslant \frac{\alpha - 1}{\mu}\rho(x - Tp) + \frac{\beta}{\mu}\rho(Tx - y) + (1 - \beta)\rho(x - y) \\
&\leqslant \frac{\alpha - 1}{\mu}\mu\bigl[\rho(x - Tx) + \rho(Tx - Tp)\bigr] \\
&\quad + \frac{\beta}{\mu}\mu\bigl[\rho(Tx - x) + \rho(x - y)\bigr] + (1 - \beta)\rho(x - y) \\
&\leqslant (\alpha + \beta - 1)\rho(x - Tx) + (\alpha - 1)\rho(Tx - Tp) + \rho(x - y),
\end{aligned}$$

therefore,

$$\rho(Tx - Tp) \leqslant (\alpha + \beta - 1)\rho(x - Tx) + \rho(x - y). \tag{1}$$

Case 2. Suppose that $\rho(x - Tx)/(2\mu) > \rho(x - y)$. Then we must have instead $\rho(Tx - T^2x)/(2\mu) \leqslant \rho(Tx - y)$ and, since T is a (α, β)-Suzuki mapping, we find

$$\alpha\rho\bigl(T^2x - Ty\bigr) \leqslant \frac{\alpha - 1}{\mu}\rho(Tx - Ty) + \frac{\beta}{\mu}\rho\bigl(T^2x - y\bigr) + (1 - \beta)\rho(Tx - y).$$

It follows that

$$\begin{aligned}
&\alpha\rho(Tx - Ty) \\
&\quad \leqslant \mu\bigl(\alpha\rho\bigl(Tx - T^2x\bigr) + \alpha\rho\bigl(T^2x - Ty\bigr)\bigr) \\
&\quad \leqslant \mu\left[\alpha\rho\bigl(Tx - T^2x\bigr) + \frac{\alpha - 1}{\mu}\rho(Tx - Ty) + \frac{\beta}{\mu}\rho\bigl(T^2x - y\bigr) + (1 - \beta)\rho(Tx - y)\right]
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \mu\biggl[\alpha\rho(Tx - T^2x) + \frac{\alpha-1}{\mu}\rho(Tx - Ty) + \frac{\beta}{\mu}\mu\bigl[\rho(T^2x - Tx) + \rho(Tx - y)\bigr] \\
&\quad + (1-\beta)\rho(Tx - y)\biggr] \\
&\leqslant \mu\biggl[(\alpha+\beta)\rho\bigl(Tx - T^2x\bigr) + \frac{\alpha-1}{\mu}\rho(Tx - Ty) + \rho(Tx - y)\biggr] \\
&\leqslant \mu\biggl[(\alpha+\beta)\rho\bigl(Tx - T^2x\bigr) + \frac{\alpha-1}{\mu}\rho(Tx - Ty) + \mu\bigl[\rho(Tx - x) + \rho(x - y)\bigr]\biggr] \\
&\leqslant \mu\biggl[(\alpha+\beta+\mu)\rho\bigl(Tx - T^2x\bigr) + \frac{\alpha-1}{\mu}\rho(Tx - Ty) + \mu\rho(x - y)\biggr],
\end{aligned}
$$

that is,

$$\rho(Tx - Ty) \leqslant \mu(\alpha+\beta+\mu)\rho\bigl(Tx - T^2x\bigr) + \mu^2\rho(x - y). \tag{2}$$

From the initial assumption of Case 2 we have $\rho(x - y) < \rho(x - Tx)/(2\mu)$. Also, from Lemma 8(i) we have $\rho(Tx - T^2x) \leqslant \rho(x - Tx)$. Using these inequalities in relation (2), this leads to

$$\rho(Tx - Ty) \leqslant \mu\biggl(\frac{1}{2} + \alpha + \beta + \mu\biggr)\rho(x - Tx). \tag{3}$$

Denoting $\lambda = \mu(1/2 + \alpha + \beta + \mu)$ and comparing inequalities (1) and (3), we obtain

$$\rho(Tx - Ty) \leqslant \lambda\rho(x - Tx) + \rho(x - y) \quad \forall x, y \in S.$$

By considering property (b) from Definition 1 we also obtain

$$\rho(Tx - Ty) \leqslant \lambda\rho(y - Ty) + \rho(x - y) \quad \forall x, y \in S$$

ultimately leading to the conclusion. □

We previously had the opportunity to test the fact that, on what concerns Banach spaces, condition (E) is equivalent to condition (CDE). In the following, we provide an example to prove that this equivalence does not hold anymore for arbitrary modular spaces.

