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The stability of a stochastic discrete SIVS epidemic model with general nonlinear incidence*

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Abstract. In this paper, based on Euler–Marryama method and theory of stochastic processes, a stochastic discrete SIVS epidemic model with general nonlinear incidence and vaccination is proposed by adding random perturbation and then discretizing the corresponding stochastic differential equation model. Firstly, the basic properties of continuous and discrete deterministic SIVS epidemic models are obtained. Then a criterion on the asymptotic mean-square stability of zero solution for a general linear stochastic difference system is established. As the applications of this criterion, the sufficient conditions on the stability in probability of the disease-free and endemic equilibria for the stochastic discrete SIVS epidemic model are obtained. The numerical simulations are given to illustrate the theoretical results.

Keywords: stochastic discrete SIVS epidemic model, nonlinear incidence, vaccination, mean-square stability, stability in probability.

1 Introduction

Infectious diseases have always been the enemy of human health. The repeated epidemic of infectious diseases has brought great disasters to human survival and the national economy and people's livelihood. It has been confirmed that vaccination is an important strategy for the control and elimination of infectious diseases. Many scholars have

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investigated various types of epidemic models with vaccination (see e.g. [3–5, 13, 15, 16, 30–33, 35]). Especially, in [13] the authors investigated the following SIS epidemic model with vaccination:

$$\begin{aligned}\dot S(t) &= (1-q)A - \beta S(t)I(t) - (\mu+p)S(t) + \gamma I(t) + \varepsilon V(t),\\ \dot I(t) &= \beta S(t)I(t) - (\mu+\gamma+\alpha)I(t),\\ \dot V(t) &= qA + pS(t) - (\mu+\varepsilon)V(t).\end{aligned} \tag{1}$$

It is well known that the incidence rate of diseases is an important part of epidemic models. In many practicality the nonlinear incidence is frequently used for achieving more exact results. Many epidemic models with nonlinear incidence have been widely studied (see e.g. [2–4, 8, 10, 12, 14–17, 24, 26, 28]). Particularly, as an extension of model (1), we consider the following epidemic model with general nonlinear incidence:

$$\begin{aligned}\dot S(t) &= (1-q)A - \beta f\bigl(S(t), I(t)\bigr) - (\mu+p)S(t) + \gamma I(t) + \varepsilon V(t),\\ \dot I(t) &= \beta f\bigl(S(t), I(t)\bigr) - (\mu+\gamma+\alpha)I(t),\\ \dot V(t) &= qA + pS(t) - (\mu+\varepsilon)V(t).\end{aligned} \tag{2}$$

where $S(t)$, $I(t)$ and $V(t)$ represent the numbers of susceptible, infectious and immune at time t, respectively; A is the recruitment of new numbers into the population; q is a fraction of vaccinated for new number; β represents the disease transmission coefficient; μ stands for the natural death rate of the population; γ is the recovery of infectious; p represents the proportional coefficient of vaccinated for the susceptible; ε denotes the rate of losing their immunity for vaccinated individuals; α denotes the disease-caused death rate of infectious; $f(S, I)$ represents the nonlinear incidence.

In the real world, epidemic models are always affected by the environmental white noise, which is an important component in an ecosystem. In [29,34] the authors concluded that climate change and natural disasters, including floods, locusts, earthquakes, wind and frost, have different impacts on infectious diseases (cholera, typhoid, malaria, etc.) in different periods and regions of China. All these factors can be described by random in biological mathematics. Therefore, lots of scholars have studied the stochastic epidemic models (see e.g. [1–4, 6, 7, 9, 10, 12, 14–18, 22–24, 26, 28, 35]).

Let $E^+ = (S^+, I^+, V^+)$ be an nonnegative equilibrium of deterministic model (2). Consider the environmental white noise effects on model (2). We assume that the disturbance of white noise around equilibrium E^+ and the degree of disturbance for each component S, I and V is proportional to $S(t)-S^+$, $I(t)-I^+$ and $V(t)-V^+$, respectively. Thus, in this paper, we propose the following stochastic SVIS epidemic model:

$$\begin{aligned}\mathrm{d}S(t) &= \bigl[(1-q)A - \beta f\bigl(S(t), I(t)\bigr) - (\mu+p)S(t) + \gamma I(t) + \varepsilon V(t)\bigr]\mathrm{d}t\\ &\quad + \sigma_1\bigl(S(t)-S^+\bigr)\mathrm{d}B_1(t),\\ \mathrm{d}I(t) &= \bigl[\beta f\bigl(S(t), I(t)\bigr) - (\mu+\gamma+\alpha)I(t)\bigr]\mathrm{d}t + \sigma_2\bigl(I(t)-I^+\bigr)\mathrm{d}B_2(t),\\ \mathrm{d}V(t) &= \bigl[qA + pS(t) - (\mu+\varepsilon)V(t)\bigr]\mathrm{d}t + \sigma_3\bigl(V(t)-V^+\bigr)\mathrm{d}B_3(t),\end{aligned} \tag{3}$$

where parameters q, A, β, μ, p, γ, ε and α are defined as in model (2). $B_k(t)$ $(k = 1, 2, 3)$ are independent standard Brownian motions with $B_k(0) = 0$, and σ_i denotes the intensity of $B_k(t)$.

According to the discretization characteristics of computer numerical calculation, the statistic data about infectious disease is collected by day, week, month or year, then it is more direct, more convenient and more accurate to describe the epidemic by using the discrete-time models than the continuous-time models. Currently, there are various discretization methods to discretize a continuous model, including the standard methods, such as Euler method, Runge–Kutta method, some other standard and nonstandard finite difference scheme (see e.g. [6, 7, 11, 19–23, 27]).

By using the Euler–Marryama discretization method (see e.g. [6, 7, 22]) we can discretize stochastic model (3) in the following pattern. In model (3), we choose a time step size $i > 0$. For any $t \geqslant 0$, since $\dot{S}(t) = \lim_{i\to 0}(S(t+i) - S(t))/i$, $\dot{I}(t) = \lim_{i\to 0}(I(t+i) - I(t))/i$, $\dot{V}(t) = \lim_{i\to 0}(V(t+i) - V(t))/i$, we can assume that for small enough $i > 0$, take $\mathrm{d}t = i$, and

$$\begin{aligned} \mathrm{d}S(t) &\doteq S(t+i) - S(t), \\ \mathrm{d}I(t) &\doteq I(t+i) - I(t), \\ \mathrm{d}V(t) &\doteq V(t+i) - V(t). \end{aligned} \tag{4}$$

In addition, for the standard Brownian motions $B_k(t)$ $(k = 1, 2, 3)$, we also can assume that $\mathrm{d}B_k(t) \doteq B_k(t+i) - B_k(t)$ and $B_k(t+1) - B_k(t) \sim N(0, i)$, that is, $B_k(t+i) - B_k(t)$ satisfies the normal distribution with mean value 0 and variance i. Through standardization, we can transform normal distribution $N(0, i)$ into standard normal distribution $N(0, 1)$. Then we can obtain that $(B_k(t+i) - B_k(t))/\sqrt{i} \sim N(0, 1)$. Let $\eta_{t+1}^{(k)}$ $(k = 1, 2, 3)$ is a family of independent random sequences obeying normal distribution $N(0, 1)$. Then $B_k(t+i) - B_k(t) = \sqrt{i}\eta_{t+1}^{(k)}$ $(k = 1, 2, 3)$. Therefore, we further have

$$\mathrm{d}B_k(t) \doteq B_k(t+i) - B_k(t) = \sqrt{i}\eta_{t+1}^{(k)}, \quad k = 1, 2, 3. \tag{5}$$

Furthermore, for the convenience of statements, we denote

$$\begin{array}{ccc} S(t+i) = S_{t+1}, & I(t+i) = I_{t+1}, & V(t+i) = V_{t+1}, \\ S(t) = S_t, & I(t) = I_t, & V(t) = V_t. \end{array} \tag{6}$$

Substituting (4)–(6) into model (3), we finally establish the following discretization model:

$$\begin{aligned} S_{t+1} &= S_t + \big[(1-q)A - \beta f(S_t, I_t) - (\mu + p)S_t + \gamma I_t + \varepsilon V_t\big]i \\ &\quad + \sigma_1\sqrt{i}\big(S_t - S^+\big)\eta_{t+1}^{(1)}, \\ I_{t+1} &= I_t + \big[\beta f(S_t, I_t) - (\mu + \gamma + \alpha)I_t\big]i + \sigma_2\sqrt{i}\big(I_t - I^+\big)\eta_{t+1}^{(2)}, \\ V_{t+1} &= V_t + \big[qA + pS_t - (\mu + \varepsilon)V_t\big]i + \sigma_3\sqrt{i}\big(V_t - V^+\big)\eta_{t+1}^{(3)}, \end{aligned} \tag{7}$$

where $t \in \mathbb{Z} = \{0, 1, 2, \dots\}$, the set of all nonnegative integers.

In this paper, our main purpose is to investigate the dynamical behavior of model (7). This paper is organized as follows. In Section 2, we introduce some useful lemmas and definitions. In Section 3, we first state and prove a criterion on the asymptotic mean-square stability for a general stochastic discrete linear system. Then the sufficient conditions of stability in probability for disease-free and endemic equilibria of stochastic discrete SIVS epidemic model (7) are established. Furthermore, the stability in probability for stochastic discrete SIVS epidemic models with standard incidence, Beddington–DeAngelis incidence and a nonmonotonic incidence also are discussed. In Section 4 the numerical examples are presented. Finally, in Section 5 a brief conclusion is given.

2 Preliminaries

Throughout this paper, we assume that model (7) is defined on a complete probability space $(\Omega, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbf{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions. $\eta_{t+1}^{(k)}$ $(k = 1, 2, 3)$ is a family of independent random sequences obeying normal distribution $N(0, 1)$, and $\mathcal{F}_t$ is produced by $\eta_{t+1}^{(k)}$ $(k = 1, 2, 3)$ (see [22]). For stochastic variables $\eta_t^{(k)}$ $(k = 1, 2, 3)$, the expectation $\mathbf{E}$ satisfies

$$\mathbf{E}\eta_t^{(k)} = 0, \qquad \mathbf{E}\big(\eta_t^{(k)}\big)^2 = 1, \qquad \mathbf{E}\eta_t^{(k)}\eta_t^{(j)} = 0 \quad (k \neq j). \tag{8}$$

We consider model (7) with an $\mathcal{F}_0$-adapted initial function

$$S_0 = \phi_1, \qquad I_0 = \phi_2, \qquad V_0 = \phi_3. \tag{9}$$

In addition, we introduce the following assumption.

