On the non-closure under convolution for strong subexponential distributions*

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Abstract. In this paper, we consider the convolution closure problem for the class of strong subexponential distributions, denoted as \( S^* \). First, we show that, if \( F, G \in L \), then inclusions of \( F \ast G, FG \), and \( pF + (1 - p)G \) for all (some) \( p \in (0, 1) \) into the class \( S^* \) are equivalent. Then, using examples constructed by Klüppelberg and Villasenor [The full solution of the convolution closure problem for convolution-equivalent distributions, J. Math. Anal. Appl., 41:79–92, 1991], we show that \( S^* \) is not closed under convolution.

Keywords: class of strong subexponential distributions, class of subexponential distributions, convolution closure.

1 Introduction and the main result

Throughout the paper, we will say that a distribution \( F \) is on \( \mathbb{R} := (-\infty, \infty) \) if \( \overline{F}(x) := 1 - F(x) > 0 \) for all \( x \); we will say that a distribution \( F \) is on \( \mathbb{R}_+ := [0, \infty) \) if its support is contained in \( \mathbb{R}_+ := [0, \infty) \) and \( \overline{F}(x) > 0 \) for all \( x \). For two positive functions \( a(x) \) and \( b(x) \), we write \( a(x) \sim b(x) \) if \( \lim_{x \to \infty} a(x)/b(x) = 1 \), we write \( a(x) \asymp b(x) \) if \( 0 < \liminf_{x \to \infty} a(x)/b(x) \leq \limsup_{x \to \infty} a(x)/b(x) < \infty \). For any two distribution \( F \) and \( G \), by \( F \ast G \) we denote their convolution:

\[
F \ast G(x) = \int_{-\infty}^{\infty} F(x - y) \, dG(y), \quad x \in \mathbb{R}.
\]

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We say that distribution \( F \) on \( \mathbb{R} \) belongs to the class of \textit{long-tailed} distributions, denoted \( \mathcal{L} \), if its right tail \( F = 1 - F \) satisfies
\[
\lim_{x \to \infty} \frac{F(x-y)}{F(x)} = 1
\]
for any \( y > 0 \). We say that distribution \( F \) on \( \mathbb{R} \) belongs to the \textit{subexponential} class of distributions, denoted \( \mathcal{S} \), if \( F \in \mathcal{L} \) and
\[
\lim_{x \to \infty} \frac{F \ast F(x)}{F(x)} = 2.
\]
(1)
The class of distributions, characterized by (1), was introduced by Chistyakov [2] and later, in more general setup, by Athreya and Ney [1] and Chover et al. [3, 4].

A distribution \( F \) on \( \mathbb{R} \) is said to belong to the \textit{strong subexponential} class of distributions, introduced by Klüppelberg [9] and denoted \( \mathcal{S}^* \), if \( \mu_F = \int_0^\infty F(y) \, dy \in (0, \infty) \) and
\[
\lim_{x \to \infty} \frac{1}{F(x)} \int_0^x F(x-y) F(y) \, dy = 2 \mu_F.
\]
(2)
The properties of class \( \mathcal{S}^* \) and related classes were studied in [6, Sect. 3.4], [8–14], [18–20], and other papers. In particular, it is well known that, under \( \mu_F < \infty \), it holds that \( \mathcal{S}^* \subset \mathcal{S} \subset \mathcal{L} \).

In the following theorem, we present equivalent conditions for the convolution \( F \ast G \) to be in the class \( \mathcal{S}^* \) under the initial assumption \( F, G \in \mathcal{L} \).