Example 1. On $X = \mathbb{R}$ consider the modular

$$\rho : X \to [0, \infty], \quad \rho(x) = |x|\bigl(|x| + 1\bigr)$$

and the mapping

$$T : [0, \infty) \to [0, \infty), \quad Tx = \frac{x^2}{x+1}.$$

It can be easily checked that ρ is a convex modular with Δ_2-property and modular factor $\mu = 2$. We prove next that T satisfies condition (ρCDE). Indeed, we find

$$\rho(Tx - Ty) = \rho\biggl(\frac{x^2}{x+1} - \frac{y^2}{y+1}\biggr) = \rho\biggl(\frac{xy + x + y}{xy + x + y + 1}(x - y)\biggr)$$

$$= \frac{xy+x+y}{xy+x+y+1}|x-y|\left(\frac{xy+x+y}{xy+x+y+1}|x-y|+1\right)$$
$$\leqslant |x-y|\bigl(|x-y|+1\bigr).$$

On the other hand,

$$\rho(x-y) = |x-y|\bigl(|x-y|+1\bigr).$$

Therefore,

$$\rho(Tx-Ty) \leqslant \rho(x-y) \quad \forall x,y \in X,$$

hence, T is modular nonexpansive, therefore, is also modular Suzuki and ultimately satisfies condition (ρCDE).

Let us prove now that T does not satisfy condition (ρE). Assume the contrary, and let $\lambda \geqslant 1$ be such that

$$\rho(x-Ty) \leqslant \lambda\rho(x-Tx) + \rho(x-y) \quad \forall x,y \in [0,\infty).$$

In particular, for $y=1$ and $x \geqslant 1$, the above inequality comes to

$$\rho\left(x-\frac{1}{2}\right) \leqslant \lambda\rho(x-Tx) + \rho(x-1),$$

that is,

$$\lambda \geqslant \frac{(x-\frac{1}{2})(x+\frac{1}{2})-(x-1)x}{\frac{x(2x+1)}{(x+1)^2}} = \frac{(x-\frac{1}{4})(x+1)^2}{x(2x+1)}.$$

Taking the limit $x \to \infty$, we find $\lambda = \infty$, which is not acceptable.

In the following we present our main result regarding condition (ρCDE).

Theorem 1. *Let X_ρ be a ρ-complete modular vector space. Assume that ρ takes only finite values, satisfies the Δ_2-condition, the Fatou and* (UUC1) *properties. Let S be a nonempty, ρ-closed and convex subset of X_ρ, and let $T: S \to S$ be a mapping satisfying condition* (ρCDE)*. Consider the sequence $\{x_n\}$ defined by the Mann iterative process $x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n$, $x_0 \in S$, for $\{\alpha_n\}$ a real sequence convergent to α^* such that $0 < a \leqslant \alpha_n \leqslant b < 1$. Then $\operatorname{Fix}(T) \neq \emptyset$ if and only if $\{x_n\}$ has finite asymptotic radius relative to S and $\lim_{n\to\infty} \rho(Tx_n - x_n) = 0$.*

Proof. We start with the direct implication. Let $p \in \operatorname{Fix}(T)$. Applying the definition regarding condition (ρCDE), one obtains

$$\rho(p-Tx) \leqslant \lambda\min\bigl\{0,\, \rho(x-Tx)\bigr\} + \rho(p-x) = \rho(p-x) \quad \forall x \in S. \tag{4}$$