(H) $f(S, I)$ is nonnegative and continuously differentiable for any $S \geqslant 0$, $I \geqslant 0$ and $S + I > 0$. For each fixed $I \geqslant 0$, $f(S, I)$ is increasing for $S > 0$, and $\sup_{S>0,\, I\geqslant 0}\{f(S, I)\} \leqslant B < \infty$. For each fixed $S \geqslant 0$, $f(S, I)/I$ is decreasing for $I > 0$. In addition, $f(S, 0) = f(0, I) = 0$ for any $S > 0$ and $I > 0$.

Remark 1. If $f(S, I) = SI/(S+I)$ (standard incidence), $f(S, I) = SI/(1+\omega_1 I+\omega_2 S)$ (Beddington–DeAngelis incidence) with constants $\omega_1 \geqslant 0$ and $\omega_2 \geqslant 0$, and $f(S, I) = SI/(1 + \omega I^2)$ (nonmonotonous incidence) with constant $\omega \geqslant 0$, then (H) is satisfied.

Let $(S(t), I(t), V(t))$ be any solution of model (2) with positive initial value $(S(0), I(0), V(0))$. We easily prove that the solution $(S(t), I(t), V(t))$ is defined for all $t \geqslant 0$ and is positive. Model (2) always has a disease-free equilibrium

$$E^0 = \big(S^0, 0, V^0\big) = \left(\frac{A((1-q)\mu + \varepsilon)}{\mu(\mu + p + \varepsilon)}, 0, \frac{A(q\mu + p)}{\mu(\mu + p + \varepsilon)}\right). \tag{10}$$

Following the next generation matrix method (see [25]), we calculate the basic reproduction number R_0 of model (2). Let

$$\mathcal{F} = \begin{pmatrix} \beta f(S(t), I(t)) \\ qA + pS(t) \end{pmatrix}, \qquad \widetilde{\mathcal{V}} = \begin{pmatrix} (\mu + \gamma + \alpha)I(t) \\ (\mu + \varepsilon)V(t) \end{pmatrix}.$$

Calculating the Jacobian matrices of $\mathcal{F}$ and $\mathcal{V}$ at equilibrium E^0, we have

$$F = \begin{pmatrix} \beta\frac{\partial f(S^0,0)}{\partial I} & 0 \\ 0 & 0 \end{pmatrix}, \qquad \widetilde{V} = \begin{pmatrix} \mu+\gamma+\alpha & 0 \\ 0 & \mu+\varepsilon \end{pmatrix}.$$

Then we can obtain the basic reproduction number R_0 of model (2) as follows:

$$R_0 = \rho\big(F\widetilde{V}^{-1}\big) = \frac{\beta\frac{\partial f(S^0,0)}{\partial I}}{\mu+\gamma+\alpha}.$$

On the existence of endemic equilibrium for model (2), we give the following lemma.

Lemma 1. *If $R_0 > 1$, then model* (2) *has a unique endemic equilibrium $E^* = (S^*, I^*, V^*)$, where*

$$\begin{aligned} S^* &= \frac{(\mu+\varepsilon-q\mu)A}{\mu(\mu+\varepsilon+p)} - \frac{(\mu+\alpha)(\mu+\varepsilon)}{\mu(\mu+\varepsilon+p)}I^*, \\ V^* &= \frac{qA}{\mu+\varepsilon} + \frac{p(\mu+\varepsilon-q\mu)A}{(\mu+\varepsilon)\mu(\mu+\varepsilon+p)} - \frac{p(\mu+\alpha)}{\mu(\mu+\varepsilon+p)}I^*, \end{aligned} \tag{11}$$

and I^* is the unique positive solution of the equation

$$\beta f\left(\frac{(\mu+\varepsilon-q\mu)A - (\mu+\alpha)(\mu+\varepsilon)I^*}{\mu(\mu+\varepsilon+p)}, I^*\right) - (\mu+\gamma+\alpha)I^* = 0. \tag{12}$$

Proof. The endemic equilibrium $E^* = (S^*, I^*, V^*)$ of model (2) satisfies the equations

$$\begin{aligned} &(1-q)A - \beta f(S^*, I^*) - (\mu+p)S^* + \gamma I^* + \varepsilon V^* = 0, \\ &\beta f(S^*, I^*) - (\mu+\gamma+\alpha)I^* = 0, \\ &qA + pS^* - (\mu+\varepsilon)V^* = 0. \end{aligned}$$

By calculating we obtain

$$V^* = \frac{qA+pS^*}{\mu+\varepsilon}, \qquad S^* = S^0 - \frac{(\mu+\alpha)(\mu+\varepsilon)}{\mu(\mu+\varepsilon+p)}I^*,$$

and I^* satisfies

$$\beta f\left(S^0 - \frac{(\mu+\alpha)(\mu+\varepsilon)}{\mu(\mu+\varepsilon+p)}I^*,\, I^*\right) - (\mu+\gamma+\alpha)I^* = 0.$$

Let

$$H(I) = \beta f\left(S^0 - \frac{(\mu+\alpha)(\mu+\varepsilon)}{\mu(\mu+\varepsilon+p)}I,\, I\right)\frac{1}{I} - (\mu+\gamma+\alpha).$$

Assumption (H) implies that $H(I)$ is decreasing for $I > 0$. When $R_0 > 1$,

$$\lim_{I\to 0} H(I) = \beta\frac{\partial f(S^0,0)}{\partial I} - (\mu+\gamma+\alpha) > 0.$$

Since

$$H\left(\frac{A(\mu+\varepsilon-\mu q)}{(\mu+\alpha)(\mu+\varepsilon)}\right) = -(\mu+\gamma+\alpha) < 0,$$

there exists a unique $I = I^* > 0$ such that $H(I^*) = 0$. Therefore, model (2) has a unique endemic equilibrium $E^* = (S^*, I^*, V^*)$ satisfying (11) and (12). This completes the proof. □

Furthermore, on the stability of equilibria for model (2), we have the following result.

Lemma 2.

(i) *If $R_0 < 1$, then disease-free equilibrium E^0 of model* (2) *is locally asymptotically stable, otherwise, if $R_0 > 1$, then E^0 is unstable.*

(ii) *If $R_0 > 1$, then endemic equilibrium E^* is locally asymptotically stable.*

Proof. For equilibrium E^0, we have the Jacobian matrix at E^0 as follows:

$$J(E^0) = \begin{pmatrix} -(p+\mu) & -\beta\frac{\partial f(S^0,0)}{\partial I} + \gamma & \varepsilon \\ 0 & \beta\frac{\partial f(S^0,0)}{\partial I} - (\alpha+\gamma+\mu) & 0 \\ p & 0 & -(\varepsilon+\mu) \end{pmatrix}.$$

Then the characteristic equation of $J(E^0)$ is

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{13}$$

where

$$\begin{aligned} a_1 &= p + \mu + (\alpha+\gamma+\mu)(1-R_0) + \varepsilon + \mu, \\ a_2 &= (p+\mu)(\varepsilon+\mu) + (p+\varepsilon+2\mu)(\alpha+\gamma+\mu)(1-R_0), \\ a_3 &= \big[(p+\mu)(\varepsilon+\mu) + p\varepsilon\big](\alpha+\gamma+\mu)(1-R_0). \end{aligned}$$

Clearly, if $R_0 < 1$, then $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and

$$\begin{aligned} a_1a_2 - a_3 = {}& \big[(\varepsilon+\mu)^2 + a_2\big](p+\mu) + (\alpha+\gamma+\mu)^2(1-R_0)^2(\varepsilon+p+2\mu) \\ &+ (\alpha+\gamma+\mu)(1-R_0)\big[(\varepsilon+\mu)^2 + \mu(\varepsilon+p+\mu)\big] > 0. \end{aligned}$$

Hence, all eigenvalues of $J(E^0)$ have negative real parts by the Routh–Hurwitz criterion. This implies that E^0 is locally asymptotically stable.

If $R_0 > 1$, then $a_3 < 0$. Hence, Eq. (13) has a positive root. This implies that E^0 is unstable.

For equilibrium E^*, we have the Jacobian matrix at E^* as follows:

$$J(E^*) = \begin{pmatrix} -\beta\frac{\partial f(S^*,I^*)}{\partial S} - (p+\mu) & -\beta\frac{\partial f(S^*,I^*)}{\partial I} + \gamma & \varepsilon \\ \beta\frac{\partial f(S^*,I^*)}{\partial S} & \beta\frac{\partial f(S^*,I^*)}{\partial I} - (\alpha+\gamma+\mu) & 0 \\ p & 0 & -(\varepsilon+\mu) \end{pmatrix}.$$

Then the characteristic equation of $J(E^*)$ is

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0,$$

where

$$\begin{aligned} b_1 &= -(b_{11} + b_{22} + b_{33}), \\ b_2 &= b_{11}b_{22} + b_{11}b_{33} + b_{22}b_{33} - b_{12}b_{21}, \\ b_3 &= b_{33}(b_{12}b_{21} - b_{11}b_{22}) - b_{22}b_{13}b_{31} \end{aligned}$$

with $b_{11} = -(\beta\partial f(S^*, I^*)/\partial S + p + \mu)$, $b_{12} = -\beta\partial f(S^*, I^*)/\partial I + \gamma$, $b_{13} = \varepsilon$, $b_{21} = \beta\partial f(S^*, I^*)/\partial S$, $b_{22} = \beta\partial f(S^*, I^*)/\partial I - (\alpha + \gamma + \mu)$, $b_{31} = p$ and $b_{33} = -(\varepsilon + \mu)$.

