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Global dynamics for a class of reaction–diffusion multigroup SIR epidemic models with time fractional-order derivatives*

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Abstract. This paper investigates the global dynamics for a class of multigroup SIR epidemic model with time fractional-order derivatives and reaction–diffusion. The fractional order considered in this paper is in $(0, 1]$, which the propagation speed of this process is slower than Brownian motion leading to anomalous subdiffusion. Furthermore, the generalized incidence function is considered so that the data itself can flexibly determine the functional form of incidence rates in practice. Firstly, the existence, nonnegativity, and ultimate boundedness of the solution for the proposed system are studied. Moreover, the basic reproduction number R_0 is calculated and shown as a threshold: the disease-free equilibrium point of the proposed system is globally asymptotically stable when $R_0 \leqslant 1$, while when $R_0 > 1$, the proposed system is uniformly persistent, and the endemic equilibrium point is globally asymptotically stable. Finally, the theoretical results are verified by numerical simulation.

Keywords: SIR epidemic model, multigroup, reaction–diffusion, fractional order, asymptotic stability.

1 Introduction

As we all know, mathematical models play an important role in researching the dynamical behavior of infectious diseases. In the classical epidemic model, it is generally considered that individuals are completely mixed, and everyone has the same possibility of infection. However, due to the differences in age, geographical distribution, and other factors, it is more realistic to divide the total population into several different populations, that is,

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to establish a multigroup epidemic model. Lajmanovich et al. first proposed the SIS multigroup systems and researched the stability of the endemic equilibrium point [11]. Subsequently, there are many research efforts devoted to investigating the importance of multigroup epidemic models [6, 8, 14]. Guo et al. were the first to successfully establish the complete global dynamics of the multigroup epidemic model based on the basic reproduction number [7]. Boosted by the work of Guo et al., many researchers discussed the stability of various multigroup systems [3, 15, 20, 21, 25].

Meanwhile, individual diffusion behavior is widespread in the actual propagation of infectious diseases. With the development of global transportation, individuals in incubation period can easily travel from one place to another, which is thought to be one of the main reasons of the global pandemic of infectious diseases. For instance, SARS first appeared in China's Guangdong Province in November 2002 and then quickly spread to other parts of China and even the world [26]. Also, COVID-19 was first detected at the end of December 2019 with successive cases occurring worldwide. Therefore, in order to better understand the impact of population mobility on the spread of infectious diseases, it is necessary to incorporate human movement into epidemic model to provide more theoretical guidance for epidemic control. Li et al. analyzed the stability and the uniform persistence of a SIRS epidemic model with diffusion [12]. Xu et al. studied the stability and the existence of traveling wave solutions of a SIS epidemic model with diffusion [28]. Recently, many diffusive epidemic models have been used to model within-group and inter-group interactions in spatially environments, for example, Wu et al. investigated a multigroup epidemic model with nonlocal diffusion and obtained the asymptotic behavior of traveling wave fronts [27].

It is worth noting in real life that the spread of infectious diseases not only depends on its current state, but also on its past state. Actually, it can be achieved that current state of fractional-order epidemic models depends on the past information since any fractional derivative contains a kernel function [30]. Furthermore, Smethurst et al. found that the patient waits for the doctor's time to follow a power law model $P[J_n > t] = Bt^{-\alpha}$ [24]. More importantly, Angstmann et al. proposed a infectivity SIR model with fractional-order derivative, and they showed how fractional-order derivative arise naturally by continuous time random walk [2]. As generalized of classical integers ones, Hethcote firstly proposed a fractional-order SIR model with a constant population [8]. Then Almeida et al. considered the local stability of two equilibrium points of a fractional SEIR epidemic model [1].

Typically, the reaction term describes a birth-death reaction occurring in a habitat or reactor. The diffusion term simulates the movement of the individual in the environment in real-world applications. The diffusion is often described by a power law $\langle x^2(t)\rangle - \langle x(t)\rangle^2 \sim Dt^{\alpha}$, where D is the diffusion coefficient, and t is the elapsed time. In normal diffusion, the order $\alpha = 1$. But if $\alpha > 1$, particle undergoes superdiffusion, which mainly describes the process of active cell transport; if $\alpha < 1$, this phenomenon is called subdiffusion, which can be the diffusion of proteins within cells or the diffusion of viruses between individuals [29]. And it results in a Caputo time-fractional reaction–diffusion system with fractional order $\alpha < 1$. Meanwhile, it is pointed out in [19] that long waiting times model particle sticking, and the density of this process spreads slower than normal diffusion. Also, as shown in [19], Caputo time-fractional reaction–diffusion curve has a sharper peak and heavier tails, which can be used to describe the ability to control the transmission of the disease when only a small number of people are infected, such as COVID-19. The study of subdiffusion system has attracted widespread interest in recent years. Mahmoud et al. studied the Cauchy problem of the fractional-order evolution equation and obtained the expression of the solution of the time fractional-order reaction diffusion system [18]. The subdiffusive predator–prey system is discussed, and the analytical solution of the system is studied in [29]. However, few works have been devoted to studying the subdiffusion epidemic model. Motivated by this, in this work, we focus on time-fractional reaction–diffusion epidemic system, which means the spread of infectious diseases is slower than a Brown motion.

Based on the above discussion, the dynamics of the multigroup SIR epidemic model with generally incidence rates is investigated in this paper. Particularly, the susceptible individuals, infective individuals, and recovered individuals are assumed to follow Fickian diffusion.

The organization of this paper is as follows. A class of diffusive SIR epidemic model with time fractional-order derivatives is formulated and some preliminaries are introduced in Section 2. In Section 3, global dynamics of the proposed model are studied, and numerical simulations are presented to illustrate theoretical results in Section 4. Finally, a brief discussion is given in Section 5.

2 Model development

Before presenting a class of multigroup reaction–diffusion SIR epidemic model with time fractional-order derivatives, some necessary preliminaries are presented.

2.1 Preliminaries

This section begins with some notations, definitions, and results.

Notation. Let $\mathbb{Y} = C(\overline{\Omega}, \mathbb{R}^m)$ be a continuous function; $\mathbb{Y}_+ = C(\overline{\Omega}, \mathbb{R}^m_+)$ be the positive cone of $\mathbb{Y}$; $\mathbb{X} = \mathbb{Y} \times \mathbb{Y}$ with the norm $\|\phi\|_{\mathbb{X}} = \max\{\|\phi_1\|_{\mathbb{Y}}, \|\phi_2\|_{\mathbb{Y}}\}$, where $\phi = (\phi_1, \phi_2)$ and $\phi_i \in \mathbb{Y}$ ($i = 1, 2$); $\mathbb{X}^0$ be a open set of $\mathbb{X}$ such that $\mathbb{X} = \mathbb{X}^0 \cup \partial\mathbb{X}$, where $\partial\mathbb{X}$ is the boundary of $\mathbb{X}$; $\mathbb{X}_+ = \mathbb{Y}_+ \times \mathbb{Y}_+$ be the positive cone of $\mathbb{X}$.

Definition 1. (See [22].) Caputo fractional derivative of order α ($\alpha \notin \mathbb{N}_0$) for a function $f \in \mathbb{AC}^n([0, \infty], \mathbb{R})$ is defined by

$$ {}_0^C D_t^\alpha f(t) = \left(I_{0+}^{n-\alpha} D^n f\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} \, \mathrm{d}\tau, $$

where $\Re(\alpha) \geqslant 0$ and $n = [\Re(\alpha)] + 1$.

Lemma 1. *Let $a_i(x) \in L^2(\Omega)$ and $F_i(t,x) \in L^\infty((0,T];L^2(\Omega))$ be nonnegative and $y_i(x,t)$ $(i=1,2)$ be the solution to the following system, respectively*:

$$\begin{aligned}
&{}_0^C D_t^\alpha y_i(x,t) = d\Delta y_i + F_i(t,x), \quad x \in \Omega,\ t \in (0,T],\\
&\frac{\partial y_i}{\partial \upsilon} = 0, \quad x \in \partial\Omega,\ t \in (0,T],\\
&y_i(0,x) = a_i(x), \quad x \in \bar{\Omega},\ i = 1,2.
\end{aligned}$$

If $a_1(x) \geqslant a_2(x)$ and $F_1(x,t) \geqslant F_2(x,t)$, then $y_1(x,t) \geqslant y_2(x,t)$ for $\Omega \times (0,T]$.