\textbf{Theorem 1.} Suppose that \( F \) and \( G \) are two distributions on \( \mathbb{R} \). Let \( F, G \in \mathcal{L} \). Then the following statements are equivalent:
\begin{enumerate}
    \item \( F \ast G \in \mathcal{S}^* \),
    \item \( FG \in \mathcal{S}^* \),
    \item \( pF + (1-p)G \in \mathcal{S}^* \) for some \( 0 < p < 1 \),
    \item \( pF + (1-p)G \in \mathcal{S}^* \) for all \( 0 < p < 1 \).
\end{enumerate}
Moreover, any of these equivalent statements implies the relations
\[
\frac{F \ast G(x)}{F(x)} \sim \frac{F(x)}{F(x)} + \frac{G(x)}{F(x)},
\]
(3)
\[
\int_0^x F(x-y) G(y) \, dy \sim \mu_G F(x) + \mu_F G(x),
\]
(4)
where \( \mu_F := \int_0^\infty F(y) \, dy, \mu_G := \int_0^\infty G(y) \, dy \).

In the corollary below the assumption \( F, G \in \mathcal{L} \) of Theorem 1 is replaced by a stricter assumption \( F, G \in \mathcal{S}^* \). In this case, the asymptotic relation (4) is equivalent to any of statements (i)–(iv) of Theorem 1 (see also [7, Thm. 3], which refers to [10]).

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Corollary 1. Suppose that $F$ and $G$ are two distributions on $\mathbb{R}$. Let $F, G \in \mathcal{S}^*$. Then any of statements (i)–(iv) of Theorem 1 is equivalent to (4).

Proof. We need only to prove that (4) implies (iii) of Theorem 1. Obviously,

$$I := \int_0^x \frac{1}{2}(F + G)(x - y) \frac{1}{2}(F + G)(y) \, dy$$

$$= \frac{1}{4} \int_0^x F(x - y) F(y) \, dy + \frac{1}{2} \int_0^x F(x - y) G(y) \, dy + \frac{1}{4} \int_0^x G(x - y) G(y) \, dy$$

$$=: I_1 + I_2 + I_3.$$

Since $F, G \in \mathcal{S}^*$, by relation (4), we get

$$\limsup_{x \to \infty} \frac{I}{(\mu_F + \mu_G)(F(x) + G(x))} \leq \limsup_{x \to \infty} \max \left\{ \frac{I_1}{\mu_F F(x) + \mu_G G(x)}, \frac{I_2}{\mu_F G(x)}, \frac{I_3}{\mu_G G(x)} \right\} \leq \frac{1}{2}.$$

Similarly, we obtain that

$$\liminf_{x \to \infty} \frac{I}{(\mu_F + \mu_G)(F(x) + G(x))} \geq \frac{1}{2}.$$

The derived estimates imply that

$$\int_0^x \frac{1}{2}(F + G)(x - y) \frac{1}{2}(F + G)(y) \, dy$$

$$\sim (\mu_F + \mu_G) \frac{1}{2}(F + G)(x) = 2\mu_{(F+G)/2}(F + G)/2(x),$$

and, consequently, $(F + G)/2 \in \mathcal{S}^*$ by definition. \hfill \Box

We use Theorem 1 for proving main Theorem 2 on the convolution non-closure of class $\mathcal{S}^*$. Indeed, by Theorem 1, for distributions $F, G \in \mathcal{S}^*$, their convolution $F \ast G$ is not in $\mathcal{S}^*$ if and only if $pF + (1 - p)G$ is not in $\mathcal{S}^*$ for some $p \in (0, 1)$. Hence, in order to prove Theorem 2, we construct two distributions $F$ and $G$ such that $F/2 + G/2 \notin \mathcal{S}^*$ or, equivalently, $F \ast G \notin \mathcal{S}^*$.

Theorem 2. Distribution class $\mathcal{S}^*$ is not closed under convolution, i.e. there exist distributions $F, G \in \mathcal{S}^*$ such that $F \ast G \notin \mathcal{S}^*$.

Remark 1. The first counterexample for the closure of the subexponential class with respect to convolution was provided by Leslie [16]. Another counterexample for the closure of the convolution equivalent class of distributions with respect to convolution was given a few years later by Klüppelberg and Villasenor [10].
2 Proof of Theorem 1

2.1 Auxiliary lemmas

Before proving our main result, we state two auxiliary lemmas.