By (H) we easily obtain $\beta\partial f(S^*, I^*)/\partial I < \alpha + \gamma + \mu$. Hence, $b_{22} < 0$. Since

$$\begin{aligned} &b_{11}b_{22} - b_{12}b_{21} \\ &\quad = (p + \mu)\left(\alpha + \gamma + \mu - \beta\frac{\partial f(S^*, I^*)}{\partial I}\right) + \beta\frac{\partial f(S^*, I^*)}{\partial S}(\alpha + \mu) > 0, \end{aligned}$$

we further have $b_i > 0$ ($i = 1, 2, 3$). Furthermore, we have

$$b_{11}b_{33} - b_{13}b_{31} = \left(\beta\frac{\partial f(S^*, I^*)}{\partial S} + \mu\right)(\varepsilon + \mu) + p\mu > 0,$$

and then

$$\begin{aligned} b_1b_2 - b_3 = {} & -b_{11}b_2 - (b_{11} + b_{22})b_{33}^2 \\ & - b_{22}\left[(b_{11}b_{22} - b_{12}b_{21}) + b_{22}b_{33} + (b_{11}b_{33} - b_{13}b_{31})\right] \\ > {} & 0. \end{aligned}$$

Therefore, all eigenvalues of $J(E^*)$ have negative real parts by the Routh–Hurwitz criterion. This implies that E^* is locally asymptotically stable. This completes the proof. □

By Remark 1 and Lemma 1, we have the following conclusions.

(i) If $f(S, I) = SI/(S + I)$, then $R_0 = \beta/(\mu + \gamma + \alpha)$. When $R_0 > 1$, there is a unique endemic equilibrium $E^* = (S^*, I^*, V^*)$, where $S^* = (\alpha + \mu + \gamma)\Delta$, $I^* = [\beta - (\mu + \gamma + \alpha)]\Delta$ and $V^* = (Aq + p(\alpha + \mu + \gamma)\Delta)/(\mu + \varepsilon)$ with $\Delta = A[(1 - q)\mu + \varepsilon]/((\beta - (\alpha + \mu + \gamma))\alpha(\mu + \varepsilon) + \mu((\alpha + \mu + \gamma)p + \beta(\mu + \varepsilon)))$.

(ii) If $f(S, I) = SI/(1 + \omega_1 I + \omega_2 S)$ with constants $\omega_1 \geqslant 0$ and $\omega_2 \geqslant 0$, then $R_0 = \beta S^0/((\mu + \gamma + \alpha)(1 + \omega_2 S^0))$. When $R_0 > 1$, there is a unique endemic equilibrium $E^* = (S^*, I^*, V^*)$, where

$$\begin{aligned} I^* &= \frac{A[(1 - q)\mu + \varepsilon][\beta - (\alpha + \mu + \gamma)\omega_2] - \mu(\alpha + \mu + \gamma)(\mu + \varepsilon + p)}{[\beta - (\alpha + \mu + \gamma)\omega_2](\mu + \alpha)(\mu + \varepsilon) + \mu\omega_1(\alpha + \mu + \gamma)(\mu + \varepsilon + p)}, \\ S^* &= \frac{(\alpha + \mu + \gamma)(1 + \omega_1 I^*)}{\beta - (\alpha + \mu + \gamma)}, \\ V^* &= \frac{1}{(\mu + \varepsilon)}\left(Aq + \frac{p(\alpha + \mu + \gamma)(1 + \omega_1 I^*)}{\beta - (\alpha + \mu + \gamma)}\right). \end{aligned}$$

(iii) If $f(S,I) = SI/(1+\omega I^2)$ with constant $\omega \geqslant 0$, we have $R_0 = \beta S^0/(\mu+\gamma+\alpha)$. When $R_0 > 1$, there is a unique endemic equilibrium $E^* = (S^*, I^*, V^*)$, where $S^* = \Gamma$, $I^* = (-b + \sqrt{b^2 - 4ac})/(2a)$, $V^* = (Aq + p\Gamma)/(\mu + \varepsilon)$ with $\Gamma = (2a^2 + \omega(b^2 - 2ac - b\sqrt{b^2 - 4ac}))/(2\beta\mu\omega(\mu + \varepsilon + p)a)$, $a = \mu\omega(\alpha + \mu + \gamma)(\mu + \varepsilon + p)$, $b = \beta(\mu + \alpha)(\mu + \varepsilon)$ and $c = \mu(\alpha + \mu + \gamma)(\mu + \varepsilon + p) - A\beta[\mu(1 - q) + \varepsilon]$.

From the biological background of model (3) any solution of model (3) with the positive initial value must be positive with probability one defined for all $t \geqslant 0$. However, it is regrettable that at present, we do not have the ability to prove this conclusion. Although, any solution of the corresponding deterministic model (2) with the positive initial value is positive defined for all $t \geqslant 0$. The main reason is that in model (3) the stochastic perturbation terms are $\sigma_1(S(t) - S^+)\mathrm{d}B_1(t)$, $\sigma_2(I(t) - I^+)\mathrm{d}B_2(t)$ and $\sigma_3(V(t) - V^+)\mathrm{d}B_3(t)$, respectively, and $S(t) - S^+$, $I(t) - I^+$ and $V(t) - V^+$ can change the sign along with time t. Here we will leave this problem in the future study.

When $\sigma_i = 0$ $(i = 1, 2, 3)$ in model (7), we can obtain the corresponding deterministic discrete model as follows:

$$\begin{aligned} S_{t+1} &= S_t + \big[(1 - q)A - \beta f(S_t, I_t) - (\mu + p)S_t + \gamma I_t + \varepsilon V_t\big]i, \\ I_{t+1} &= I_t + \big[\beta f(S_t, I_t) - (\mu + \gamma + \alpha)I_t\big]i, \\ V_{t+1} &= V_t + \big[qA + pS_t - (\mu + \varepsilon)V_t\big]i. \end{aligned} \tag{14}$$

It is clear that the equilibria E^0 and E^* calculated in (10) and Lemma 1 are also the disease-free and endemic equilibria of model (14), respectively. On the positivity of solutions for model (14) we can obtain the following result.

Lemma 3. *Assume that* (H) *holds and step size* $i \leqslant \min\{1/(\beta B + \mu + p), 1/(\mu + \gamma + \alpha), 1/(\mu + \varepsilon)\}$, *where constant* $B > 0$ *is given in* (H). *Then solution* (S_t, I_t, V_t) *of model* (14) *with initial values* $S_0 > 0$, $I_0 > 0$ *and* $V_0 > 0$ *is positive for all* $t > 0$.

Proof. Let $\bar{i} = \min\{1/(\beta B + \mu + p), 1/(\mu + \gamma + \alpha), 1/(\mu + \varepsilon)\}$. By $0 < i \leqslant \bar{i}$, then $0 < i \leqslant 1/(\beta B + \mu + p)$, $0 < i \leqslant 1/(\mu + \varepsilon)$ and $0 < i \leqslant 1/(\mu + \gamma + \alpha)$. As initial values $S_0 > 0$, $I_0 > 0$ and $V_0 > 0$, from model (14) we directly have

$$\begin{aligned} S_1 &\geqslant S_0\big[1 - (\beta B + \mu + p)i\big] + (1 - q)Ai + \gamma I_0 i + \varepsilon V_0 i > 0, \\ I_1 &= I_0\big[1 - (\mu + \gamma + \alpha)i\big] + \beta f(S_0, I_0)i > 0, \\ V_1 &= V_0\big[1 - (\mu + \varepsilon)i\big] + qAi + pS_0 i > 0. \end{aligned}$$

Using the induction method, we assume $S_t > 0$, $I_t > 0$ and $V_t > 0$ for any integer $t > 0$, then we have

$$\begin{aligned} S_{t+1} &\geqslant S_t\big[1 - (\beta B + \mu + p)i\big] + (1 - q)Ai + \gamma I_t i + \varepsilon V_t i > 0, \\ I_{t+1} &= I_t\big[1 - (\mu + \gamma + \alpha)i\big] + \beta f\big(S_t, I_t\big)i > 0, \\ V_{t+1} &= V_t\big[1 - (\mu + \varepsilon)i\big] + qAi + pS_t i > 0. \end{aligned}$$

Therefore, (S_t, I_t, V_t) is the positive solution of model (14) with initial values $S_0 > 0$, $I_0 > 0$ and $V_0 > 0$. The proof is completed. $\square$

Although, in Lemma 3 we have proved that any solution of the corresponding deterministic discrete model (14) with the positive initial value is positive defined for all $t \geqslant 0$. However, for discrete stochastic model (7), it is regrettable that at present, we do not have the ability to prove that any solution of model (7) with the positive initial value is positive with probability one defined for all $t \geqslant 0$. It will be an interesting open problem.