Proof. Let $y(x,t) = y_1(x,t) - y_2(x,t)$, then $y(x,t)$ satisfies the following system:

$$\begin{aligned}
&{}_0^C D_t^\alpha y(x,t) = d\Delta y + F_1(t,x) - F_2(t,x), \quad x \in \Omega,\ t \in (0,T],\\
&\frac{\partial y}{\partial \upsilon} = 0, \qquad x \in \partial\Omega,\ t \in (0,T],\\
&y(0,x) = a_1(x) - a_2(x), \quad x \in \bar{\Omega}.
\end{aligned}$$

Based on $a_1(x) - a_2(x) \geqslant 0$ and $F_1(x,t) - F_2(x,t) \geqslant 0$, we have $y(x,t) \geqslant 0$. Therefore, it can be deduced that $y_1(x,t) \geqslant y_2(x,t)$. □

Lemma 2. (*See* [29].) *Consider the following system*:

$$\begin{aligned}
&{}_0^C D_t^\alpha y_i(x,t) = d_i\Delta y_i + f_i(t,x,y_i), \quad x \in \Omega,\ i = 1,2,\dots,n,\\
&\frac{\partial y_i}{\partial \upsilon} = 0, \quad x \in \partial\Omega,\ i = 1,2,\dots,n,\\
&y_i(0,x) = a_i(x), \quad x \in \Omega,\ i = 1,2,\dots,n.
\end{aligned} \tag{1}$$

Suppose f_i is mixed quasimonotonous and satisfies the local Lipschitz condition

$$\bigl|f_i(u_1,u_2,u_3) - f_i(v_1,v_2,v_3)\bigr| \leqslant L\bigl(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|\bigr),$$

where L is constant, and $|u_i - v_i| < \varepsilon_0$, where ε_0 is a given constant. If the upper solution $U(x,t)$ and the lower solutions $V(x,t)$ satisfy $V(x,t) < U(x,t)$, system (1) *has a unique solution in $[V(x,t),U(x,t)]$.*

Lemma 3. *The system with time fractional-order derivatives*

$$\begin{aligned}
&{}_0^C D_t^\alpha u(x,t) = d\Delta u + b - \mu u, \quad x \in \Omega,\ t \in (0,T],\\
&\frac{\partial u}{\partial \upsilon} = 0, \quad x \in \partial\Omega,\ t \in (0,T],\\
&u(0,x) = a(x), \quad x \in \bar{\Omega},
\end{aligned} \tag{2}$$

has a unique global asymptotic stability of constant equilibrium $u^* = b/\mu$.

Proof. Define the Lyapunov function

$$V(u) = \frac{1}{2}\int_{\Omega} (u - u^*)^2 \,\mathrm{d}x.$$

Calculating the fractional derivative of $V(u)$ along the trajectories of system (2), one has

$${}_0^C D_t^{\alpha} V \leqslant \int_{\Omega} (u - u^*) {}_0^C D_t^{\alpha} u \,\mathrm{d}x = -\biggl(\int_{\Omega} \mu(u - u^*)^2 \,\mathrm{d}x + \int_{\Omega} \|\nabla u\|^2 \,\mathrm{d}x\biggr).$$

Let

$$V_1 = \int_{\Omega} \mu(u - u^*)^2 \,\mathrm{d}x + \int_{\Omega} \|\nabla u\|^2 \,\mathrm{d}x \geqslant 0,$$

and $V_1 = 0$ if and only if $u = u^*$. Then according to [4], there exists a unique global asymptotic stability of constant equilibrium $u^* = b/\mu$ for system (2). □

Lemma 4. *Consider the following system*:

$${}_0^C D_t^{\alpha} w(t) = g\bigl(w(t)\bigr), \qquad w(0) = w_0, \tag{3}$$

where $g(x)$ satisfies the local Lipschitz condition, and $0 < \alpha \leqslant 1$. Then for $t_1 \geqslant 0$, one has $w(t + t_1, w_0) = w(t, w_1)$ with $w_1 = w(t_1, w_0)$.

Proof. According to the definition of Caputo fractional-order derivative, one has

$$\frac{1}{\Gamma(\alpha)} \int_0^s \frac{\dot{w}(\tau)}{(s - \tau)^{\alpha}} \,\mathrm{d}\tau = g\bigl(w(s)\bigr).$$

Let $s = t + t_1$, then

$$\frac{1}{\Gamma(\alpha)} \int_0^{t+t_1} \frac{\dot{w}(\tau)}{(t + t_1 - \tau)^{\alpha}} \,\mathrm{d}\tau = g\bigl(w(t + t_1)\bigr).$$

By calculation, ${}_0^C D_{t+t_1}^{\alpha} w(t + t_1) = g(w(t + t_1))$. Hence, $w(t + t_1, w_0)$ is a solution of system (9). Further, $w(t + t_1, w_0)|_{t=0} = w(t_1, w_0) = w_1$. By the uniqueness of the solution it is deduced that $w(t + t_1, w_0) = w(t, w_1)$. □

2.2 System description

In [10], Korobrinikov et al. studied a multigroup SIR model as follows:

$$\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= \lambda - \mu S - S \sum_{j=1}^{n} \beta_j I_j, \\
\frac{\mathrm{d}I_k}{\mathrm{d}t} &= p_k S \sum_{j=1}^{n} \beta_j I_j - \delta I_k, \\
\frac{\mathrm{d}R_k}{\mathrm{d}t} &= r_k I_k - \mu R_k, \quad k = 1, 2 \ldots, n.
\end{aligned} \tag{4}$$

But individual movement is not be considered in system (4) that is unrealistic, then Wu et al. considered the following SIR epidemic model with diffusion [27]:

$$\begin{aligned}\frac{dS_k}{\mathrm{d}t} &= d_{1k}\Delta S_k + b_k - \mu_{1k}S_k - \sum_{j=1}^{n}\beta_{kj}S_k g_j(I_j),\\ \frac{\mathrm{d}I_k}{\mathrm{d}t} &= d_{2k}\Delta I_k + \sum_{j=1}^{n}\beta_{kj}S_k g_j(I_j) - (\mu_{2k}+r_k)I_k,\\ \frac{\mathrm{d}R_k}{\mathrm{d}t} &= d_{1k}\Delta S_k + r_k I_k - \mu_{3k}R_k, \quad k = 1,2\dots,n.\end{aligned} \tag{5}$$

Based on the previous analysis, since fractional order has the long-term memory, which can describe the spread of infectious diseases more accurate. In addition, it is traditionally assumed that the incidence of disease transmission is bilinear with respect to the number of susceptible individuals and the number of infected individuals. But in reality, it is often difficult to obtain detailed information on the spread of infectious diseases because they may change with the surrounding environment. Therefore, the general incidence rates will be chose in this paper. Motivated by the above work, as an extension of system (5), a class of multigroup SIR epidemic model are investigated as follows:

$$\begin{aligned}&{}_0^C D_t^\alpha S_k = d_{1k}\Delta S_k + b_k - \mu_{1k}S_k - \sum_{j=1}^{n}\beta_{kj}f_k(S_k)g_j(I_j), \quad x\in\Omega,\ t\geqslant 0,\\ &{}_0^C D_t^\alpha I_k = d_{2k}\Delta I_k + \sum_{j=1}^{n}\beta_{kj}f_k(S_k)g_j(I_j) - (\mu_{2k}+r_k)I_k, \quad x\in\Omega,\ t\geqslant 0,\\ &{}_0^C D_t^\alpha R_k = d_{3k}\Delta R_k + r_k I_k - \mu_{3k}R_k, \quad x\in\Omega,\ t\geqslant 0,\\ &\frac{\partial S_k}{\partial \upsilon} = \frac{\partial I_k}{\partial \upsilon} = \frac{\partial R_k}{\partial \upsilon} = 0, \quad x\in\partial\Omega,\ t\geqslant 0,\\ &\big(S_k(0,x), I_k(0,x), R_k(0,x)\big) = \big(\phi_{1k}(x),\phi_{2k}(x),\phi_{3k}(x)\big),\\ &\quad x\in\Omega,\ k=1,2,\dots,n,\end{aligned} \tag{6}$$

where ${}_0^C D_t^\alpha$ implies Caputo fractional-order operator ($0<\alpha\leqslant 1$); $\Delta = \partial^2/\partial x^2$ denotes the Laplace operator; $\partial/\partial\upsilon$ denotes the outward normal derivative on the smooth boundary $\partial\Omega$; $S_k(x,t)$, $I_k(x,t)$, and $R_k(x,t)$ represent the number of the susceptible, infective, and recovered individuals in kth group at time t and spatial location x, respectively; μ_{ik} ($i=1,3$) imply the nature death rates of S_k and R_k in kth, respectively; μ_{2k} denotes the disease-related death rates of I_k in kth; b_k represents the recruitment rate of the total population; r_k implies the recovery rate of the infected individuals in kth group; d_{ik} ($k=1,2,3$) denotes the diffusion rate of S_k, I_k, and R_k in kth group; β_{kj} represents the infection rate of S_k infected by I_j. Furthermore, d_{ik}, b_k, μ_{ik} ($i=1,2,3$), β_{kk} and r_k are positive constants for $k=1,2,\dots,n$, and $\beta_{kj}(k\neq j)$ are nonnegative constants for $k=1,2,\dots,n$.