Lemma 1. Suppose that \( F \) and \( G \) are two distributions on \( \mathbb{R} \).

(i) If \( F \in \mathcal{S}^* \), \( G \in \mathcal{L} \), and \( G(x) \sim F(x) \), then \( G \in \mathcal{S}^* \).

(ii) If \( F \in \mathcal{L} \), \( G \in \mathcal{L} \), then \( F \ast G \in \mathcal{L} \).

(iii) If \( F \in \mathcal{L} \), \( G \in \mathcal{L} \), and \( F \ast G \in \mathcal{S} \), then \( F \ast G(x) \sim F(x) + G(x) \).

Proof. For the proof of part (i), see [9, Thm. 2.1(b)] (see also [6, Thm. 3.25]). For part (ii), see [5, Thm. 3(b)] (see also [6, Cor. 2.42] or [17, Lemma 4.2]). Part (iii) can be found in Theorem 1.1 of Leipus and Šiaulys [15].

Lemma 2. Suppose \( F \) is distribution on \( \mathbb{R} \) with finite \( \mu_F \). The following statements are equivalent:

(i) \( F \in \mathcal{S}^* \).

(ii) \( F \in \mathcal{L} \) and

\[
\lim_{v \to \infty} \limsup_{x \to \infty} \int_v^x \frac{F(x-y)F(y)}{F(x)} \, dy = 0.
\]

Proof. An equivalent assertion by choosing a special form function instead of the additional variable \( v \) is given in [6, Them. 3.24]. For the sake of completeness, we briefly present the main steps of the lemma proof. According to considerations in [6] (see the proof of Theorem 3.24), [7] (see the proof of Lemma 4), and [9] (see the proof of Theorem 3.2(b)), the assertion of the lemma follows from the estimate

\[
\int_0^{x/2} \frac{F(x-z)F(z)}{F(x)} \, dz \geq \int_0^y \frac{F(z)}{F(x)} \, dz + \frac{F(x-y)}{F(x)} \int_y^{x/2} F(z) \, dz, \quad x > 2y > 0,
\]

implying that

\[
1 \leq \frac{F(x-y)}{F(x)} \leq \left( \frac{F(x)}{F(x)} \right)^{-1} \left( \int_0^{x/2} F(x-z)F(z) \, dz - \int_0^y F(z) \, dz \right) \frac{F(x-y)}{F(x)} \int_0^{x/2} F(z) \, dz - \int_0^y F(z) \, dz,
\]

and from equality

\[
\int_0^x F(x-z)F(z) \, dz = 2 \int_0^v F(x-z)F(z) \, dz + \int_v^{x-v} F(x-z)F(z) \, dz,
\]

where \( x > 2v > 0 \).
2.2 Proof of the theorem

(ii) ⇒ (i) Assume that $FG \in \mathcal{I}^*$. Lemma 1(ii) implies $F \ast G \in \mathcal{L}$, thus the proof will follow from

$$F \ast G(x) \asymp FG(x) \quad (5)$$

and Lemma 1(i). To prove (5), assume that $X$ and $Y$ are independent random variables with distributions $F$ and $G$, correspondingly, and write

$$F \ast G(x) = P(X + Y > x) \geq P\left(\{X > x, Y > 0\} \cup \{X > 0, Y > x\}\right)$$
$$= \mathcal{G}(0) F(x) + F(0) \mathcal{G}(x) - F(x) G(x),$$
$$FG(x) = F(x) + \mathcal{G}(x) - F(x) \mathcal{G}(x).$$

Thus,

$$\liminf_{x \to \infty} \frac{F \ast G(x)}{FG(x)} \geq \liminf_{x \to \infty} \frac{\mathcal{G}(0) F(x) + F(0) \mathcal{G}(x) - F(x) G(x)}{F(x) + \mathcal{G}(x) - F(x) \mathcal{G}(x)} \geq \min\{F(0), \mathcal{G}(0)\} > 0. \quad (6)$$

On the other hand,

$$\limsup_{x \to \infty} \frac{F \ast G(x)}{FG(x)} = \limsup_{x \to \infty} \frac{P(X_1 + Y_2 > x)}{P(X \lor Y > x)} \leq \limsup_{x \to \infty} \frac{P(X_1 \lor Y_1 + X_2 \lor Y_2 > x)}{P(X \lor Y > x)}, \quad (7)$$

where $(X_1, Y_1)$ and $(X_2, Y_2)$ are independent copies of $(X, Y)$.