Let $E^+ = (S^+, I^+, V^+)$ be any nonnegative equilibrium of model (2). Take the transformation $u_t = S_t - S^+$, $\nu_t = I_t - I^+$ and $\omega_t = V_t - V^+$, then model (7) takes the following form:

$$\begin{aligned} u_{t+1} &= u_t + \big[(1-q)A - \beta f(u_t + S^+,\, \nu_t + I^+) - (\mu + p)\big(u_t + S^+\big) \\ &\quad + \gamma(\nu_t + I^+) + \varepsilon\big(\omega_t + V^+\big)\big]i + \sigma_1\sqrt{i}u_t\eta_{t+1}^{(1)}, \\ \nu_{t+1} &= \nu_t + \big[\beta f\big(u_t + S^+,\, \nu_t + I^+\big) - (\mu + \gamma + \alpha)\big(\nu_t + I^+\big)\big]i \\ &\quad + \sigma_2\sqrt{i}\nu_t\eta_{t+1}^{(2)}, \\ \omega_{t+1} &= \omega_t + \big[qA + p\big(u_t + S^+\big) - (\mu + \varepsilon)\big(\omega_t + V^+\big)\big]i + \sigma_3\sqrt{i}\omega_t\eta_{t+1}^{(3)}. \end{aligned} \tag{15}$$

Clearly, system (15) has equilibrium $(0, 0, 0)$ corresponding to $E^+ = (S^+, I^+, V^+)$. Linearizing system (15) at $(0, 0, 0)$, we get the linearized system as follows:

$$\begin{aligned} X_{t+1} &= \left[1 - \beta\frac{\partial f(S^+, I^+)}{\partial u}i - (\mu + p)i\right]X_t + \left(\gamma - \beta\frac{\partial f(S^+, I^+)}{\partial \nu}\right)iY_t \\ &\quad + \varepsilon iZ_t + \sigma_1\sqrt{i}X_t\eta_{t+1}^{(1)}, \\ Y_{t+1} &= \beta\frac{\partial f(S^+, I^+)}{\partial u}X_t + \left[1 + \beta\frac{\partial f(S^+, I^+)}{\partial \nu}i - (\mu + \gamma + \alpha)i\right]Y_t \\ &\quad + \sigma_2\sqrt{i}Y_t\eta_{t+1}^{(2)}, \\ Z_{t+1} &= piX_t + \big[1 - (\mu + \varepsilon)i\big]Z_t + \sigma_3\sqrt{i}Z_t\eta_{t+1}^{(3)}. \end{aligned} \tag{16}$$

For disease-free equilibrium $E^0 = (S^0, 0, V^0)$, from system (16) we obtain the linearized system at E^0 as follows:

$$\begin{aligned} X_{t+1} &= \big[1 - (\mu + p)i\big]X_t + \left(\gamma - \beta\frac{\partial f(S^0, 0)}{\partial \nu}\right)iY_t + \varepsilon iZ_t \\ &\quad + \sigma_1\sqrt{i}X_t\eta_{t+1}^{(1)}, \\ Y_{t+1} &= \left[1 + \beta\frac{\partial f(S^0, 0)}{\partial \nu}i - (\mu + \gamma + \alpha)i\right]Y_t + \sigma_2\sqrt{i}Y_t\eta_{t+1}^{(2)}, \\ Z_{t+1} &= piX_t + \big[1 - (\mu + \varepsilon)i\big]Z_t + \sigma_3\sqrt{i}Z_t\eta_{t+1}^{(3)}. \end{aligned} \tag{17}$$

For endemic equilibrium $E^* = (S^*, I^*, V^*)$, from system (16) we obtain the linearized system at E^* as follows:

$$
\begin{aligned}
X_{t+1} &= \left[1 - \beta\frac{\partial f(S^*, I^*)}{\partial u} i - (\mu + p)i\right] X_t + \left(\gamma - \beta\frac{\partial f(S^*, I^*)}{\partial \nu}\right) iY_t \\
&\quad + \varepsilon i Z_t + \sigma_1\sqrt{i} X_t \eta_{t+1}^{(1)}, \\
Y_{t+1} &= \beta\frac{\partial f(S^*, I^*)}{\partial u} X_t + \left[1 + \beta\frac{\partial f(S^*, I^*)}{\partial \nu} i - (\mu + \gamma + \alpha)i\right] Y_t \\
&\quad + \sigma_2\sqrt{i} Y_t \eta_{t+1}^{(2)}, \\
Z_{t+1} &= piX_t + \bigl[1 - (\mu + \varepsilon)i\bigr] Z_t + \sigma_3\sqrt{i} Z_t \eta_{t+1}^{(3)}.
\end{aligned}
\tag{18}
$$

For the convenience, we denote the solutions of systems (15) and (16) with initial value $\phi = (\phi_1, \phi_2, \phi_3)$ by $U_t(\phi) = (u_t(\phi), \nu_t(\phi), \omega_t(\phi))$ and $x_t(\phi) = (X_t(\phi), Y_t(\phi), Z_t(\phi))$, respectively.

Since we do not obtain the positivity of solutions of model (7) satisfying any positive initial condition, the global dynamics of solutions for model (7) at present cannot be investigated. Therefore, we here consider the local dynamical behavior of solutions for model (7) around the nonnegative equilibrium of corresponding deterministic model (14). Especially, we will mainly discuss the local stability in probability. For this purpose, we introduce the following definitions for the stability in probability, the mean-square stability and the asymptotic mean-square stability for systems (15) and (16), respectively.

Definition 1. (See [22].) The zero solution of system (15) is said to be stable in probability if for any $\varepsilon > 0$ and $\varepsilon_1 > 0$ there exists $\delta > 0$ such that the solution $U_t(\phi)$ of system (15) satisfies the inequality $\mathbf{P}\{\sup_{t\in\mathbb{Z}} |U_t(\phi)| > \varepsilon\} < \varepsilon_1$ for any initial function (9) such that $\mathbf{P}\{|\phi| < \delta\} = 1$.

Definition 2. (See [22].)

(i) The zero solution of system (16) is called to be mean-square stable if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathbf{E}|x_t(\phi)|^2 < \varepsilon$, $t \in \mathbb{Z}$, for any initial function (9) such that $\mathbf{E}|\phi|^2 < \delta$.

(ii) The zero solution of system (16) is called to be asymptotically mean-square stable if the zero solution is mean-square stable and each solution $x_t(\phi)$ of system (16) has $\lim_{t\to\infty} \mathbf{E}|x_t(\phi)|^2 = 0$.

For any nonnegative function $V_t = V(t, x_t)$, $t \in \mathbb{Z}$, we define the difference $\Delta V_t = V_{t+1} - V_t = V(t+1, x_{t+1}) - V(t, x_t)$ (see [23]). We have the following Lyapunov function type criteria for the stability in probability, the mean-square stability and the asymptotic mean-square stability for systems (15) and (16), respectively.

Lemma 4. (*See* [22, 23].) *For system* (15)*, there exists a function* $V_t = V(t, U_t)$*,* $t \in \mathbb{Z}$*, satisfying the conditions*

$$V(t, U_t) \geqslant c_0|U_t|^2, \tag{19}$$

$$\mathbf{E}V(0, \phi) \leqslant c_1|\phi|^2, \tag{20}$$

where c_0 and c_1 are positive constants, and

$$\mathbf{E}\Delta V_t \leqslant 0, \quad t \in \mathbb{Z}. \tag{21}$$

Then the zero solution of system (15) *is stable in probability.*

Lemma 5. (*See* [22, 23].) *For linear system* (16)*, there exists a nonnegative function $V_t = V(t, x_t)$, $t \in \mathbb{Z}$, satisfying conditions* (20) *and*

$$\mathbf{E}\Delta V_t \leqslant -c_2 \mathbf{E}|x_t|^2, \quad t \in \mathbb{Z}, \tag{22}$$

where $c_2 > 0$ is a constant. Then the zero solution of system (16) *is asymptotically mean-square stable.*

Furthermore, from Remark 7.9 given in [22] we easily see that if there exists a function V_t, which satisfies conditions (19), (20) and (22) for the corresponding linearized system (16), then this function V_t also satisfies condition (21) for original system (15). This shows that in order to obtain the stability in probability of the zero solution for original system (15), it is enough by virtue of some function V_t, which satisfies conditions (19), (20) and (22) to get sufficient conditions for asymptotic mean-square stability of the zero solution for the corresponding linearized system (16).

3 Stability in probability

In this section, we consider the general forms of system (16) in the following linear stochastic difference system:

$$\begin{aligned} X_{t+1} &= b_{11}X_t + b_{12}Y_t + b_{13}Z_t + \sigma_1\sqrt{i}X_t\eta_{t+1}^{(1)}, \\ Y_{t+1} &= b_{21}X_t + b_{22}Y_t + \sigma_2\sqrt{i}Y_t\eta_{t+1}^{(2)}, \\ Z_{t+1} &= b_{31}X_t + b_{33}Z_t + \sigma_3\sqrt{i}Z_t\eta_{t+1}^{(3)}. \end{aligned} \tag{23}$$

Let

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & 0 \\ b_{31} & 0 & b_{33} \end{pmatrix}, \qquad \varphi(\eta_t) = \begin{pmatrix} \sigma_1\sqrt{i}\eta_t^{(1)} & 0 & 0 \\ 0 & \sigma_2\sqrt{i}\eta_t^{(2)} & 0 \\ 0 & 0 & \sigma_3\sqrt{i}\eta_t^{(3)} \end{pmatrix}. \tag{24}$$

Denote $x_t = (X_t, Y_t, Z_t)$, then system (23) can be rewritten in the following vector form:

$$x_{t+1} = \bigl(B + \varphi(\eta_{t+1})\bigr)x_t.$$

For any symmetric matrices E and F, we define that $E > F$ if $E - F$ is a positive definite matrix. Let

$$D = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{12} & d_{22} & d_{23} \\ d_{13} & d_{23} & d_{33} \end{pmatrix}$$

be a semipositive definite matrix. Then we have $d_{kk} \geqslant 0$ for $k = 1, 2, 3$.