Before giving the main results, hypothesis in terms of generalized incidence rates $f_k(S_k)$ and $g_k(I_k)$ is made as follows:

(H) (i) $f_k(S_k)$ and $g_k(I_k)$ satisfy the local Lipschitz condition and $f_k(0) = 0$, $g_k(0) = 0$ for $k = 1, 2, \dots, n$;
(ii) $f_k(S_k)$ is strictly monotone increasing on $S_k \in [0, \infty)$ and $g_k(I_k)$ is strictly monotone increasing on $I_k \in [0, \infty)$ for all $k = 1, 2, \dots, n$;
(iii) $g_k(I_k) \leqslant c_k I_k$ for all $I_k \geqslant 0$, where $c_k = \dot{g}_k(0)$;
(iv) $(\beta_{kj})_{1\leqslant k,j\leqslant n}$ is nonnegative and irreducible. Furthermore, $\beta_{kj} > 0$ if and only if $g_j(I_j) > 0$ for $I_j > 0$, and $\beta_{kj} = 0$ if and only if $g_j(I_j) \equiv 0$;
(v) $\mu_{1k} \leqslant \mu_{2k} + r_k$ for all $k = 1, 2, \dots, n$.

Remark 1. Note that under hypothesis (H), many existing models can be regarded as a special form of system (6), such as $f_k(S_k) = c_k S_k$, $f_k(S_k) = c_k S_k/(1 + a_k S_k)$, $g_k(I_k) = c_k I_k$, $g_k(I_k) = c_k I_k/(1 + b_k I_k)$, and other nonlinear incidence rate in [16].

3 Model analysis

Some dynamical behavior of system (6) are investigated in this section. Here it can be found that the susceptible class S_k and the infected class I_k are not effected by the recovered class R_k of system (6). Hence, we will focus our attention on the following reduced system:

$$\begin{aligned}
&{}_0^C D_t^\alpha S_k = d_{1k}\Delta S_k + b_k - \mu_{1k} S_k - \sum_{j=1}^{n} \beta_{kj} f_k(S_k) g_j(I_j), \quad x \in \Omega,\\
&{}_0^C D_t^\alpha I_k = d_{2k}\Delta I_k + \sum_{j=1}^{n} \beta_{kj} f_k(S_k) g_j(I_j) - (\mu_{2k} + r_k) I_k, \quad x \in \Omega,\\
&\frac{\partial S_k}{\partial \upsilon} = \frac{\partial I_k}{\partial \upsilon} = 0, \quad x \in \partial\Omega,\ t > 0,\\
&\big(S_k(0,x), I_k(0,x)\big) = \big(\phi_{1k}(x), \phi_{2k}(x)\big), \quad x \in \overline{\Omega},\ 1 \leqslant k \leqslant n.
\end{aligned} \tag{7}$$

Then some basic properties of system (7) are discussed in following parts.

3.1 Nonnegative and boundedness

It is significant to demonstrate the existence, uniqueness, and boundedness of a nonnegative solutions for system (7) before implementing its stable process. Thus, this subsection moves to the discussion of proprieties mentioned above.

Theorem 1. *Under hypothesis* (H)*, there exists a unique nonnegative solution* $(S(x,t), I(x,t))$ *of system* (7)*, and it is also ultimately bounded for any given initial function* $\phi_k(x) = (\phi_{1k}(x), \phi_{2k}(x)) \in \mathbb{X}_+$ $(k = 1, 2, \dots, n)$*, where* $S(x,t) = (S_1(x,t), S_2(x,t), \dots, S_n(x,t))$*, and* $I(x,t) = (I_1(x,t), I_2(x,t), \dots, I_n(x,t))$*.*

Proof. Consider these two function: $F_{1k}(S_k, I_k) = b_k - \mu_{1k} S_k - \sum_{j=1}^{n} \beta_{kj} f_k(S_k) g_j(I_j)$ and $F_{2k}(S_k, I_k) = \sum_{j=1}^{n} \beta_{kj} f_k(S_k) g_j(I_j) - (\mu_{2k} + r_k) I_k$. According to condition (i) in hypothesis (H), it is obvious that F_{1k} and F_{2k} are mixed quasimonotonous.

Consider the following auxiliary system:

$$\begin{aligned}
{}_0^C D_t^\alpha \underline{S}_k(t) &= -\mu_{1k} \underline{S}_k - \sum_{j=1}^{n} \beta_{kj} f(\underline{S}_k) g_j(\underline{I}_j),\\
{}_0^C D_t^\alpha \underline{I}_k(t) &= \sum_{j=1}^{n} \beta_{kj} f(\underline{S}_k) g_j(\underline{I}_j) - (\mu_{2k} + r_k) \underline{I}_k,\\
\underline{S}_k(0) &= \underline{I}_k(0) = 0.
\end{aligned}$$

It is obvious that $(\underline{S}_k, \underline{I}_k) = (0, 0)$ is a pair of the lower solution to system (7). Then, according to Lemma 1, one has $S_k \geqslant 0$ and $I_k \geqslant 0$.

Furthermore, the following auxiliary system is introduced:

$$ {}_0^C D_t^\alpha \overline{S}_k(t) = b_k - \mu_{1k} \overline{S}_k, \qquad \overline{S}_k(0) = \left\| S_k(x, 0) \right\|_\Omega, \tag{8}$$

then the above system (8) has a solution as follows:

$$\overline{S}_k(t) = \left(\left\| S_k(x, 0) \right\|_\Omega - \frac{b_k}{\mu_{1k}} \right) E_\alpha\left(-\mu_{1k} t^\alpha\right) + \frac{b_k}{\mu_{1k}}.$$

Therefore, $\limsup_{t\to\infty} \overline{S}_k(t) = b_k/\mu_{1k}$, then there exists a constant T_0 satisfied $S_k(x,t) \leqslant b_k/\mu_{1k}$ for $t > T_0$. Further, consider the following auxiliary system:

$$\begin{aligned}
{}_0^C D_t^\alpha \overline{I}_k &= \left[\beta_{kk} c_k f_k(\overline{I}_k) - (\mu_{2k} + r_k)\right] \overline{I}_k + \sum_{j=1,\, j\neq k}^{n} \beta_{kj} f_k(\overline{S}_k) g_j(\overline{I}_k),\\
\overline{I}_k(0) &= \left\| I_k(x, 0) \right\|_\Omega,
\end{aligned} \tag{9}$$

then the solution for the above system (9) is

$$\begin{aligned}
\overline{I}_k = {} & E_\alpha\left[\left(\beta_{kk} c_k f_k(\overline{S}_k) - (\mu_{2k} + r_k)\right) t^\alpha\right]\\
& \times \left[\left\| I_k(x, 0) \right\|_\Omega - \frac{\sum_{j=1,\, j\neq k}^{n} \beta_{kj} f_k(\overline{S}_k) g_j(\overline{I}_k)}{\beta_{kk} c_k f_k(\overline{S}_k) - (\mu_{2k} + r_k)} \right]\\
& + \frac{\sum_{j=1,\, j\neq k}^{n} \beta_{kj} f_k(\overline{S}_k) g_j(\overline{I}_k)}{\beta_{kk} c_k f_k(\overline{S}_k) - (\mu_{2k} + r_k)}.
\end{aligned}$$

Similarly, $I_k(x, t) \leqslant \overline{I}_k(t)$. Since ${}_0^C D_t^\alpha \overline{S}_k(x, t) = b_k - \mu_{1k} \overline{S}_k$, then

$$\begin{aligned}
& {}_0^C D_t^\alpha \overline{S}_k(x, t) - d_{1k} \Delta \overline{S}_k - b_k + \mu_{1k} \overline{S}_k + \sum_{j=1}^{n} \beta_{kj} f_k(\overline{S}_k) g_j(\underline{I}_j)\\
& \quad \geqslant \sum_{j=1}^{n} \beta_{kj} f_k(\overline{S}_k) g_j(\underline{I}_j) \geqslant 0.
\end{aligned} \tag{10}$$