Using the convexity of the modular and inequality (4), one finds

$$\begin{aligned}\rho(x_{n+1}-p) &= \rho\bigl(\alpha_n(x_n-p) + (1-\alpha_n)(Tx_n-p)\bigr)\\ &\leqslant \alpha_n\rho(x_n-p) + (1-\alpha_n)\rho(Tx_n-p)\\ &\leqslant \alpha_n\rho(x_n-p) + (1-\alpha_n)\rho(x_n-p)\\ &= \rho(x_n-p).\end{aligned}$$

It follows that $\{\rho(x_n - p)\}$ is a decreasing nonnegative sequence in $\mathbb{R}_+$. Therefore, $\{\rho(x_n - p)\}$ is convergent to a nonnegative real number. Let

$$r = \lim_{n\to\infty} \rho(x_n - p). \tag{5}$$

One immediate consequence of this is that $\tau(S) \leqslant r < \infty$, where τ denotes the ρ-type function corresponding to $\{x_n\}$; therefore, $\{x_n\}$ has finite asymptotic radius.

By denoting $y_n = Tx_n$ one finds, according to inequality (4), $\rho(y_n - p) = \rho(Tx_n - p) \leqslant \rho(x_n - p)$. Therefore,

$$\limsup_{n\to\infty} \rho(y_n - p) \leqslant r. \tag{6}$$

In addition,

$$\lim_{n\to\infty} \rho\bigl(\alpha_n(x_n - p) + (1-\alpha_n)(y_n - p)\bigr) = \lim_{n\to\infty} \rho\bigl(x_{n+1} - p\bigr) = r. \tag{7}$$

Using inequalities (5), (6) and (7) and the fact that ρ satisfies the (UUC1) property, it follows, according to Lemma 1, that $\lim_{n\to\infty} \rho(x_n - y_n) = \lim_{n\to\infty} \rho(x_n - Tx_n) = 0$, which ends this part of the proof.

In the following, we test the converse statement. Let $\tau, \bar\tau : S \to [0,\infty]$ denote the ρ-type functions corresponding to sequences $\{x_n\}$ and $\{y_n = Tx_n\}$, respectively. We shall prove first that, for each $p \in S$, $\bar\tau(Tp) \leqslant \tau(p)$. Indeed, for each $n \in \mathbb{N}$,

$$\begin{aligned}\rho(y_n - Tp) &= \rho(Tx_n - Tp)\\ &\leqslant \lambda \min\bigl\{\rho(x_n - Tx_n), \rho(p - Tp)\bigr\} + \rho(x_n, p).\end{aligned}$$

Taking $\limsup_{n\to\infty}$ on the above inequality and using Lemma 3(i), one finds

$$\bar\tau(Tp) \leqslant \lambda \min\bigl\{0,\ \rho(p - Tp)\bigr\} + \tau(p) = \tau(p). \tag{8}$$

Also,

$$\begin{aligned}\rho(x_{n+1} - p) &= \rho\bigl(\alpha_n(x_n - p) + (1-\alpha_n)(p - y_n)\bigr)\\ &\leqslant \alpha_n\rho(x_n - p) + (1-\alpha_n)\rho(p - y_n).\end{aligned}$$

Again, from Lemma 3(ii) it follows $\tau(p) \leqslant \alpha^*\tau(p) + (1-\alpha^*)\bar\tau(p)$, where $\alpha^* \in (0,1)$ is the limit of the sequence $\{\alpha_n\}$, thus

$$\tau(p) \leqslant \bar\tau(p). \tag{9}$$

Combining relations (8) and(9), one finds

$$\tau(Tp) \leqslant \bar\tau(Tp) \leqslant \tau(p) \leqslant \bar\tau(p) \quad \forall p \in S.$$

Let $\{c_n\}$ be a minimizing sequence of τ. Then $\lim_{n\to\infty} \tau(c_n) = r(S)$. Since, as we just have proved, $\tau(Tc_n) \leqslant \tau(c_n)$, it follows that $\{Tc_n\}$ is also a minimizing sequence of τ. According to Lemma 2, all the minimizing sequences are ρ-convergent to the same limit c, i.e.