Since, by (ii), $F_{X \lor Y} \in \mathcal{I}^* \subset \mathcal{I}$, we have

$$P(X_1 \lor Y_1 + X_2 \lor Y_2 > x) \sim 2P(X \lor Y > x). \quad (8)$$

Hence, by (6)–(8),

$$\min\{F(0), \mathcal{G}(0)\} \leq \liminf_{x \to \infty} \frac{F \ast G(x)}{FG(x)} \leq \limsup_{x \to \infty} \frac{F \ast G(x)}{FG(x)} \leq 2$$

and (5) follows.

(i) ⇒ (ii) Let $F \ast G \in \mathcal{I}^*$. Since $\mathcal{I}^* \subset \mathcal{I}$, by Lemma 1(iii),

$$F \ast G(x) \sim F(x) + \mathcal{G}(x) \sim FG(x),$$

which further implies $FG \in \mathcal{L}$ by the above second equivalence. Therefore, $FG \in \mathcal{I}^*$ follows from Lemma 1(i) immediately.

(ii) ⇔ (iii) ⇔ (iv) follows because of Lemma 1(i).
Finally, relation (3) holds by Lemma 1(iii). It remains to prove relation (4). First, observe that any of the equivalent statements in (i)--(iv) from Theorem 1 implies the existence of finite $\mu_F$ and $\mu_G$. Further, for $M > 0$ and $x > 2M$, we have

\[
\int_0^x F(x - y)G(y) \, dy = \int_0^M F(x - y)G(y) \, dy + \int_M^{x-M} F(x - y)G(y) \, dy + \int_0^M F(y)G(x - y) \, dy
\]

\[=: J_1 + J_2 + J_3.\]

Therefore,

\[
\liminf_{x \to \infty} \frac{\int_0^x F(x - y)G(y) \, dy}{\mu_G F(x) + \mu_F G(x)} \geq \liminf_{x \to \infty} \frac{J_1 + J_3}{\mu_G F(x) + \mu_F G(x)} \\
\geq \liminf_{x \to \infty} \min \left\{ \frac{J_1}{\mu_G F(x)}, \frac{J_3}{\mu_F G(x)} \right\} \\
\geq \min \left\{ \frac{\int_0^M G(y) \, dy}{\mu_G}, \frac{\int_0^M F(y) \, dy}{\mu_F} \right\}. \tag{9}
\]

Letting $M \to \infty$, we get from (9) that

\[
\liminf_{x \to \infty} \frac{\int_0^x F(x - y)G(y) \, dy}{\mu_G F(x) + \mu_F G(x)} \geq 1. \tag{10}
\]

For the corresponding upper bound, we obtain

\[
\limsup_{x \to \infty} \frac{\int_0^x F(x - y)G(y) \, dy}{\mu_G F(x) + \mu_F G(x)} \leq \limsup_{x \to \infty} \max \left\{ \frac{J_1}{\mu_G F(x)}, \frac{J_3}{\mu_F G(x)} \right\} \\
+ \limsup_{x \to \infty} \frac{J_2}{\mu_G F(x) + \mu_F G(x)}. \tag{11}
\]

By condition $F \in \mathcal{L}$, we get

\[
\limsup_{x \to \infty} \frac{J_1}{\mu_G F(x)} \leq \limsup_{x \to \infty} \frac{F(x - M)}{F(x)} \frac{1}{\mu_G} \int_0^M G(y) \, dy = \frac{1}{\mu_G} \int_0^M G(y) \, dy.
\]