However, it is easy to see that

$$
\begin{aligned}
&{}_0^C D_t^\alpha \underline{S}_k(x,t) - d_{1k}\Delta \underline{S}_k - b_k + \mu_{1k}\underline{S}_k + \sum_{j=1}^{n} \beta_{kj} f_k(\underline{S}_k) g_j(\overline{I}_j) \\
&\quad = -b_k < 0.
\end{aligned} \tag{11}
$$

It can be deduced from Eqs. (10) and (11) that

$$
\begin{aligned}
&{}_0^C D_t^\alpha \overline{S}_k(x,t) - d_{1k}\Delta \overline{S}_k - F_{1k}(\overline{S}_k, \underline{I}_k) \\
&\quad \geqslant 0 >{}_0^C D_t^\alpha \underline{S}_k(x,t) - d_{1k}\Delta \underline{S}_k - F_{1k}(\underline{S}_k, \overline{I}_k).
\end{aligned}
$$

Similar to the above analysis, it can be obtained the following equation:

$$
\begin{aligned}
&{}_0^C D_t^\alpha \overline{I}_k(x,t) - d_{2k}\Delta \overline{I}_k - F_{2k}(\overline{S}_k, \overline{I}_k) \\
&\quad \geqslant 0 >{}_0^C D_t^\alpha \underline{I}_k(x,t) - d_{2k}\Delta \underline{I}_k - F_{2k}(\underline{S}_k, \underline{I}_k).
\end{aligned}
$$

Based on the above analysis and Lemma 2, system (7) has a unique nonnegative global solution. Furthermore, the expression for the solution of system (7) is

$$
\begin{aligned}
S_k(x,t) &= T_\alpha^{(1k)}(t)\phi_{1k} + \int_0^t M_\alpha^{(1k)}(t-s) F_{1k}(S_k, I_k)\,\mathrm{d}s, \\
I_k(x,t) &= T_\alpha^{(2k)}(t)\phi_{2k} + \int_0^t M_\alpha^{(2k)}(t-s) F_{2k}(S_k, I_k)\,\mathrm{d}s,
\end{aligned}
$$

where

$$
\begin{aligned}
T_\alpha^{(ik)}(t) &= \int_0^\infty \zeta_\alpha(\theta) G_{ik}\bigl(t^\alpha \theta\bigr)\,\mathrm{d}\theta, \\
M_\alpha^{(ik)}(t) &= \alpha \int_0^\infty \theta t^{\alpha-1} \zeta_\alpha(\theta) G_{ik}\bigl(t^\alpha \theta\bigr)\,\mathrm{d}\theta, \quad i = 1,2,\ k = 1,2,\dots,n, \\
\zeta_\alpha(\theta) &= \frac{1}{2\pi z} \int_\Gamma \mathrm{e}^{\lambda\theta} E_{\alpha,1}(-\lambda)\,\mathrm{d}\lambda, \quad z \text{ is an imaginary number,}
\end{aligned}
$$

with $\zeta_\alpha(\theta)$ represents a probability density; $G_{1k}(t)_{t\geqslant 0}$ represent generated strong continuous operator semigroups by $d_{1k}\Delta$; $G_{2k}(t)_{t\geqslant 0}$, denoting generated strong continuous operator semigroups by $d_{2k}\Delta - \mu_{ik} - r_k$ ($k = 1,2,\dots,n$), can be rewritten by [23]

$$
G_{2k}(t)\phi_{2k} = \int_\Omega T_{2k}(t,x,z)\phi_{2k}(z)\,\mathrm{d}z,
$$

where T_{2k} is the Green function yielded

$$T_{2k}(t,x,z)=\sum_{j\geqslant 1}\mathrm{e}^{\tau_j^k t}\varphi_j^k(x)\varphi_j^k(z),\quad 1\leqslant k\leqslant n,$$

with τ_j^k be the eigenvalue of $d_{2k}\Delta-\mu_{ik}-r_k$ with the eigenfunction $\varphi_j^k(x)$ satisfying

$$0>\tau_1^k=-\mu_{2k}-r_k\geqslant\tau_2^k\geqslant\dots\tau_j^k\geqslant\cdots.$$

Hence, by the boundedness of the eigenfunction $\varphi_j^k(x)$ one has

$$T_{2k}(t,x,y)\leqslant\omega_{2k}\sum_{j\geqslant 1}\mathrm{e}^{\tau_j^k t}\leqslant\omega_{2k}\mathrm{e}^{\tau_1^k t}=\omega_{2k}\mathrm{e}^{-(\mu_{2k}+r_k)t}.$$

According the upper solution $\overline{S}_k$ of system (7), one has $\limsup_{t\to\infty}S_k=b_k/\mu_{1k}$, which implies S_k is ultimate bounded. Further, the ultimate bounded of I_k will be analyzed. Let $N_k=S_k+I_k$ and $P_k=\int_\Omega N_k\,\mathrm{d}x$. Adding the first two equations of system (7) and integrating it on Ω, one has

$${}_0^C D_t^\alpha P_k\leqslant b_k|\Omega|-\mu_{1k}P_k.$$

Therefore, by [13] one has

$$P_k\leqslant\biggl(\int_\Omega(\phi_{1k}+\phi_{2k})\,\mathrm{d}x-\frac{b_k|\Omega|}{\mu_{1k}}\biggr)E_\alpha\bigl[-\mu_{1k}t^\alpha\bigr]+\frac{b_k|\Omega|}{\mu_{1k}}\to\frac{b_k|\Omega|}{\mu_{1k}}\quad(t\to\infty),$$

then there exist two constants $M>0$ and $T_1>0$ satisfying $P_k\leqslant M$ for $t\geqslant T_1$. According to [17], the operator families $\{T_\alpha^{2k}\}$ is uniformly bounded. Hence, there exist two constants $M_1>0$ and $T_2>T_1>0$ satisfying $T_\alpha^{2k}\leqslant M_1$ for $t\geqslant T_2$. Finally, the uniformly boundedness of the infected group $I_k(x,t)$ can be studied as follows:

$$\begin{aligned}I_k(x,t)&=T_\alpha^{(2k)}(t)\phi_2+\int_{T_2}^t M_\alpha^{(2k)}(t-s)F_{2k}(S_k,I_k)\,\mathrm{d}s\\&\leqslant M_1\|\phi_2\|_\Omega+\alpha\int_{T_2}^t\int_0^\infty\theta(t-s)^{\alpha-1}\xi_\alpha(\theta)G_{2k}\bigl(\theta(t-s)^\alpha\bigr)F_{2k}\,\mathrm{d}\theta\,\mathrm{d}s\\&\leqslant M_1\|\phi_2\|_\Omega+\sum_{j=1}^n\frac{\beta_{kj}\omega_{2k}f_k(S_k^0)c_jM}{\mu_{2k}+r_k},\end{aligned}$$

where $S_k^0=b_k/\mu_{1k}$, thus I_k is ultimate bounded. Therefore, there exists a unique positive global solution $(S(x,t),I(x,t))$ of system (7), and it is also ultimately bounded. □

3.2 Stability analysis

In this subsection, the global stability analysis of system (7) will be discussed. It is easy to find that the disease-free equilibrium point $E^0 = (S_1^0, 0, S_2^0, 0, \dots, S_n^0, 0)$ of system (7) always exists where $S_k^0 = b_k/\mu_{1k}$ $(k = 1, 2, \dots, n)$. Define the following function:

$$\widetilde{R}_0 = \rho\bigl(M\bigl(S^0\bigr)\bigr),$$

where $M(S^0) = (\beta_{kj} f_k(S_k^0)c_j/(\mu_{k2} + r_k))_{n\times n}$, and $\rho(M(S^0))$ is the spectral radiuses of the matrix $M(S^0)$.

Lemma 5. *The basic reproduction number* $R_0 = \widetilde{R}_0$.