Theorem 1. *Suppose that there exists a semipositive definite matrix D such that*

$$D - B^{\mathrm{T}} D B > \begin{pmatrix} \sigma_1^2 i d_{11} & 0 & 0 \\ 0 & \sigma_2^2 i d_{22} & 0 \\ 0 & 0 & \sigma_3^2 i d_{33} \end{pmatrix}. \tag{25}$$

Then the zero solution of system (23) *is asymptotically mean-square stable.*

Proof. We first denote

$$P = D - B^{\mathrm{T}} D B = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{pmatrix}. \tag{26}$$

Define Lyapunov function $W_t = x_t^{\mathrm{T}} D x_t$. We have

$$\Delta W_t = W_{t+1} - W_t = x_{t+1}^{\mathrm{T}} D x_{t+1} - x_t^{\mathrm{T}} D x_t.$$

By (3) we can obtain

$$\begin{aligned} \Delta W_t &= x_t^{\mathrm{T}} \big(\big(B + \varphi(\eta_{t+1})\big)^{\mathrm{T}} D \big(B + \varphi(\eta_{t+1})\big) - D \big) x_t \\ &= x_t^{\mathrm{T}} \big(-P + \varphi^{\mathrm{T}}(\eta_{t+1}) D \varphi(\eta_{t+1}) \big) x_t, \end{aligned}$$

where

$$\begin{aligned} &\varphi^{\mathrm{T}}(\eta_{t+1}) D \varphi(\eta_{t+1}) \\ &\quad = \begin{pmatrix} \sigma_1^2 i (\eta_{t+1}^{(1)})^2 d_{11} & \sigma_1 \sigma_2 i \eta_{t+1}^{(1)} \eta_{t+1}^{(2)} d_{12} & \sigma_1 \sigma_3 i \eta_{t+1}^{(1)} \eta_{t+1}^{(3)} d_{13} \\ \sigma_1 \sigma_2 i \eta_{t+1}^{(1)} \eta_{t+1}^{(2)} d_{12} & \sigma_2^2 i (\eta_{t+1}^{(2)})^2 d_{22} & \sigma_2 \sigma_3 i \eta_{t+1}^{(2)} \eta_{t+1}^{(3)} d_{23} \\ \sigma_1 \sigma_3 i \eta_{t+1}^{(1)} \eta_{t+1}^{(3)} d_{13} & \sigma_2 \sigma_3 i \eta_{t+1}^{(2)} \eta_{t+1}^{(3)} d_{23} & \sigma_3^2 i (\eta_{t+1}^{(3)})^2 d_{33}. \end{pmatrix}. \end{aligned}$$

Furthermore, we have

$$-P + \varphi^{\mathrm{T}}(\eta_{t+1}) D \varphi(\eta_{t+1}) = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{pmatrix},$$

where

$$\begin{aligned} c_{11} &= -p_{11} + \sigma_1^2 i \big(\eta_{t+1}^{(1)}\big)^2 d_{11}, & c_{12} &= -p_{12} + \sigma_1 \sigma_2 i \eta_{t+1}^{(1)} \eta_{t+1}^{(2)} d_{12}, \\ c_{13} &= -p_{13} + \sigma_1 \sigma_3 i \eta_{t+1}^{(1)} \eta_{t+1}^{(3)} d_{13}, & c_{22} &= -p_{22} + \sigma_2^2 i \big(\eta_{t+1}^{(2)}\big)^2 d_{22}, \\ c_{23} &= -p_{23} + \sigma_2 \sigma_3 i \eta_{t+1}^{(2)} \eta_{t+1}^{(3)} d_{23}, & c_{33} &= -p_{33} + \sigma_3^2 i \big(\eta_{t+1}^{(3)}\big)^2 d_{33}. \end{aligned}$$

Then we have

$$\begin{aligned} \Delta W_t &= \begin{pmatrix} X_t & Y_t & Z_t \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{pmatrix} \begin{pmatrix} X_t \\ Y_t \\ Z_t \end{pmatrix} \\ &= c_{11} X_t^2 + 2c_{12} X_t Y_t + 2c_{13} X_t Z_t + c_{22} Y_t^2 + 2c_{23} Y_t Z_t + c_{33} Z_t^2. \end{aligned}$$

Calculating the expectation of ΔW_t, we can obtain

$$\mathbf{E}\Delta W_t = \mathbf{E}\bigl(c_{11}X_t^2 + 2c_{12}X_tY_t + 2c_{13}X_tZ_t + c_{22}Y_t^2 + 2c_{23}Y_tZ_t + c_{33}Z_t^2\bigr).$$

By (8) we have $\mathbf{E}c_{11} = -p_{11} + \sigma_1^2 id_{11}$, $\mathbf{E}c_{12} = -p_{12}$, $\mathbf{E}c_{13} = -p_{13}$, $\mathbf{E}c_{22} = -p_{22} + \sigma_2^2 id_{22}$, $\mathbf{E}c_{23} = -p_{23}$ and $\mathbf{E}c_{33} = -p_{33} + \sigma_3^2 id_{33}$. Let

$$m = \min\bigl\{p_{11} - \sigma_1^2 id_{11},\, p_{12},\, p_{13},\, p_{22} - \sigma_2^2 id_{22},\, p_{23},\, p_{33} - \sigma_3^2 id_{33}\bigr\}.$$

Then

$$\mathbf{E}\Delta W_t \leqslant -m\mathbf{E}|x_t|^2.$$

This completes the proof. □

Remark 2. Based on (26), we further obtain that condition (25) is equivalent to

$$\begin{aligned}
&p_{11} - \sigma_{12}^2 id_{11} > 0, \qquad \bigl(p_{11} - \sigma_1^2 id_{11}\bigr)\bigl(p_{22} - \sigma_2^2 id_{22}\bigr) - p_{12}^2 > 0,\\
&\bigl(p_{11} - \sigma_1^2 id_{11}\bigr)\bigl(p_{22} - \sigma_2^2 id_{22}\bigr)\bigl(p_{33} + \sigma_3^2 id_{33}\bigr) - p_{23}^2\bigl(p_{11} - \sigma_1^2 id_{11}\bigr)\\
&\quad - p_{12}^2\bigl(p_{33} - \sigma_3^2 id_{33}\bigr) - p_{13}^2\bigl(p_{22} - \sigma_2^2 id_{22}\bigr) + 2p_{12}p_{13}p_{23} > 0.
\end{aligned} \tag{27}$$

Assume $R_0 > 1$. We consider linearized system (18). According to system (23), we can obtain that matrix B in (24) is given by

$$B = \begin{pmatrix} b_1 & b_3 & \varepsilon i\\ b_0 & b_2 & 0\\ pi & 0 & b_\varepsilon \end{pmatrix}, \tag{28}$$

where $b_0 = \beta(\partial f(S^*, I^*)/\partial u)i$, $b_1 = 1 - b_0 - (\mu + p)i$, $b_2 = 1 + \beta(\partial f(S^*, I^*)/\partial \nu)i - (\mu + \gamma + \alpha)i$, $b_3 = (\gamma - \beta(\partial f(S^*, I^*)/\partial \nu))i$ and $b_\varepsilon = 1 - (\mu + \varepsilon)i$. From (26) we further get

$$\begin{aligned}
-p_{11} &= b_0^2 d_{22} + 2b_1b_0d_{12} + 2b_0pid_{23} + \bigl(b_1^2 - 1\bigr)d_{11} + 2b_1pid_{13} + p^2i^2d_{33},\\
-p_{12} &= b_1b_3d_{11} + (b_2b_1 + b_3b_0 - 1)d_{12} + b_0b_2d_{22} + b_2pid_{23} + b_3pid_{13}\\
&\quad + b_0b_3d_{33},\\
-p_{13} &= b_1\varepsilon id_{11} + \bigl(b_1b_\varepsilon + p\varepsilon i^2 - 1\bigr)d_{13} + pib_\varepsilon d_{33} + b_0\varepsilon id_{12} + b_0b_\varepsilon d_{23},\\
-p_{22} &= b_3^2d_{11} + 2b_2b_3d_{12} + \bigl(b_2^2 - 1\bigr)d_{22},\\
-p_{23} &= b_3\varepsilon id_{11} + b_2\varepsilon id_{12} + b_3b_\varepsilon d_{13} + (b_\varepsilon b_2 - 1)d_{23},\\
-p_{33} &= \varepsilon^2i^2d_{11} + 2b_\varepsilon\varepsilon id_{13} + \bigl(b_\varepsilon^2 - 1\bigr)d_{33}.
\end{aligned} \tag{29}$$

Furthermore, based on condition (27), we can obtain conclusion as follows.

Theorem 2. *For the coefficient matrix* (28) *of system* (18)*, assume that* $R_0 > 1$*, and there is a semidefinite* D *such that condition* (27) *is satisfied with coefficients* p_{ij} $(i, j = 1, 2, 3)$ *defined in* (29)*. Then the zero solution of system* (18) *is asymptotically mean-square stable. Moreover, endemic equilibrium* $E^* = (S^*, I^*, V^*)$ *of model* (7) *is stable in probability.*

Particularly, when $f(S,I)=SI/(S+I)$ (standard incidence), we have

$$B=\begin{pmatrix} b_4 & b_6 & \varepsilon i\\ b_{0s} & b_5 & 0\\ pi & 0 & b_\varepsilon \end{pmatrix}, \tag{30}$$

where $b_{0s}=\beta(I^*/(S^*+I^*))^2 i$, $b_4=1-\beta(I^*/(S^*+I^*))^2 i-(\mu+p)i$, $b_5=1+\beta(S^*/(S^*+I^*))^2 i-(\mu+\gamma+\alpha)i$ and $b_6=(\gamma-\beta(S^*/(S^*+I^*))^2)i$. From (29) it follows that

$$\begin{aligned}
-p_{11}&=\bigl(b_4^2-1\bigr)d_{11}+b_{0s}^2d_{22}+2b_4b_{0s}d_{12}+2b_4pid_{13}+2b_{0s}pid_{23}+p^2i^2d_{33},\\
-p_{12}&=b_4b_6d_{11}+b_{0s}b_5d_{22}+[b_4b_5+b_6b_{0s}-1]d_{12}+b_6pid_{13}\\
&\quad+b_5pid_{23}+b_{0s}b_6d_{33},\\
-p_{13}&=b_4\varepsilon id_{11}+\bigl(b_4b_\varepsilon+p\varepsilon i^2-1\bigr)d_{13}+pib_\varepsilon d_{33}+b_{0s}\varepsilon id_{12}+b_{0s}b_\varepsilon d_{23},\\
-p_{22}&=b_6^2d_{11}+\bigl(b_5^2-1\bigr)d_{22}+2b_5b_6d_{12},\\
-p_{23}&=b_6\varepsilon id_{11}+(b_\varepsilon b_5-1)d_{23}+b_5\varepsilon id_{12}+b_6b_\varepsilon d_{13},\\
-p_{33}&=\varepsilon^2i^2d_{11}+2b_\varepsilon\varepsilon id_{23}+\bigl(b_\varepsilon^2-1\bigr)d_{33}.
\end{aligned} \tag{31}$$