Now, letting $M \to \infty$, we obtain

\[
\lim_{M \to \infty} \limsup_{x \to \infty} \frac{J_1}{\mu_G F(x)} \leq 1. \tag{12}
\]

Similarly, condition $G \in \mathcal{L}$ implies

\[
\lim_{M \to \infty} \limsup_{x \to \infty} \frac{J_3}{\mu_F G(x)} \leq 1. \tag{13}
\]
Further, according to Theorem 1(iii), we have that \((F + G)/2 \in \mathcal{S}^*\). Hence, due to Lemma 2,

\[
\lim_{M \to \infty} \limsup_{x \to \infty} \frac{J_2}{\mu_G F(x) + \mu_F G(x)} \leq \frac{1}{\min\{\mu_F, \mu_G\}} \lim_{M \to \infty} \limsup_{x \to \infty} \frac{\int_{M}^{x-M} F(x-y) G(y) \, dy}{F(x) + G(x)}
\]

\[
\leq \frac{2}{\min\{\mu_F, \mu_G\}} \lim_{M \to \infty} \limsup_{x \to \infty} \int_{M}^{x-M} \frac{1/2(F + G)(x-y)1/2(F + G)(y)}{1/2(F + G)(x)} \, dy
\]

\[
= 0.
\]

Estimates (11)–(14) imply

\[
\limsup_{x \to \infty} \int_{0}^{x} \frac{F(x-y) G(y)}{\mu_G F(x) + \mu_F G(x)} \, dy \leq 1.
\]

Hence, the desired relation (4) of the theorem follows immediately from (10) and (15). Theorem 1 is proved.

\section{Proof of Theorem 2}

\subsection{Auxiliary lemmas}

In this subsection, we present two additional lemmas, which play a crucial role in the proof of Theorem 2. The statement of the first lemma is similar to that in Lemma 2. Note that equivalent condition for \(F \in \mathcal{S}^*\) does not require additional condition \(F \in \mathcal{L}\), comparing to Lemma 2.

\lemmas[3]{F\text{ is distribution of } \mathbb{R} \text{ such that } \mu_F < \infty. Then } F \in \mathcal{S}^* \text{ if and only if }

\[
\lim_{x \to \infty} \int_{0}^{x/2} F(x-y) - F(x) \frac{F(y)}{F(x)} \, dy = 0.
\]

\proof The proof is similar to the proof of Lemma 3 from [10]. Obviously, equality (2) is equivalent to

\[
\int_{0}^{x/2} \frac{F(x-y) F(y)}{F(x)} \, dy = \mu_F.
\]

Thus,

\[
F \in \mathcal{S}^* \iff \lim_{x \to \infty} \int_{0}^{x/2} \frac{F(x-y) F(y)}{F(x)} \, dy = \lim_{x \to \infty} \int_{0}^{x/2} F(y) \, dy
\]

\[
\iff \lim_{x \to \infty} \int_{0}^{x/2} \left( \frac{F(x-y)}{F(x)} - 1 \right) F(y) \, dy = 0. \qed
\]
The second lemma is a technical result about behaviour of the special sequences.

**Lemma 4.** Let \( \{a_n, n \geq 1\} \) be an unboundedly increasing sequence of positive numbers, and let
\[
h_n := \max\{ k: (k + 1)! \leq a_n (\log a_n)^\beta \}
\]
with some positive \( \beta > 0 \). Then, for all sufficiently large \( n \),
\[
\frac{\log a_n}{\log \log a_n} \leq h_n \leq \frac{2 \log a_n}{\log \log a_n}.
\]

**(16)**

**Proof.** The proof is constructed along to similar lines as in Lemma 5 from [10]. Namely, the Stirling’s formula implies that
\[
\log(k + 1)! = (k + 1) \log(k + 1) - (k + 1) - O(\log k)
\]
for \( k \to \infty \). Define
\[
\hat{h}_n = \frac{2 \log a_n}{\log \log a_n}.
\]