Proof. Linearizing system (7) at the disease-free equilibrium point E^0, one has

$$\begin{aligned}
{}_0^C D_t^\alpha u_{1k} &= d_{1k}\Delta u_{1k} - \mu_{1k}u_{1k} - \sum_{j=1}^n \beta_{kj} f_k\bigl(S_k^0\bigr)c_j u_{2j} \\
{}_0^C D_t^\alpha u_{2k} &= d_{2k}\Delta u_{2k} - (\mu_{2k} + r_k)u_{2k} + \sum_{j=1}^n \beta_{kj} f_k\bigl(S_k^0\bigr)c_j u_{2j}.
\end{aligned}$$

Let $F = (\beta_{kj} f_k(S_k^0)c_j)_{n\times n}$ and $V = \operatorname{diag}(\mu_{21} + r_1, \dots, \mu_{2n} + r_n)$. Then it is easy to find $M(S^0) = V^{-1}F$. Obviously, we have $\widetilde{R}_0 = \rho(V^{-1}F)$. By the definition of the basic reproduction number [5] one has $R_0 = \rho(FV^{-1})$. Thus, by the properties of matrix eigenvalues it can be deduced that $R_0 = \widetilde{R}_0$. □

Therefore, $\widetilde{R}_0$ is considered as a threshold parameter in place of R_0. In the following, the uniqueness and the global stability of $E^0 = (S_1^0, 0, S_2^0, 0, \dots, S_n^0, 0)$ are studied.

Theorem 2. *Under hypothesis* (H) *and* $\widetilde{R}_0 \leqslant 1$, *there exists the unique equilibrium point* E^0 *of system* (7), *and it is globally asymptotically stable in domain* Γ, *where*

$$\Gamma = \Gamma_1 \times \Gamma_2 \times \dots \times \Gamma_n, \quad \Gamma_k = \bigl\{(S_k, I_k) \in \mathbb{R}_+ \colon S_k \leqslant S_k^0\bigr\}.$$

Proof. Let $S = (S_1, S_2, \dots, S_n)^{\mathrm{T}}$, $S^0 = (S_1^0, S_2^0, \dots, S_n^0)^{\mathrm{T}}$, and $I = (I_1, I_2, \dots, I_n)^{\mathrm{T}}$, where $S_k^0 = b_k/\mu_{1k}$. Define

$$M(S) = \left(\frac{\beta_{kj} f_k(S_k)c_j}{\mu_{2k} + r_k}\right)_{1\leqslant k,j\leqslant n}.$$

It is clear to find from $(S, I) \in \Gamma$ that $0 \leqslant S_k \leqslant S_k^0$ $(k = 1, 2, \dots, n)$. Then one has $0 \leqslant M(S) \leqslant M(S^0)$. Since B is irreducible, it can be obtained that $M(S)$ and $M(S^0)$ are irreducible. So $M(S^0) + M(S)$ is also irreducible. If $S \neq S^0$, the inequality $\rho(M(S)) < \rho(M(S^0))$ holds. Further, it can be deduced that $\rho(M(S)) < 1$ if $S \leqslant S^0$ and $S \neq S^0$. Thus, $M(S)I = I$ has a only trivial solution $I = 0$. This shows that E^0 is the unique equilibrium of system (7) when $\widetilde{R}_0 \leqslant 1$. Further, $M(S^0)$ is positive, then $\widetilde{R}_0 = \rho(M(S^0))$ is an eigenvalue of the matrix $M(S^0)$, and $M(S^0)$ has a nonnegative eigenvector corresponding to $\rho(M(S^0))$. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ be the positive left eigenvector of $M(S^0)$ corresponding to the spectral radius $\rho(M(S^0))$, that is, $(\alpha_1, \alpha_2, \dots, \alpha_n)\rho(M(S^0)) = (\alpha_1, \alpha_2, \dots, \alpha_n)M(S^0)$. Define Lyapunov function

$$L_1(t) = \sum_{k=1}^{n} \int_{\Omega} \frac{\alpha_k}{\mu_{2k} + r_k} I_k \, \mathrm{d}x.$$

Calculating the time fractional derivative of L_1 along the trajectories of system (7), one has

$$\begin{aligned} {}_0^C D_t^\alpha L_1(t) &\leqslant \sum_{k=1}^{n} \int_{\Omega} \frac{\alpha_k}{\mu_{2k} + r_k} \left(\sum_{j=1}^{n} \beta_{kj} f_k(S_k) c_j I_j - (\mu_{2k} + r_k) I_k \right) \mathrm{d}x \\ &= \int_{\Omega} \rho\big(M\big(S^0\big) - 1\big) \alpha I \, \mathrm{d}x. \end{aligned}$$

Let $L_2(I) = \int_\Omega (1 - \rho(M(S^0))\alpha I \, \mathrm{d}x$, which is a positive definition function in Γ. Then it is concluded from [4] that E^0 is globally asymptotically stable in domain Γ. □

Theorem 3. *Under hypothesis* (H) *and* $\widetilde{R}_0 > 1$, *system* (7) *is uniform persistence, that is, for any initial value* $\phi_k(x) = (\phi_{1k}(x), \phi_{2k}(x)) \in \mathbb{X}_+$ *with* $\phi_{2k} \neq 0$ ($k = 1, 2, \dots, n$), *the solution* $(S(t, \phi), I(t, \phi))$ *satisfies*

$$\liminf_{t\to\infty} S_k(t, \phi) \geqslant \delta, \quad \liminf_{t\to\infty} I_k(t, \phi) \geqslant \delta, \quad 1 \leqslant k \leqslant n,$$

where $\delta > 0$ *is a constant.*

Proof. Define a set

$$X_0 = \big\{\phi = (\phi_1, \phi_2) \in \mathbb{X}_+ \colon \phi_{2k} \not\equiv 0, \ 1 \leqslant k \leqslant n\big\},$$

and

$$\partial X_0 = \mathbb{X}_+ / \mathbb{X}_0 = \big\{\phi = (\phi_1, \phi_2) \in \mathbb{X}_+ \colon \phi_{2k} = 0, \ 1 \leqslant k \leqslant n\big\}.$$

Let $Y(t, x) = (S, I)$ be the solution of system (7) under the initial value $Y_0 = Y(0, x) = (\phi_1(x), \phi_2(x)) \in \mathbb{X}_+$. For any $t \geqslant 0$, it can be known that all nonnegative solutions $(S(t, \phi), I(t, \phi))$ generate a solution semiflow $T(t) : \mathbb{X}_+ \to \mathbb{X}_+$ with $T(t)Y_0 = Y(t, x, \phi)$. Thus, we have $T(0)Y_0 = Y(0)$, and it is obvious that $T(0) = E$, where E is the identity matrix. It can be deduced from Lemma 4 that

$$T(t + s)Y_0 = Y(t + s, \, x, \, \phi) = T(t)Y(s, x, \phi) = T(t)T(s)Y_0.$$

Then $T(t + s) = T(t)T(s)$. Based on the above analysis, one has $T(t)$ is C^0-semigroup on $\mathbb{X}_+$. Obviously, $T(t)$ is compact for $t \geqslant 0$ and point dissipative in $\mathbb{X}_+$. The following system is considered:

$${}_0^C D_t^\alpha S_k = d_{1k} \Delta S_k + b_k - \mu_{1k} S_k.$$

It can be found from Lemma 3 that S_k^0 is globally asymptotically stable. Thus, system (7) is globally asymptotically stable at the disease-free equilibrium point $E^0 = (S_1^0, 0, \dots, S_n^0, 0)$. It can be deduced that the disease-free equilibrium E^0 in $\partial\mathbb{X}_+$ is a global attractor of $T(t)$, which implies $\Omega(\partial\mathbb{X}_0) = \{E^0\}$. Let $M = \{M_1\}$, where $M_1 = \{E^0\}$. Considering $\widetilde{R}_0 > 1$, there exists a sufficiently small constant $\varepsilon_0 > 0$ such that $\rho(M(S^0, \varepsilon_0)) > 1$, where $\rho(M(S^0, \varepsilon_0)) = (\beta_{kj} f_k S_k^0 - \varepsilon_0 c_j/(\mu_{2k} + r_k))_{n\times n}$. If $W^s(E^0) \cap X_0 \neq \emptyset$, there exists a solution (S_k, I_k) of system (7) with the initial value such that $(S_k, I_k) \to (S_k^0, 0)$ $(k = 1, 2, \dots, n)$ as $t \to \infty$, then there exists a constant $\tau > 0$ such that $S_k > S_k^0 - \varepsilon_0$ and $I_k > \varepsilon_0$ for $t \geqslant \tau$. Since B is irreducible, $M(S^0, \varepsilon_0)$ is irreducible. Let $(\alpha_1, \alpha_2, \dots, \alpha_n)$ be the positive left eigenvector of $M(S^0, \varepsilon_0)$ corresponding to the spectral radius $\rho(M(S^0, \varepsilon_0))$, that is,

$$(\alpha_1, \alpha_2, \dots, \alpha_n)\rho\bigl(M\bigl(S^0, \varepsilon_0\bigr)\bigr) = (\alpha_1, \alpha_2, \dots, \alpha_n)M\bigl(S^0, \varepsilon_0\bigr).$$

Define the following arbitrary function:

$$L_1(t) = \sum_{k=1}^{n} \int_{\Omega} \frac{\alpha_k}{\mu_{2k} + r_k} I_k \,\mathrm{d}x.$$

Calculating the time fractional derivative of L_1 along the trajectories of system (7), one has

$${}_0^C D_t^\alpha L_1(t) \geqslant \int_{\Omega} \rho\bigl(M\bigl(S^0, \varepsilon_0\bigr) - 1\bigr)\alpha I \,\mathrm{d}x > 0,$$

which leads to a contradiction with $\lim_{t\to\infty} I_k(t) = 0$. Therefore, $W^s(E^0) \cap X_0 = \emptyset$. Thus, it can be deduced from [25] that $T(t)$ is uniformly persistent. It is concluded that system (7) is uniformly persistent. □

The ultimate boundedness and the uniform persistence imply the existence of a positive equilibrium point of system (7). Therefore, the existence and global stability of the positive endemic equilibrium point of system (7) can be further discussed.