$$\lim_{n\to\infty} \rho(c_n - c) = \lim_{n\to\infty} \rho(Tc_n - c). \tag{10}$$

On the other hand, using again the definition of condition (ρCDE), we find that

$$\begin{aligned}\rho(Tc_n - Tc) &\leqslant \min\bigl\{\rho(c_n - Tc_n), \rho(c - Tc)\bigr\} + \rho(c_n - c)\\ &\leqslant \min\bigl\{\mu\rho(c_n - c) + \mu\rho(c - Tc_n), \rho(c - Tc)\bigr\}\\ &\quad + \rho(c_n - c),\end{aligned}$$

which, by taking n to infinity, leads to the conclusion $\lim_{n\to\infty}\rho(Tc_n - Tc) = 0$ meaning that $\{Tc_n\}$ is ρ-convergent to Tc. On the other hand, equation (10) states that it is also convergent to c, and, since the ρ-limit is unique, it follows that $Tc = c$. □

5 Extension to Ishikawa iterative processes

In the following, let us consider an Ishikawa two-step iterative process

$$\begin{aligned}&x_0 \in X_\rho;\\ &y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad 0 < a \leqslant \alpha_n \leqslant b < 1;\\ &x_{n+1} = \beta_n x_n + (1 - \beta_n)Ty_n, \quad 0 \leqslant \beta_n \leqslant c < 1, \quad \forall n \geqslant 0.\end{aligned} \tag{11}$$

The aim of this section is to phrase a new necessary and sufficient condition for the existence of fixed points of a given mapping T satisfying condition (ρCDE) via the iterative procedure listed above. In order to reach this outcome, a stronger definition of modular uniform convexity is required as follows.

Definition 15. Let $D_1(r,\epsilon)$ and $\delta_1(r,\epsilon)$ be as in Definition 3. The convex modular ρ is said to satisfy property (UUC1$'$) if for every $s \geqslant 0$ and $\epsilon > 0$, there exists $\eta_1(s,\epsilon) > 1/2$ such that $\delta_1(r,\epsilon) > \eta_1(s,\epsilon) > 1/2$ for $r > s$.

Obviously, if ρ satisfies property (UUC1$'$), it also satisfies condition (UUC1).

The next Lemma will provide an important tool for the proof of the main outcome.

Lemma 9. *Let X_ρ be a ρ-complete modular space. Let S be a nonempty ρ-closed convex subset of X_ρ, and $\{x_n\}$, $\{y_n\}$ be two sequences in X_ρ. Assume that τ and $\bar\tau$ denote their ρ-type functions and $\tau_0 < \infty$ and $\bar\tau_0 < \infty$, respectively, are their finite asymptotic radii. For $a, b > 0$, define the mixed ρ-type function*

$$\varphi : S \to [0,\infty], \quad \varphi(x) = a\tau(x) + b\bar\tau(x).$$

If ρ satisfies property (UUC1$'$)*, then all the minimizing sequences of φ are ρ-convergent to the same limit.*

Proof. Let us denote $\varphi_0 = \inf_{x\in S}\varphi(x)$, and let $\{c_n\}$ be a minimizing sequence for φ. Assume that $\{c_n\}$ is not ρ-Cauchy. Then there exists two subsequences $\{c_{k_n}\}$ and $\{c_{m_n}\}$, $k_n \neq m_n \geqslant n$, and $\epsilon_0 > 0$ such that $\rho(c_{k_n} - c_{m_n}) \geqslant \epsilon_0$ for all n. Let us analyze next the first subsequence. Since $\{c_n\}$ is minimizing, one also has

$$\lim_{n\to\infty}\varphi(c_{k_n}) = a\lim_{n\to\infty}\tau(c_{k_n}) + b\lim_{n\to\infty}\bar\tau(c_{k_n}) = \varphi_0.$$

Fix $\epsilon \in (0,1)$. Then there exists n_0 such that, for all $n \geqslant n_0$,

$$a\tau(c_{k_n}) + b\bar{\tau}(c_{k_n}) < \varphi_0 + \epsilon.$$

This leads to

$$\bar{\tau}(c_{k_n}) < \frac{1}{b}(\varphi_0 - a\tau_0 + \epsilon), \tag{12}$$

$$\tau(c_{k_n}) < \frac{1}{a}(\varphi_0 - b\bar{\tau}_0 + \epsilon). \tag{13}$$

We focus next on inequality (12) meaning

$$\limsup_{l\to\infty} \rho(c_{k_n} - y_l) < \frac{1}{b}(\varphi_0 - a\tau_0 + \epsilon).$$