When $f(S,I)=SI/(1+\omega_1 I+\omega_2 S)$ with constants $\omega_1\geqslant 0$ and $\omega_2\geqslant 0$ (Beddington–DeAngelis incidence), we have

$$B=\begin{pmatrix} b_{\Omega_{I^*}} & (\gamma-\beta S^*\Omega_{S^*})i & \varepsilon i\\ \beta I^*\Omega_{I^*}i & b_{\Omega_{S^*}} & 0\\ pi & 0 & 1-(\mu+\varepsilon)i \end{pmatrix}, \tag{32}$$

where $b_{\Omega_{I^*}}=1-\beta I^*\Omega_{I^*}i-(\mu+p)i$, $b_{\Omega_{S^*}}=1+\beta S^*\Omega_{S^*}i-(\mu+\gamma+\alpha)i$, $\Omega_{S^*}=(\omega_2S^*+1)/(1+\omega_1I^*+\omega_2S^*)^2$ and $\Omega_{I^*}=(\omega_1I^*+1)/(1+\omega_1I^*+\omega_2S^*)^2$. From (29) it follows that

$$\begin{aligned}
-p_{11}&=\bigl(b_{\Omega_{I^*}}^2-1\bigr)d_{11}+2b_{\Omega_{I^*}}\beta I^*\Omega_{I^*}id_{12}+2b_{\Omega_{I^*}}pid_{13}+(\beta I^*\Omega_{I^*}i)^2d_{22}\\
&\quad+2\beta I^*\Omega_{I^*}pi^2d_{23}+p^2i^2d_{33},\\
-p_{12}&=b_{\Omega_{I^*}}\bar\gamma id_{11}+(\gamma-\beta S^*\Omega_{S^*})pi^2d_{13}+\beta I^*\Omega_{I^*}ib_{\Omega_{S^*}}d_{22}+b_{\Omega_{S^*}}pid_{23}\\
&\quad+\bigl(b_{\Omega_{I^*}}b_{\Omega_{S^*}}+\bar\gamma\beta I^*\Omega_{I^*}i^2-1\bigr)d_{12}+\beta I^*\Omega_{I^*}i^2\bar\gamma d_{33},\\
-p_{13}&=b_{\Omega_{I^*}}\varepsilon id_{11}+\beta I^*\Omega_{I^*}\varepsilon i^2d_{12}+pib_\varepsilon d_{33}+\bigl(b_{\Omega_{I^*}}b_\varepsilon+p\varepsilon i^2-1\bigr)d_{13}\\
&\quad+\beta I^*\Omega_{I^*}ib_\varepsilon d_{23},\\
-p_{22}&=\bar\gamma^2i^2d_{11}+\bigl(b_{\Omega_{S^*}}^2-1\bigr)d_{22}+2b_{\Omega_{S^*}}\bar\gamma id_{12},\\
-p_{23}&=\bar\gamma i^2\varepsilon d_{11}+(b_\varepsilon b_{\Omega_{S^*}}-1)d_{23}+b_{\Omega_{S^*}}\varepsilon id_{12}+\bar\gamma ib_\varepsilon d_{13},\\
-p_{33}&=\varepsilon^2i^2d_{11}+2b_\varepsilon\varepsilon id_{23}+\bigl(b_\varepsilon^2-1\bigr)d_{33},
\end{aligned} \tag{33}$$

where $\bar\gamma=\gamma-\beta S^*\Omega_{S^*}$.

When $f(S,I) = SI/(1+\omega I^2)$ with constant $\omega \geqslant 0$ (a nonmonotonic incidence), we have

$$B = \begin{pmatrix} b_7 & b_9 & \varepsilon i \\ b_{0n} & b_8 & 0 \\ pi & 0 & b_\varepsilon \end{pmatrix},$$

where $b_{0n} = (\beta I^*/(1+\omega I^{*2}))i$, $b_7 = 1 - (\beta I^*/(1+\omega I^{*2}))i - (\mu+p)i$, $b_8 = 1 + (\beta S^*(1-\omega I^{*2})/(1+\omega I^{*2})^2)i - (\mu+\gamma+\alpha)i$ and $b_9 = (\gamma - \beta S^*(1-\omega I^{*2})/(1+\omega I^{*2})^2)i$. From (29) it follows that

$$\begin{aligned}
-p_{11} &= \bigl(b_7^2 - 1\bigr)d_{11} + 2b_7 pi d_{13} + p^2 i^2 d_{33} + 2b_7 b_{0n} d_{12} + b_{0n}^2 d_{22} \\
&\quad + 2pi b_{0n} d_{23}, \\
-p_{12} &= b_{0n} b_8 d_{22} + b_{0n} b_9 d_{33} + (b_7 b_8 + b_9 b_{0n} - 1)d_{12} + b_7 b_9 d_{11} \\
&\quad + b_9 pi d_{13} + b_8 pi d_{23}, \\
-p_{13} &= b_7 \varepsilon i d_{11} + \varepsilon i b_{0n} d_{12} + pi b_\varepsilon d_{33} + \bigl(b_\varepsilon b_7 + p\varepsilon i^2 - 1\bigr)d_{13} + b_\varepsilon b_{0n} d_{23}, \\
-p_{22} &= b_9^2 d_{11} + \bigl(b_8^2 - 1\bigr)d_{22} + 2b_8 b_9 d_{12}, \\
-p_{23} &= b_9 \varepsilon i d_{11} + [b_\varepsilon b_8 - 1]d_{23} + b_8 \varepsilon i d_{12} + b_9 b_\varepsilon d_{13}, \\
-p_{33} &= \varepsilon^2 i^2 d_{11} + 2b_\varepsilon \varepsilon i d_{23} + (b_\varepsilon^2 - 1)d_{33}.
\end{aligned}$$

Next, we assume $R_0 < 1$ and consider linearized system (17). According to system (23), we can obtain that matrix B in (24) is given by

$$B = \begin{pmatrix} b_p & b_{11} & \varepsilon i \\ 0 & b_{10} & 0 \\ pi & 0 & b_\varepsilon \end{pmatrix}, \tag{34}$$

where $b_p = 1 - (\mu+p)i, b_{10} = 1 + \beta(\partial f(S^0,0)/\partial\nu)i - (\mu+\gamma+\alpha)i$, $b_{11} = (\gamma - \beta\partial f(S^0,0)/\partial\nu)i$. By (26) we can get

$$\begin{aligned}
-p_{11} &= \bigl(b_p^2 - 1\bigr)d_{11} + 2b_p pi d_{13} + p^2 i^2 d_{33}, \\
-p_{12} &= b_p b_{11} d_{11} + b_{11} pi d_{13} + (b_p b_{10} - 1)d_{12} + b_{10} pi d_{23}, \\
-p_{13} &= b_p \varepsilon i d_{11} + pi b_\varepsilon d_{33} + \bigl[(\mu+p)(\mu+\varepsilon)i - (2\mu+\varepsilon+p)\bigr] i d_{13}, \\
-p_{22} &= b_{11}^2 d_{11} + 2b_{10} b_{11} d_{12} + \bigl(b_{10}^2 - 1\bigr)d_{22}, \\
-p_{23} &= b_{11} \varepsilon i d_{11} + (b_\varepsilon b_{10} - 1)d_{23} + b_{10} \varepsilon i d_{12} + b_{11} b_\varepsilon d_{13}, \\
-p_{33} &= \varepsilon i^2 d_{11} + 2b_\varepsilon \varepsilon i d_{23} + \bigl(b_\varepsilon^2 - 1\bigr)d_{33}.
\end{aligned} \tag{35}$$

Thus, based on condition (27), we can obtain the following conclusion.

Theorem 3. *For the coefficient matrix* (34) *of system* (17)*, assume that* $R_0 < 1$*, and there is a semidefinite matrix* D *such that condition* (27) *is satisfied with coefficients* p_{ij} $(i,j = 1,2,3)$ *defined in* (35)*. Then the zero solution of system* (17) *is asymptotically mean-square stable. Moreover, disease-free equilibrium* $E^0 = (S^0, 0, V^0)$ *of model* (7) *is stable in probability.*

Particularly, when $f(S,I) = SI/(S+I)$, we have

$$B = \begin{pmatrix} b_p & (\gamma-\beta)i & \varepsilon i \\ 0 & b_{12} & 0 \\ pi & 0 & b_\varepsilon \end{pmatrix},$$

where $b_{12} = 1 + \beta i - (\mu+\gamma+\alpha)i$. From (35) it follows that

$$\begin{aligned}
-p_{11} &= \bigl(b_p^2-1\bigr)d_{11} + 2b_p pid_{13} + p^2i^2d_{33},\\
-p_{12} &= b_p(\gamma-\beta)id_{11} + (\gamma-\beta)i^2pd_{13} + (b_pb_{12}-1)d_{12} + b_{12}pid_{23},\\
-p_{13} &= b_p\varepsilon id_{11} + pib_\varepsilon d_{33} + \bigl[(\mu+p)(\mu+\varepsilon)i - (2\mu+\varepsilon+p)\bigr]id_{13},\\
-p_{22} &= (\gamma-\beta)^2i^2d_{11} + 2b_{12}(\gamma-\beta)id_{12} + \bigl(b_{12}^2-1\bigr)d_{22},\\
-p_{23} &= (\gamma-\beta)\varepsilon i^2d_{11} + (b_\varepsilon b_{12}-1)d_{23} + b_{12}\varepsilon id_{12} + (\gamma-\beta)ib_\varepsilon d_{13},\\
-p_{33} &= \varepsilon i^2d_{11} + 2b_\varepsilon\varepsilon id_{23} + \bigl(b_\varepsilon^2-1\bigr)d_{33}.
\end{aligned}$$