Theorem 4. *Under hypothesis* (H) *and* $\widetilde{R}_0 > 1$, *system* (7) *has at least one endemic equilibrium* $E^* = (S_1^*, I_1^*, \dots, S_n^*, I_n^*)$ *satisfying*

$$b_k = \mu_{1k} S_k^* + \sum_{j=1}^{n} \beta_{kj} f_k(S_k^*) g_j(I_j^*), \tag{12}$$

$$\sum_{j=1}^{n} \beta_{kj} f_k(S_k^*) g_j(I_j) = (\mu_{2k} + r_k) I_k^*. \tag{13}$$

Furthermore, if

$$\left(\frac{f_k(S_k)g_j(I_j)}{S_k} - \frac{f_k(S_k^*)g_j(I_j^*)}{S_k^*}\right)\left(\frac{f_k(S_k)g_j(I_j)}{S_k I_j} - \frac{f_k(S_k^*)g_j(I_j^*)}{S_k^* I_j^*}\right) \leqslant 0, \tag{14}$$

system (7) *is globally asymptotically stable at the endemic equilibrium point* E^*.

Proof. According to Theorem 1, for any given initial condition $\phi_k(x) = (\phi_{1k}(x), \phi_{2k}(x)) \in \mathbb{X}_+$ ($k = 1, 2, \dots, n$), the corresponding solution $(S_k(x,t), I_k(x,t))$ ($k = 1, 2, \dots, n$) is ultimately bounded, and system (7) is uniformly persistent when $\widetilde{R}_0 > 1$. Therefore, there exists a positive equilibrium point E^* of system (7) that satisfies Eqs. (12), (13).

Next, the global stability of $E^* = (S_1^*, I_1^*, \dots, S_n^*, I_n^*)$ will be analyzed. Define the Lyapunov function

$$L_2(t) = \sum_{k=1}^{n} c_k V_k(t),$$

where

$$V_k(t) = \int_{\Omega} \left[\left(S_k - S_k^* - S_k^* \ln \frac{S_k}{S_k^*} \right) + \left(I_k - I_k^* - I_k^* \ln \frac{I_k}{I_k^*} \right) \right] \mathrm{d}x, \quad 1 \leqslant k \leqslant n,$$

and the coefficients c_k will be determined in Eq. (20). Calculating the time fractional-order derivative of V_k along the trajectories of system (7), it can be conclude that

$$\begin{aligned}
{}_0^C D_t^\alpha V_k(t) \leqslant{}& \int_{\Omega} \left(1 - \frac{S_k^*}{S_k}\right) d_{1k} \Delta S_k \,\mathrm{d}x + \int_{\Omega} \left(1 - \frac{I_k^*}{I_k}\right) d_{2k} \Delta I_k \,\mathrm{d}x \\
&- \int_{\Omega} \left(1 - \frac{S_k^*}{S_k}\right) \mu_{1k} (S_k - S_k^*) \,\mathrm{d}x \\
&+ \int_{\Omega} \sum_{j=1}^{n} \beta_{kj} f_k(S_k^*) g_j(I_j^*) \left(1 - \frac{S_k^*}{S_k}\right) + \sum_{j=1}^{n} \beta_{kj} f_k(S_k) g_j(I_j) \frac{S_k^*}{S_k} \,\mathrm{d}x \\
&- \int_{\Omega} \sum_{j=1}^{n} \beta_{kj} f_k(S_k) g_j(I_j) \frac{I_k^*}{I_k} - (\mu_{2k} + r_k)(I_k - I_k^*) \,\mathrm{d}x. \qquad (15)
\end{aligned}$$

For each $1 \leqslant k \leqslant n$, it can be deduced from the divergence theorem that

$$\int_{\Omega} d_{1k} \Delta S_k \,\mathrm{d}x = 0, \qquad \int_{\Omega} d_{2k} \Delta I_k \,\mathrm{d}x = 0,$$

$$\int_{\Omega} \frac{d_{1k} \Delta S_k}{S_k} \,\mathrm{d}x = \int_{\Omega} \frac{d_{1k} \|\nabla S_k\|^2}{S_k^2} \,\mathrm{d}x, \qquad \int_{\Omega} \frac{d_{2k} \Delta I_k}{I_k} \,\mathrm{d}x = \int_{\Omega} \frac{d_{2k} \|\nabla I_k\|^2}{I_k^2} \,\mathrm{d}x.$$

Thus, Eq. (15) can be deduced that

$$\begin{aligned}
&{}_0^C D_t^\alpha V_k(t) \\
&\quad \leqslant - \int_{\Omega} S_k^* d_{1k} \frac{\|\nabla S_k\|^2}{S_k^2} \,\mathrm{d}x - \int_{\Omega} I_k^* d_{2k} \frac{\|\nabla I_k\|^2}{I_k^2} \,\mathrm{d}x - \int_{\Omega} \mu_{1k} \frac{(S_k - S_k^*)^2}{S_k} \,\mathrm{d}x
\end{aligned}$$

$$+\int\limits_{\Omega}\sum_{j=1}^{n}\beta_{kj}f_k(S_k^*)g_j(I_j^*)\biggl(2-\frac{S_k^*}{S_k}+\frac{f_k(S_k)g_j(I_j)S_k^*}{f_k(S_k^*)g_j(I_j^*)S_k}-\frac{I_k}{I_k^*} - \frac{f_k(S_k)g_j(I_j)I_k^*}{f_k(S_k^*)g_j(I_j^*)I_k}\biggr)\,\mathrm{d}x. \tag{16}$$

Let $h(x)=x-1-lnx$, then

$$\begin{aligned}&2-\frac{S_k^*}{S_k}+\frac{f_k(S_k)g_j(I_j)S_k^*}{f_k(S_k^*)g_j(I_j^*)S_k}-\frac{I_k}{I_k^*}-\frac{f_k(S_k)g_j(I_j)I_k^*}{f_k(S_k^*)g_j(I_j^*)I_k}\\&\quad\leqslant -h\biggl(\frac{S_k^*}{S_k}\biggr)-h\biggl(\frac{S_kI_jf_k(S_k^*)g_j(I_j^*)}{S_k^*I_j^*f_k(S_k)g_j(I_j)}\biggr)-h\biggl(\frac{f_k(S_k)g_j(I_j)I_k^*}{f_k(S_k^*)g_j(I_j^*)I_k}\biggr)\\&\qquad+h\biggl(\frac{I_j}{I_j^*}\biggr)-h\biggl(\frac{I_k}{I_k^*}\biggr).\end{aligned} \tag{17}$$