It follows that

$$\rho(c_{k_n} - y_l) < \frac{1}{b}(\varphi_0 - a\tau_0 + 2\epsilon) \quad \forall l \geqslant l_0,\ \forall n \geqslant n_0.$$

A similar inequality could be obtained for the subsequence $\{c_{m_n}\}$ too. Since also

$$\begin{aligned}&\rho\big((c_{k_n} - y_l) - (c_{m_n} - y_l)\big)\\ &\quad \geqslant \epsilon_0 \geqslant \frac{1}{b}(\varphi_0 - a\tau_0 + 2\epsilon)\frac{b\epsilon_0}{\varphi_0 - a\tau_0 + 2} \quad \forall n \neq m \geqslant n_0,\end{aligned}$$

the uniform convexity condition (UUC1$'$) leads to the conclusion that there exists $\eta_1((\varphi_0 - a\tau_0)/b, b\epsilon_0/(\varphi_0 - a\tau_0 + 2)) > 1/2$ such that

$$\begin{aligned}&\rho\left(\frac{c_{k_n} - c_{m_n}}{2} - y_l\right)\\ &\quad \leqslant \frac{1}{b}(\varphi_0 - a\tau_0 + \epsilon)\left[1 - \eta_1\left(\frac{1}{b}(\varphi_0 - a\tau_0), \frac{b\epsilon_0}{\varphi_0 - a\tau_0 + 2}\right)\right].\end{aligned}$$

Letting $\epsilon \to 0$, taking afterwards $\limsup_{l\to\infty}$ and using the fact that $\eta_1 > 1/2$, we find

$$\begin{aligned}b\bar{\tau}\left(\frac{c_{k_n} - c_{m_n}}{2}\right) &\leqslant (\varphi_0 - a\tau_0)\left[1 - \eta_1\left(\frac{1}{b}(\varphi_0 - a\tau_0), \frac{b\epsilon_0}{\varphi_0 - a\tau_0 + 2}\right)\right]\\ &< \frac{1}{2}(\varphi_0 - a\tau_0).\end{aligned} \tag{14}$$

Similar arguments for inequality (13) lead to

$$a\tau\left(\frac{c_{k_n} - c_{m_n}}{2}\right) < \frac{1}{2}(\varphi_0 - b\bar{\tau}_0). \tag{15}$$

By adding inequalities (14) and (15), we find

$$\varphi\left(\frac{c_{k_n} - c_{m_n}}{2}\right) < \varphi_0 - \frac{1}{2}(a\tau_+ b\bar{\tau}_0)$$

leading to

$$\varphi_0 < \varphi_0 - \frac{1}{2}(a\tau_+ b\bar{\tau}_0),$$

which is impossible. Therefore, $\{c_n\}$ is a Cauchy sequence, and, since S is a nonempty ρ-closed convex subset of a ρ-complete modular space, it follows that $\{c_n\}$ is convergent to a point $c \in S$. To finish the proof, we need to test that the ρ-limit is independent of the minimizing sequence. Indeed, if $\{c_n\}$ and $\{\bar{c}_n\}$ are two minimizing sequences of φ, so it is the sequence $\{\tilde{c}_n\}$, where $\tilde{c}_{2n} = c_n$ and $\tilde{c}_{2n+1} = \bar{c}_n$. Hence, its modular limit is common for both $\{c_n\}$ and $\{\bar{c}_n\}$. $\square$

We state next the main outcome of the section.