When $f(S,I) = SI/(1+\omega_1I+\omega_2S)$ with constants $\omega_1 \geqslant 0$ and $\omega_2 \geqslant 0$, we have

$$B = \begin{pmatrix} b_p & b_{14} & \varepsilon i \\ 0 & b_{13} & 0 \\ pi & 0 & b_\varepsilon \end{pmatrix}, \tag{36}$$

where $b_{14} = (\gamma-\beta S^0/(1+\omega_2S^0))i$, $b_{13} = 1-(\mu+b_{14}+\alpha)i$. From (35) it follows that

$$\begin{aligned}
-p_{11} &= \bigl(b_p^2-1\bigr)d_{11} + 2b_p pid_{13} + p^2i^2d_{33},\\
-p_{12} &= b_pb_{14}d_{11} + b_{14}pid_{13} + (b_pb_{13}-1)d_{12} + b_{13}pid_{23},\\
-p_{13} &= b_p\varepsilon id_{11} + pib_\varepsilon d_{33} + \bigl[(\mu+p)(\mu+\varepsilon)i - (2\mu+\varepsilon+p)\bigr]id_{13},\\
-p_{22} &= b_{14}^2d_{11} + 2b_{13}b_{14}d_{12} + \bigl(b_{13}^2-1\bigr)d_{22},\\
-p_{23} &= b_{14}\varepsilon id_{11} + (b_\varepsilon b_{13}-1)d_{23} + b_{13}\varepsilon id_{12} + b_{14}b_\varepsilon d_{13},\\
-p_{33} &= \varepsilon i^2d_{11} + 2b_\varepsilon\varepsilon id_{23} + \bigl(b_\varepsilon^2-1\bigr)d_{33}.
\end{aligned} \tag{37}$$

When $f(S,I) = SI/(1+\omega I^2)$ with constant $\omega \geqslant 0$, we have

$$B = \begin{pmatrix} b_p & (\gamma-\beta S^0)i & \varepsilon i \\ 0 & b_{15} & 0 \\ pi & 0 & b_\varepsilon \end{pmatrix}, \tag{38}$$

where $b_{15} = 1 + \beta S^0 i - (\mu+\gamma+\alpha)i$. From (35) it follows that

$$\begin{aligned}
-p_{11} &= \bigl(b_p^2-1\bigr)d_{11} + 2b_p pid_{13} + p^2i^2d_{33},\\
-p_{12} &= b_p\bigl(\gamma-\beta S^0\bigr)id_{11} + \bigl(\gamma-\beta S^0\bigr)i^2pd_{13} + (b_pb_{15}-1)d_{12} + b_{15}pid_{23},\\
-p_{13} &= b_p\varepsilon id_{11} + pib_\varepsilon d_{33} + \bigl[(\mu+p)(\mu+\varepsilon)i - (2\mu+\varepsilon+p)\bigr]id_{13},\\
-p_{22} &= \bigl(\gamma-\beta S^0\bigr)^2i^2d_{11} + 2b_{15}\bigl(\gamma-\beta S^0\bigr)id_{12} + \bigl(b_{15}^2-1\bigr)d_{22},\\
-p_{23} &= \bigl(\gamma-\beta S^0\bigr)\varepsilon i^2d_{11} + (b_\varepsilon b_{15}-1)d_{23} + b_{15}\varepsilon id_{12} + \bigl(\gamma-\beta S^0\bigr)ib_\varepsilon d_{13},\\
-p_{33} &= \varepsilon i^2d_{11} + 2b_\varepsilon\varepsilon id_{23} + \bigl(b_\varepsilon^2-1\bigr)d_{33}.
\end{aligned} \tag{39}$$

4 Numerical examples

In this section, we give the numerical examples to illustrate the above theoretical results. In all figures the blue line represents the trajectory of the deterministic discrete model, and the red line represents the trajectory of the stochastic discrete model. Throughout this section, we take the positive define matrix

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{40}$$

From (27) we let

$$\begin{aligned} &K_1 = p_{11} - \sigma_1^2 id_{11}, \qquad K_2 = \bigl(p_{11} - \sigma_1^2 id_{11}\bigr)\bigl(p_{22} - \sigma_2^2 id_{22}\bigr), \\ &K_3 = \bigl(p_{11} - \sigma_1^2 id_{11}\bigr)\bigl(p_{22} - \sigma_2^2 id_{22}\bigr)\bigl(p_{33} + \sigma_3^2 id_{33}\bigr). \end{aligned} \tag{41}$$

Under the same parameters, the trajectory of stochastic discrete model (such as the white noise intensities $\sigma_1 = \sigma_2 = \sigma_3 = 0.3$) will be stable at the endemic equilibrium of corresponding deterministic discrete model. All trajectories of model (7) are stable to endemic equilibrium $E^*(S^*, I^*, V^*)$ or disease-free equilibrium $E^0 = (S^0, 0, V^0)$ (see Figs. 1–8).

Example 1. We take parameters $A = 4$, $q = 0.2$, $\varepsilon = 0.5$, $\alpha = 0.2$, $p = 0.15$ and $\mu = 0.04$ in model (7). The numerical simulations of solution (S_t, I_t, V_t) with initial value $(S_0, I_0, V_0) = (80, 10, 30)$ are given in Figs. 1 and 3. The numerical simulations of solution (S_t, I_t, V_t) with different initial value (S_0, I_0, V_0) are given in Figs. 2 and 4.

(i) Choose $f(S, I) = SI/(S + I)$, and parameters $\beta = 0.6$ and $\gamma = 0.2$. By calculating we have $R_0 = 1.3636 > 1$ and endemic equilibrium $E^* = (S^*, I^*, V^*) = (28.4769, 10.3552, 9.3917)$. According to (30), we can obtain the following matrix B. Furthermore, by (26), (31), (40) and matrix B we can solve the semipositive definite matrix D as follows:

$$B = \begin{pmatrix} 0.9767 & -0.0123 & 0.0500 \\ 0.0043 & 0.9883 & 0 \\ 0.0150 & 0 & 0.9460 \end{pmatrix}, \qquad D = \begin{pmatrix} 1.0501 & 0.0086 & -0.0727 \\ 0.0086 & 1.0239 & -0.0007 \\ -0.0727 & -0.0007 & 1.1222 \end{pmatrix}.$$

Then by (41) we have $K_1 = 0.9912 > 0$, $K_2 = 0.9913 > 0$ and $K_3 = 0.9914 > 0$. Thus, all conditions in Theorem 2 are satisfied. This means that endemic equilibrium $E^* = (S^*, I^*, V^*)$ is stable in probability. The numerical simulations are given in Figs. 1 and 2.

(ii) Choose $f(S, I) = SI/(1 + \omega_2 S + \omega_1 I)$, where $\omega_2 = \omega_1 = 0.1$, and parameters $\beta = 0.2$ and $\gamma = 0.7$. By calculating we have $R_0 = 1.8834 > 1$ and endemic equilibrium $E^* = (S^*, I^*, V^*) = (19.7071, 12.2229, 6.9557)$. According to (32), we can obtain the following matrix B. Furthermore, by (26), (33), (40) and matrix B

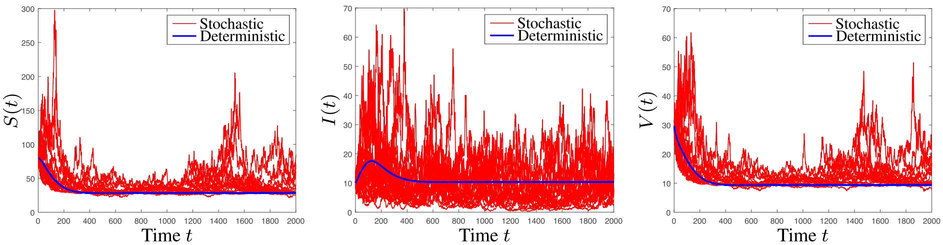

Figure 1. It shows that $\lim_{t\to\infty}(S_t, I_t, V_t) = (28.4769, 10.3552, 9.3917)$ a.s.

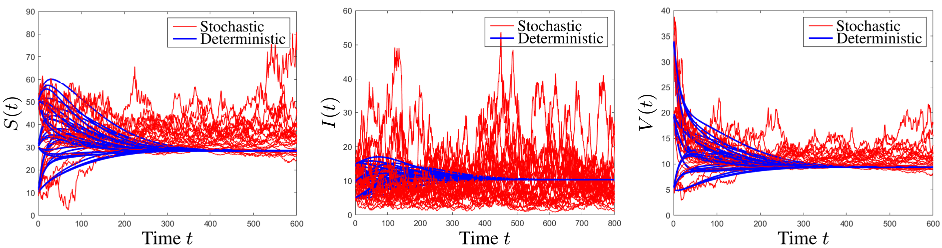

Figure 2. Numerical simulations of solution (S_t, I_t, V_t) with initial value (S_0, I_0, V_0), where $S_0 = 10$, 30 and 50, $I_0 = 5$, 10 and 15 and $V_0 = 5$, 20 and 35, respectively. It shows that $\lim_{t\to\infty}(S_t, I_t, V_t) = (28.4769, 10.3552, 9.3917)$ a.s.

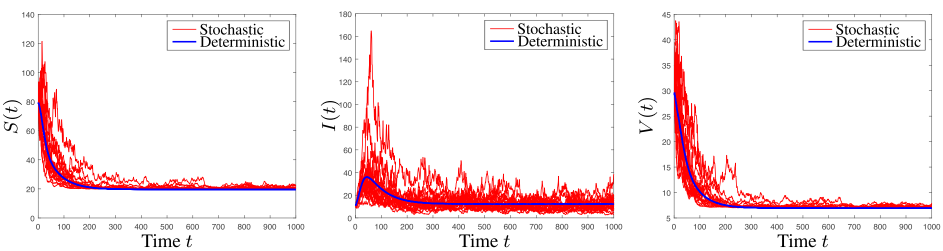

Figure 3. It shows that $\lim_{t\to\infty}(S_t, I_t, V_t) = (19.7071, 12.2229, 6.9557)$ a.s.