Substituting Eq. (17) into Eq. (16), the following inequality holds:

$$\begin{aligned}{}_0^CD_t^\alpha V_k(t)\leqslant{}&-\int\limits_{\Omega}S_k^*d_{1k}\frac{||\nabla S_k||^2}{S_k^2}\,\mathrm{d}x-\int\limits_{\Omega}I_k^*d_{2k}\frac{\|\nabla I_k\|^2}{I_k^2}\,\mathrm{d}x-\int\limits_{\Omega}\mu_{1k}\frac{(S_k-S_k^*)^2}{S_k}\,\mathrm{d}x\\&+\int\limits_{\Omega}\sum_{j=1}^{n}\beta_{kj}f_k(S_k^*)g_j(I_j^*)\biggl(-h\biggl(\frac{S_k^*}{S_k}\biggr)-h\biggl(\frac{S_kI_jf_k(S_k^*)g_j(I_j^*)}{S_k^*I_j^*f_k(S_k)g_j(I_j)}\biggr)\\&\qquad-h\biggl(\frac{f_k(S_k)g_j(I_j)I_k^*}{f_k(S_k^*)g_j(I_j^*)I_k}\biggr)\biggr)\,\mathrm{d}x\\&+\int\limits_{\Omega}\sum_{j=1}^{n}\beta_{kj}f_k(S_k^*)g_j(I_j^*)\biggl(h\biggl(\frac{I_j}{I_j^*}\biggr)-h\biggl(\frac{I_k}{I_k^*}\biggr)\biggr)\,\mathrm{d}x.\end{aligned}$$

Calculating the fractional-order derivative of $L_2(t)$ along any solution of system (7), one has

$$\begin{aligned}{}_0^CD_t^\alpha L_2\leqslant{}&\sum_{k=1}^{n}c_j\biggl(-\int\limits_{\Omega}S_k^*d_{1k}\frac{\|\nabla S_k\|^2}{S_k^2}\,\mathrm{d}x-\int\limits_{\Omega}I_k^*d_{2k}\frac{\|\nabla I_k\|^2}{I_k^2}\,\mathrm{d}x-\int\limits_{\Omega}\mu_{1k}\frac{(S_k-S_k^*)^2}{S_k}\,\mathrm{d}x\biggr)\\&+\int\limits_{\Omega}\sum_{k=1}^{n}c_k\Biggl(\sum_{j=1}^{n}\beta_{kj}f_k(S_k^*)g_j(I_j^*)\biggl(-h\biggl(\frac{S_k^*}{S_k}\biggr)-h\biggl(\frac{S_kI_jf_k(S_k^*)g_j(I_j^*)}{S_k^*I_j^*f_k(S_k)g_j(I_j)}\biggr)\\&\qquad-h\biggl(\frac{f_k(S_k)g_j(I_j)I_k^*}{f_k(S_k^*)g_j(I_j^*)I_k}\biggr)\biggr)\\&+\int\limits_{\Omega}\sum_{k=1}^{n}c_k\Biggl(\sum_{j=1}^{n}\beta_{kj}f_k(S_k^*)g_j(I_j^*)\biggl(h\biggl(\frac{I_j}{I_j^*}\biggr)-h\biggl(\frac{I_k}{I_k^*}\biggr)\biggr)\Biggr)\,\mathrm{d}x.\end{aligned} \tag{18}$$

Since E^* is the endemic equilibrium point of system (7), one has

$$\sum_{k=1}^{n} c_k\left(\sum_{j=1}^{n}\beta_{kj}f_k(S_k^*)g_j(I_j^*)\left(h\left(\frac{I_j}{I_j^*}\right)-h\left(\frac{I_k}{I_k^*}\right)\right)\right)$$
$$=\sum_{k=1}^{n}h\left(\frac{I_k}{I_k^*}\right)\sum_{j=1}^{n}c_j\beta_{jk}f_j(S_j^*)g_k(I_k^*)-c_k(\mu_{2k}+r_k)I_k^*, \tag{19}$$

where c_k denotes the cofactor of the kth diagonal entry of $\widetilde{B}$, where

$$\widetilde{B}=\begin{bmatrix}\sum_{j\neq 1}\widetilde{\beta_{1j}} & -\widetilde{\beta_{21}} & \dots & -\widetilde{\beta_{n1}}\\ -\widetilde{\beta_{12}} & \sum_{j\neq 2}\widetilde{\beta_{2j}} & \dots & -\widetilde{\beta_{n2}}\\ \dots & \dots & \dots & \dots\\ -\widetilde{\beta_{1n}} & -\widetilde{\beta_{2n}} & \dots & \sum_{j\neq n}\widetilde{\beta_{nj}},\end{bmatrix} \tag{20}$$

with $\widetilde{\beta}_{kj}=\beta_{kj}f_k(S_k^*)g_j(I_j^*)$. It can be deduced from [7] that $\widetilde{B}x=0$ exists a unique positive solution $c=(c_1,c_2,\dots,c_n)$. Therefore,

$$\sum_{j=1}^{n}c_j\widetilde{\beta_{jk}}=c_k\sum_{j=1}^{n}\widetilde{\beta_{kj}}. \tag{21}$$

Thus, substituting Eq. (19) into Eq. (21), it can be obtained that

$$\sum_{j=1}^{n}c_j\beta_{jk}f_j(S_j^*)g_k(I_k^*)=c_k\sum_{j=1}^{n}\beta_{kj}f_k(S_k^*)g_j(I_j^*)=c_k(\mu_{2k}+r_k)I_k^*. \tag{22}$$

Further, it is concluded from Eqs. (18) and (22) that

$${}_0^CD_t^\alpha L_2\leqslant -L_3,$$

where

$$L_3=\sum_{k=1}^{n}c_j\left(\int_\Omega S_k^*d_{1k}\frac{\|\nabla S_k\|^2}{S_k^2}\,\mathrm{d}x+\int_\Omega I_k^*d_{2k}\frac{\|\nabla I_k\|^2}{I_k^2}\,\mathrm{d}x+\int_\Omega \mu_{1k}\frac{(S_k-S_k^*)^2}{S_k}\,\mathrm{d}x\right)$$
$$+\int_\Omega\sum_{k=1}^{n}c_k\left(\sum_{j=1}^{n}\beta_{kj}f_k(S_k^*)g_j(I_j^*)\left(h\left(\frac{S_k^*}{S_k}\right)+h\left(\frac{S_kI_jf_k(S_k^*)g_j(I_j^*)}{S_k^*I_j^*f_k(S_k)g_j(I_j)}\right)\right.\right.$$
$$\left.\left.+h\left(\frac{f_k(S_k)g_j(I_j)I_k^*}{f_k(S_k^*)g_j(I_j^*)I_k}\right)\right)\right)\mathrm{d}x.$$

Based on [4], the endemic equilibrium point $E^*=(S_1^*,I_1^*,\dots,S_n^*,I_n^*)$ of system (7) is globally asymptotically stable. □

Corollary 1. *When $f_k(S_k)=S_k$, the endemic equilibrium point $E^*=(S_1^*,I_1^*,\dots,S_n^*,I_n^*)$ is globally asymptotically stable if hypothesis* (H) *and $\widetilde{R}_0>1$ satisfied.*

Remark 2. It can be seen that Corollary 1 is similar with Theorem 6 of [17] when $\alpha=1$.

Remark 3. Not considering infection between populations, that is, when $\beta_{kj}=0$ $(k\neq j)$, the reproduction number R_0^k of group k is $R_0^k=\beta_{kk}c_k f_k(S_k^0)/(\mu_{2k}+r_k)$. Furthermore, the disease-free equilibrium point $E_k^0=(b_k/\mu_{1k},0)$ is globally asymptotically stable when $R_0^k\leqslant 1$, and the endemic equilibrium point $E_k^*=(S_k^*,I_k^*)$ is globally asymptotically stable when $R_0^k>1$.

4 Numerical simulations

In order to verify theoretical results numerically, numerical simulations are presented in this section. We consider system (7) with two-group case, which is suitable for infectious diseases transmitted between two cities or communities. Furthermore, system (7) with two groups ($n=2$) can be calculated by the central difference method in L_1-type space and Alikhanov-type discretization in time [9]. Furthermore, we consider the following incidence rate as an example: $f_k(S_k)=S_k$, $g_k(I_k)=I_k/(1+\tau I_k)$, which τ is a positive parameter measuring the psychological or inhibitory effect. Obviously, $f_k(S_k)$ and $g_k(I_k)$ satisfy hypothesis (H). The corresponding system can be expressed as

$$\begin{aligned}
{}_0^C D_t^\alpha S_1 &= d_{11}\Delta S_1 + b_1 - \mu_{11}S_1 - \beta_{11}S_1\frac{I_1}{1+\tau I_1} - \beta_{12}S_1\frac{I_2}{1+\tau I_2},\\
{}_0^C D_t^\alpha I_1 &= d_{21}\Delta I_1 + \beta_{11}S_1\frac{I_1}{1+\tau I_1} + \beta_{12}S_1\frac{I_2}{1+\tau I_2} - (\mu_{21}+r_1)I_1,\\
{}_0^C D_t^\alpha S_2 &= d_{12}\Delta S_2 + b_2 - \mu_{12}S_1 - \beta_{21}S_2\frac{I_1}{1+\tau I_1} - \beta_{22}S_2\frac{I_2}{1+\tau I_2},\\
{}_0^C D_t^\alpha I_2 &= d_{22}\Delta I_2 + \beta_{21}S_2\frac{I_1}{1+\tau I_1} + \beta_{22}S_2\frac{I_2}{1+\tau I_2} - (\mu_{22}+r_2)I_2.
\end{aligned} \tag{23}$$