Theorem 2. *Let X_ρ be a ρ-complete modular vector space. Assume that ρ takes only finite values, satisfies the Δ_2-condition and* (UUC1′)*-property. Let S be a nonempty, ρ-closed and convex subset of X_ρ, and let $T : S \to S$ be mapping satisfying condition* (ρCDE)*. Consider the sequence $\{x_n\}$ defined by the Ishikawa iterative process* (11) *with $\{\alpha_n\}$ convergent to $\alpha^* \in (0,1)$ and $\{\beta_n\}$ convergent to $\beta^* \in [0,1)$. Then* $\operatorname{Fix}(T) \neq \emptyset$ *if and only if $\{x_n\}$ and $\{y_n\}$ have finite asymptotic radiuses relative to S and* $\lim_{n\to\infty} \rho(Tx_n - x_n) = 0$.

Proof. Assume that $p \in \operatorname{Fix}(T)$. Inequality (4) holds true, which means that T is modular quasinonexpansive. The following relations derive from the convexity of ρ:

$$\begin{aligned}
\rho(y_n - p) &= \rho\bigl(\bigl(\alpha_n x_n + (1-\alpha_n)Tx_n\bigr) - p\bigr)\\
&= \rho\bigl(\alpha_n(x_n - p) + (1-\alpha_n)(Tx_n - p)\bigr)\\
&\leqslant \alpha_n\rho(x_n - p) + (1-\alpha_n)\rho(Tx_n - p)\\
&\leqslant \rho(x_n - p),
\end{aligned}$$

and also

$$\begin{aligned}
\rho(x_{n+1} - p) &= \rho\bigl(\bigl(\beta_n x_n + (1-\beta_n)Ty_n\bigr) - p\bigr)\\
&= \rho\bigl(\beta_n(x_n - p) + (1-\alpha_n)(Ty_n - p)\bigr)\\
&\leqslant \beta_n\rho(x_n - p) + (1-\beta_n)\rho(y_n - p)\\
&\leqslant \rho(x_n - p).
\end{aligned} \tag{16}$$

We have obtained again that $\{\rho(x_n - p)\}$ is a decreasing nonnegative sequence in $\mathbb{R}_+$ and, consequently, it is convergent. Let $r \geqslant 0$ denote its ρ-limit. Using again inequality (4), it follows $\rho(Tx_n - p) \leqslant \rho(x_n - p)$. Therefore,

$$\limsup_{n\to\infty} \rho(Tx_n - p) \leqslant r.$$

From inequality (16) one also finds that

$$\frac{\rho(x_{n+1}) - \beta_n\rho(x_n - p)}{1-\beta_n} \leqslant \rho(y_n - p) \leqslant \rho(x_n - p).$$

Letting $n \to \infty$, one obtains

$$\limsup_{n\to\infty} \rho(y_n - p) = r$$

meaning

$$\lim_{n\to\infty} \rho\big(\alpha_n(x_n - p) + (1-\alpha_n)(Tx_n - p)\big) = r.$$

Using again Lemma 1, we may conclude that $\lim_{n\to\infty} \rho(x_n - Tx_n) = 0$, which ends this part of the proof.

In the following, we test the converse statement. Let $\tau, \bar{\tau} : S \to [0,\infty]$ denote the ρ-type functions corresponding to sequences $\{x_n\}$ and $\{y_n\}$, and let φ denote the mixed ρ-type function

$$\varphi : S \to [0,\infty], \quad \varphi(x) = (1-\alpha^*)\tau(x) + \bar{\tau}(x) \quad \forall x \in S.$$

We know that $\lim_{n\to\infty} \rho(x_n - Tx_n) = 0$. Then

$$\begin{aligned}
\rho(y_n - x_n) &\leqslant (1-\alpha_n)\rho(Tx_n - x_n) \to 0;\\
\rho(y_n - Tx_n) &\leqslant \alpha_n \rho(x_n - Tx_n) \to 0;\\
\rho(Ty_n - Tx_n) &\leqslant \lambda \min\big\{\rho(x_n - Tx_n), \rho(y_n - Ty_n)\big\}\\
&\quad + \rho(x_n - y_n) \to 0;\\
\rho(y_n - Ty_n) &\leqslant \mu\big[\rho(y_n - Tx_n) + \rho(Ty_n - Tx_n)\big] \to 0.
\end{aligned}$$