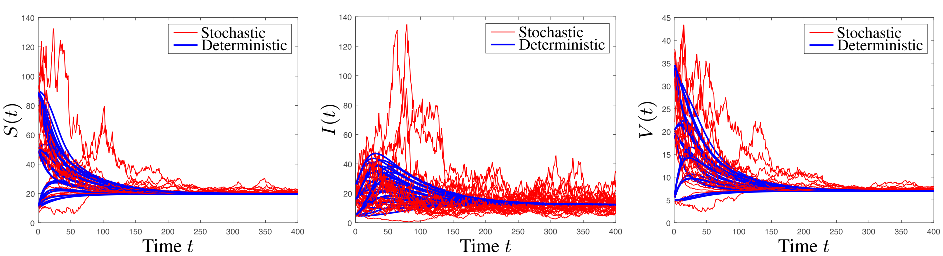

Figure 4. Numerical simulations of solution (S_t, I_t, V_t) with initial value (S_0, I_0, V_0), where $S_0 = 10$, 50 and 90, $I_0 = 5$, 15 and 25 and $V_0 = 5$, 20 and 35, respectively. It shows that $\lim_{t\to\infty}(S_t, I_t, V_t) = (19.7071, 12.2229, 6.9557)$ a.s.

we can solve semipositive definite matrix D as follows:

$$B = \begin{pmatrix} 0.9501 & 0.0034 & 0.0500 \\ 0.0309 & 0.9726 & 0 \\ 0.0150 & 0 & 0.9460 \end{pmatrix}, \qquad D = \begin{pmatrix} 1.1113 & -0.0383 & -0.0764 \\ -0.0383 & 1.0574 & 0.0021 \\ -0.0764 & 0.0021 & 1.1224 \end{pmatrix}.$$

Then by (41) we have $K_1 = 0.9914 > 0$, $K_2 = 0.9915 > 0$ and $K_3 = 0.9916 > 0$. Thus, all conditions in Theorem 2 are satisfied. This means that endemic equilibrium $E^* = (S^*, I^*, V^*)$ is stable in probability. The numerical simulations are given in Figs. 3 and 4.

In addition, choosing $f(S, I) = SI/(1 + \omega I^2)$, where $\omega = 0.1$, and parameters $\beta = 0.2$ and $\gamma = 0.7$, then by the similar arguments as in the above we can obtain that endemic equilibrium $E^* = (S^*, I^*, V^*)$ is stable in probability.

Example 2. We take parameters $A = 2$, $q = 0.2$, $\varepsilon = 0.5$, $\alpha = 0.5$, $p = 0.3$, $\beta = 0.1$, $\mu = 0.1$, $\gamma = 0.7$ and $\omega = \omega_1 = \omega_2 = 0.1$ in model (7). The numerical simulations of solution (S_t, I_t, V_t) with initial value $(S_0, I_0, V_0) = (20, 16, 6)$ are given in Figs. 5 and 7. By (10) we have disease-free equilibrium $E^0 = (S^0, 0, V^0) = (12.8889, 0, 7.1111)$. The numerical simulations of solution (S_t, I_t, V_t) with initial value (S_0, I_0, V_0), where $S_0 = 10$, 20 and 30, $I_0 = 6$, 16 and 26 and $V_0 = 1$, 6 and 11, respectively, are given in Figs. 6 and 8.

(i) Choose $f(S, I) = SI/(1+\omega_2 S+\omega_1 I)$. By calculating we have $R_0 = 0.7320 < 1$. According to (36), we can obtain the following matrix B. Furthermore, by (26), (37), (40) and matrix B we can solve semipositive definite matrix D as follows:

$$B = \begin{pmatrix} 0.9600 & 0.0137 & 0.0500 \\ 0 & 0.9263 & 0 \\ 0.0300 & 0 & 0.9400 \end{pmatrix} \qquad D = \begin{pmatrix} 1.0898 & -0.0161 & -0.0934 \\ -0.0161 & 1.1657 & 0.0014 \\ -0.0934 & 0.0014 & 1.1386 \end{pmatrix}.$$

Then by (41) we have $K_1 = 0.9914 > 0$, $K_2 = 0.9914 > 0$, $K_3 = 0.9915 > 0$. Thus, all conditions in Theorem 3 are satisfied. This means that disease-free equilibrium $E^0 = (S^0, 0, V^0)$ is stable in probability.

(ii) Choose $f(S, I) = SI/(1 + \omega I^2)$. By calculating we have $R_0 = 0.9915 < 1$. According to (38), we can obtain the following matrix B. Furthermore, by (26), (39), (40) and matrix B, we can solve semipositive definite matrix D as follows:

$$B = \begin{pmatrix} 0.9600 & -0.0589 & 0.0500 \\ 0 & 0.9989 & 0 \\ 0.0300 & 0 & 0.9400 \end{pmatrix}, \qquad D = \begin{pmatrix} 1.0898 & 0.0642 & -0.0934 \\ 0.0642 & 1.0060 & -0.0055 \\ -0.0934 & -0.0055 & 1.1386 \end{pmatrix}.$$

Then by (41) we have $K_1 = 0.9914 > 0$, $K_2 = 0.9914 > 0$, $K_3 = 0.9915 > 0$. Thus, all condition in Theorem 3 are satisfied. This means that disease-free equilibrium $E^0 = (S^0, 0, V^0)$ is stable in probability.

In addition, choosing $f(S, I) = SI/(S + I)$, by the similar arguments as in the above we can obtain that disease-free equilibrium $E^0 = (S^0, 0, V^0)$ is stable in probability.

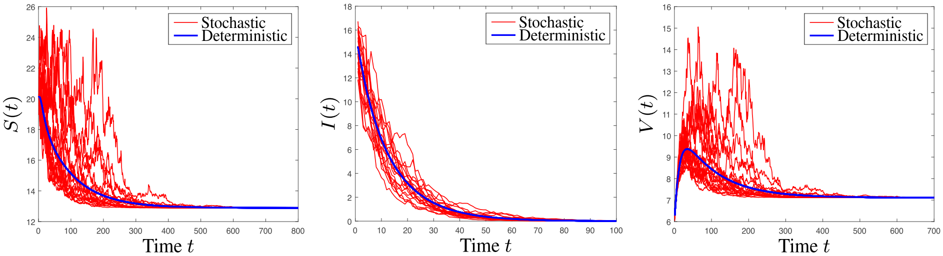

Figure 5. It shows that $\lim_{t\to\infty}(S_t, I_t, V_t) = (12.8889, 0, 7.1111)$ a.s.

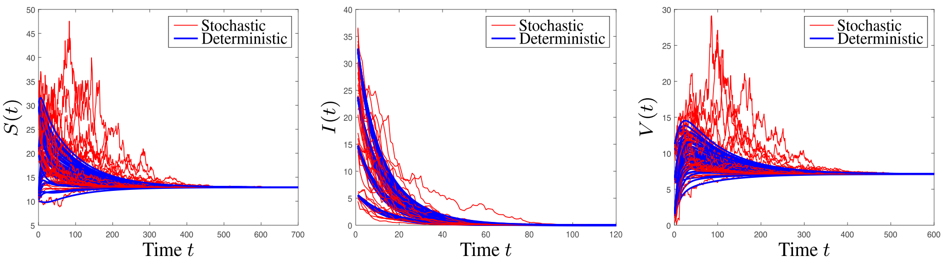

Figure 6. It shows that for different initial values, we also have $\lim_{t\to\infty}(S_t, I_t, V_t) = (12.8889, 0, 7.1111)$ a.s.

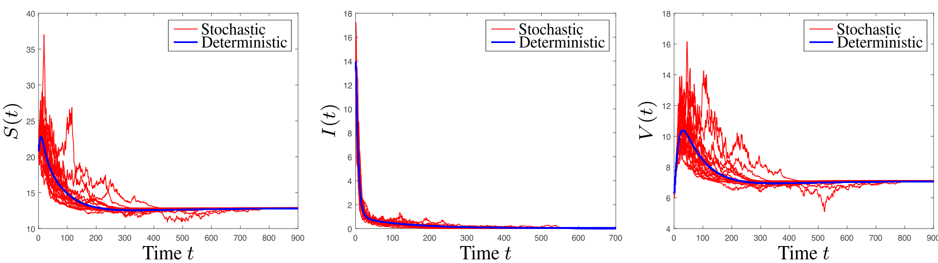

Figure 7. It shows that $\lim_{t\to\infty}(S_t, I_t, V_t) = (12.8889, 0, 7.1111)$ a.s.

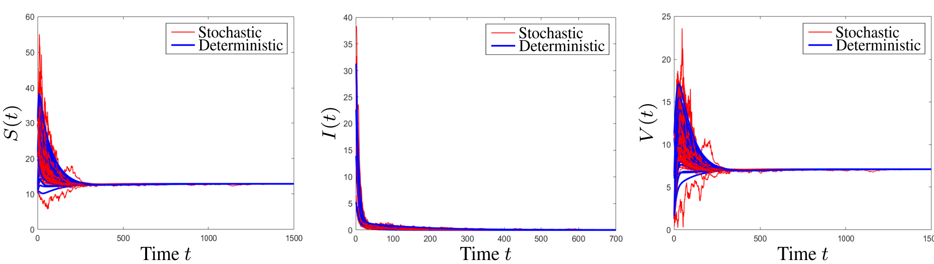

Figure 8. It shows that for different initial values, we also have $\lim_{t\to\infty}(S_t, I_t, V_t) = (12.8889, 0, 7.1111)$ a.s.

5 Conclusion

In this paper, we investigated a stochastic discrete SIVS epidemic model with general nonlinear incidence and vaccination. The model is proposed through discretizing the corresponding continuous-time stochastic differential equation model by means of Euler–Marryma method and the theory of random white noise disturbance. Firstly, the criterion on the asymptotic mean-square stability of zero solution for the general linear stochastic difference system is established. Next, as the applications of this criterion, the sufficient conditions on the stability in probability of the unique endemic and disease-free equilibria for the stochastic discrete SIVS epidemic model with general nonlinear incidence are further established. Moreover, the stability in probability of the equilibria for stochastic discrete SIVS epidemic models with some special nonlinear incidences such as standard incidence, Beddington–DeAngelis incidence and a nonmonotonic incidence also are discussed. Lastly, the numerical simulations are presented to illustrate the above theoretical results.

In the future, we can investigate the some other properties for this stochastic discrete SIVS epidemic model, such as the stochastic extinction and persistence of disease, the global stochastic stability of equilibrium and the stochastic dynamical complexity, etc. Furthermore, the method which is introduced in this paper whether can be extended to some other kind of stochastic discrete epidemic models also is interesting open problem. Moreover, according to the actual situation of the transmission of a specific infectious disease in a fixed area, the comprehensive effects of rain, wind and high temperature are fully considered, and the model proposed in this paper is tested through the actual data given by the literature or official.

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