Let assign the following values to the parameters of system (23):

$$\begin{aligned}
&b_1=0.01, \quad \mu_{11}=0.12, \quad \beta_{11}=0.55, \quad \beta_{12}=0.5, \quad \mu_{21}=0.2, \quad r_1=0.14,\\
&b_2=0.01, \quad \mu_{12}=0.7, \quad \beta_{21}=0.5, \quad \beta_{22}=0.2, \quad \mu_{22}=0.1, \quad r_2=0.2.
\end{aligned}$$

It is easy to calculate that $\widetilde{R}_0=0.1549<1$. Based on by Theorem 2, the disease-free equilibrium point E^0 of system (23) is global stable which is verified by Figs. 1 and 2. Further, the following parameters is chose:

$$\begin{aligned}
&b_1=0.1332, \quad \mu_{11}=0.15, \quad \beta_{11}=0.55, \quad \beta_{12}=0.5, \quad \mu_{21}=0.24, \quad r_1=0.1,\\
&b_2=0.057, \quad \mu_{12}=0.1, \quad \beta_{21}=0.5, \quad \beta_{22}=0.2, \quad \mu_{22}=0.15, \quad r_2=0.15.
\end{aligned}$$

System (23) has a unique equilibrium point $E^*=(S_1^*,I_1^*,S_2^*,I_2^*)$. It can be calculated that $\widetilde{R}_0=1.0101>1$, and Eq. (14) is satisfied. Based on the above analysis, the endemic equilibrium point E^* of system (23) is global stable, which is verified by Figs. 3 and 4.

Further, with regard to the disease-free equilibrium point of the first group, the influence of different fractional order α on the stability of the infected are discussed. The error

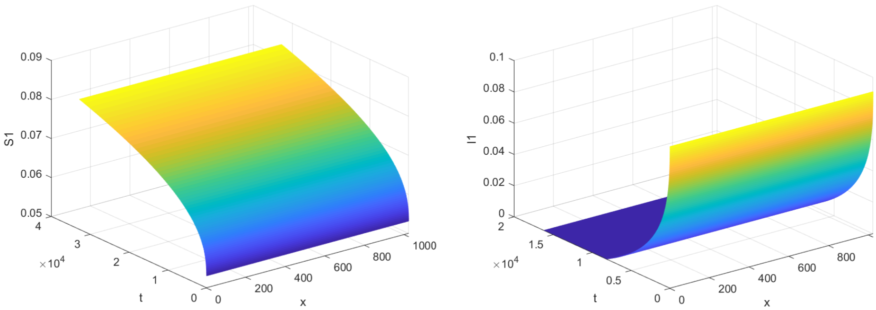

Figure 1. The first group stability of the disease-free equilibrium E^0.

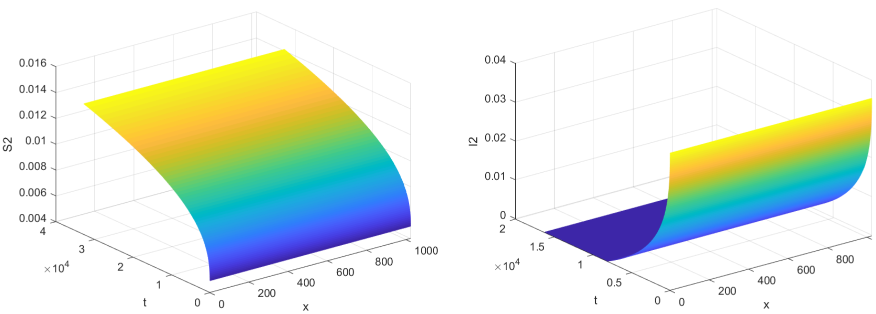

Figure 2. The second group stability of the disease-free equilibrium E^0.

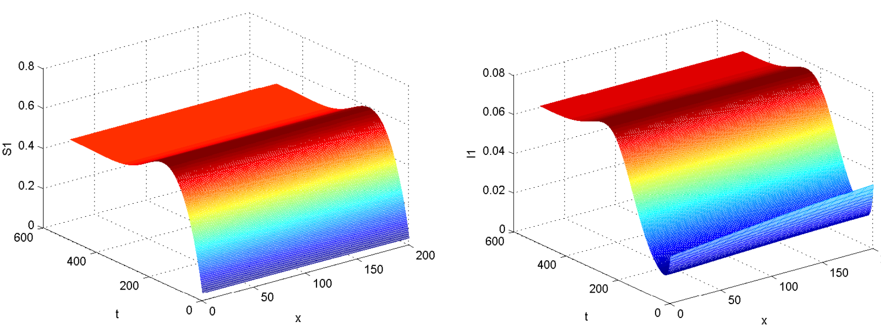

Figure 3. The first group stability of the endemic equilibrium E^*.

images of the infected of $\alpha = 0.4$ and $\alpha = 0.75$, $\alpha = 0.75$ and $\alpha = 0.98$ are described in Fig. 5, respectively. It is easy to seen from Fig. 5 that although the infected will disappear, different order α will have a sensitive effect on the change of solution. Further, when α tends to 1, the numerical solutions of system (7) are also convergent to the solutions of the classical ones [17]. But the relationship between the change of the solution for system (7), and fractional order α is not discussed.

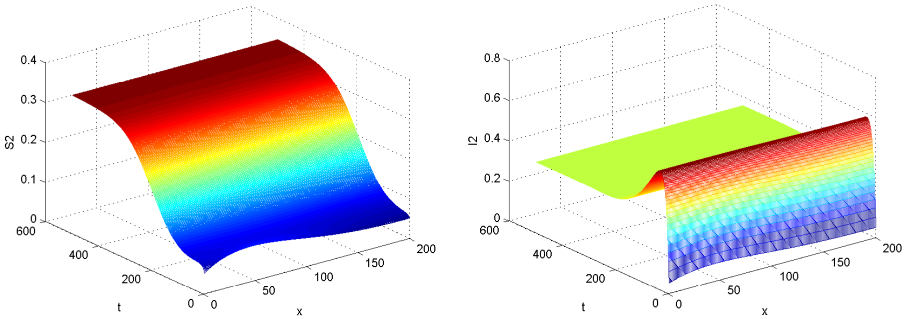

Figure 4. The second group stability of the endemic equilibrium E^*.

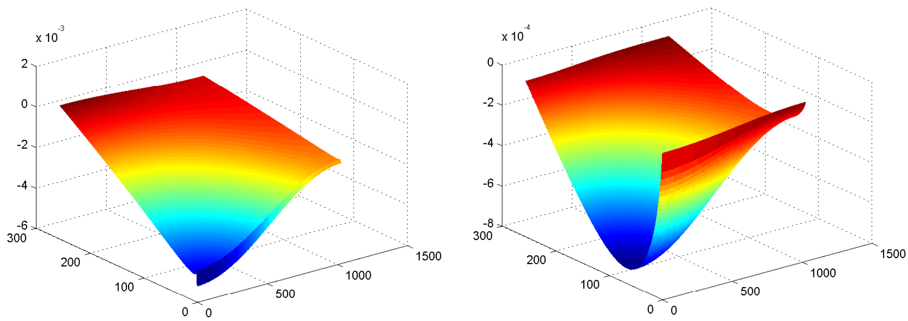

Figure 5. The error about different fractional order of the disease-free equilibrium point.

5 Discussion

In this article, incorporating the population diffusion and time fractional-order derivatives, theory analysis of a class of multigroup SIR epidemic model are investigated. Firstly, the existence and uniqueness of the nonnegative solution for system (7) are established. By using Lyapunov functions the global stability of the disease-free equilibrium point E^0 is obtained when the basic reproduction number $R_0 \leqslant 1$. Besides, when $R_0 > 1$, the uniform persistence and the global stability of the endemic equilibrium point E^* are discussed. The proposed model, a more accurate epidemic model, can help us to understand some dynamical behaviors of infectious diseases. Moreover, theoretical results may provide some useful guidance for making effective countermeasures on infectious diseases. However, the relationship between system (7) and fractional order α is still an open question, which will be our future work.

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