Let $p \in S$ be an arbitrary point. Using the iterative formulas for $\{y_n\}$ and $\{x_{n+1}\}$ from (11), we find

$$\begin{aligned}
\rho(y_n - Tp) &\leqslant \alpha_n \rho(x_n - Tp) + (1-\alpha_n)\rho(Tx_n - Tp)\\
&\leqslant \alpha_n \rho(x_n - Tp)\\
&\quad + (1-\alpha_n)\big[\lambda \min\big\{\rho(x_n - Tx_n), \rho(p - Tp)\big\} + \rho(x_n - p)\big],\\
\rho(x_{n+1} - Tp) &\leqslant \beta_n \rho(x_n - Tp) + (1-\beta_n)\rho(Ty_n - Tp)\\
&\leqslant \beta_n \rho(x_n - Tp)\\
&\quad + (1-\beta_n)\big[\lambda \min\big\{\rho(y_n - Ty_n), \rho(p - Tp)\big\} + \rho(y_n - p)\big].
\end{aligned}$$

Letting $n \to \infty$, the above inequalities lead to

$$\begin{aligned}
&\bar{\tau}(Tp) \leqslant \alpha^* \tau(Tp) + (1-\alpha^*)\tau(p);\\
&\tau(Tp) \leqslant \beta^* \tau(Tp) + (1-\beta^*)\bar{\tau}(p) \quad \iff \quad \tau(Tp) \leqslant \bar{\tau}(p).
\end{aligned}$$

Combining these two, we find

$$\bar{\tau}(Tp) + (1-\alpha^*)\tau(Tp) \leqslant \bar{\tau}(p) + (1-\alpha^*)\tau(p),$$

that is,

$$\varphi(Tp) \leqslant \varphi(p) \quad \forall p \in S.$$

Let $\{c_n\}$ be a minimizing sequence of φ. Then $\lim_{n\to\infty} \varphi(c_n) = \varphi_0$. Since, as we just have proved, $\varphi(Tc_n) \leqslant \varphi(c_n)$, it follows that $\{Tc_n\}$ is also a minimizing sequence of φ. According to Lemma 9, all the minimizing sequences are ρ-convergent to the same limit c, i.e.

$$\lim_{n\to\infty} \rho(c_n - c) = \lim_{n\to\infty} \rho(Tc_n - c) = 0.$$

On the other hand, using again condition (ρCDE), we find that

$$\begin{aligned}\rho(Tc_n - Tc) &\leqslant \lambda \min\bigl\{\rho(c_n - Tc_n), \rho(c - Tc)\bigr\} + \rho(c_n - c) \\ &\leqslant \lambda \min\bigl\{\mu\rho(c_n - c) + \mu\rho(c - Tc_n), \rho(c - Tc)\bigr\} \\ &\quad + \rho(c_n - c),\end{aligned}$$

which, by taking n to infinity, leads to the conclusion $\lim_{n\to\infty} \rho(Tc_n - Tc) = 0$ meaning that $\{Tc_n\}$ is ρ-convergent to Tc. Since the ρ-limit is unique, it follows that $Tc = c$. $\square$

6 Conclusions

The elements presented above show that the search for wider classes of nonexpansive mappings is far away from reaching to an end; we were able to define a new nonexpansiveness property, namely, the (CDE) condition, which, as it concerns Banach spaces, proves to be equivalent to condition (E), while, when going to modular vector spaces, provides a distinct class of mappings. Moreover, compared with modular condition (ρE), this new modular condition has the advantage to include the modular Suzuki mappings.

On the other hand, our approach proves that the extension of newly introduced nonexpansiveness conditions to modular vector spaces leads often to similar outcomes as on Banach spaces, despite the fact that modulars lack some properties when compared with norms. Nevertheless, the limits of such extensions are also emphasized. It seems that not all the properties from Banach spaces can be so smoothly transposed to modular setting; for instance, the equivalence between classes of mappings satisfying conditions (CDE) and (E) on Banach spaces is lost when turning to modular framework.

Moreover, the content of the last section proves that the choice of the Mann iterative process to study the fixed points for mappings on modular vector spaces is not arbitrary. When taking more elaborated iteration procedures (Ishikawa, for instance), some stronger uniform convexity related requirements are necessary.

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