For some positive constant \( c_1 \) and for sufficiently large \( n \), we have
\[
\log(\hat{h}_n + 1)! \geq (\hat{h}_n + 1) \log(\hat{h}_n + 1) - (\hat{h}_n + 1) - c_1 \log \hat{h}_n \geq \frac{9}{10} \hat{h}_n \log \hat{h}_n
\]
\[
= \frac{9}{5} \log a_n (\log a_n)^\beta \frac{\log a_n}{\log \log a_n} \frac{\log 2 + \log \log a_n}{\log a_n + \beta \log \log a_n}
\]
\[
\geq \log a_n (\log a_n)^\beta,
\]
which implies the upper bound in (16).

Similarly, using Stirling’s formula again, for
\[
\tilde{h}_n = \frac{\log a_n}{\log \log a_n},
\]
we obtain
\[
\log(\tilde{h}_n + 1)! \leq \tilde{h}_n \log \tilde{h}_n + c_2 \log \tilde{h}_n
\]
\[
= \log a_n \left( 1 - \frac{\log \log a_n}{\log \log a_n} \right) \left( 1 + \frac{c_2}{\tilde{h}_n} \right)
\]
with some positive \( c_2 \) and sufficiently large \( n \). Therefore, for large \( n \),
\[
\log(\tilde{h}_n + 1)! \leq \log \left( a_n (\log a_n)^\beta \right),
\]
which implies the lower bound in (16). Lemma is proved. \( \square \)
### 3.2 Proof of the theorem

Define two distributions $\mathcal{F}$ and $\mathcal{G}$ with tails:

$$
\mathcal{F}(x) := 1_{(-\infty, 6!)}(x) + (6!)^2 \left\{ \sum_{n=6}^{\infty} \frac{1}{(n+1)!} \left[ 1_{[n!, (n+1)!-b_n d_n]}(x) \right] \right\},
$$

$$
\mathcal{G}(x) := 1_{(-\infty, 8!)}(x) + (8!)^2 \left\{ \sum_{n=3}^{\infty} \frac{1}{((2n+1)!)^2} \left[ 1_{[(2n+1)!, (2n+1)!-\hat{b}_n \hat{d}_n]}(x) \right] \right\},
$$

where $b_n := n^2 + 2n$, $d_n := (\log b_n)^3$, $\hat{b}_n = b_{2n} = 2^n (2^n + 2)$, and $\hat{d}_n = (\log \hat{b}_n)^2$. The functions above are constructed according to the scheme presented in [10] and [16].

Because of Theorem 1, it suffices to prove that $\mathcal{F}, \mathcal{G} \in \mathcal{S}^*$ and $(\mathcal{F} + \mathcal{G})/2 \notin \mathcal{S}^*$. According to Lemma 3, we have to prove the following relations:

$$
\mu_{\mathcal{F}} < \infty, \quad \mu_{\mathcal{G}} < \infty, \quad (17)
$$

$$
\limsup_{x \to \infty} \frac{x}{2} \int_{0}^{x/2} \frac{\mathcal{F}(x-y) - \mathcal{F}(x)}{\mathcal{F}(x)} \mathcal{F}(y) \, dy = 0, \quad (18)
$$

$$
\limsup_{x \to \infty} \frac{x}{2} \int_{0}^{x/2} \frac{\mathcal{G}(x-y) - \mathcal{G}(x)}{\mathcal{G}(x)} \mathcal{G}(y) \, dy = 0, \quad (19)
$$

$$
\limsup_{x \to \infty} \frac{x}{2} \int_{0}^{x/2} \frac{(\mathcal{F} + \mathcal{G})(x-y) - (\mathcal{F} + \mathcal{G})(x)}{(\mathcal{F} + \mathcal{G})(x)} (\mathcal{F} + \mathcal{G})(y) \, dy > 0. \quad (20)
$$

Denote

$$
\Delta_{\mathcal{F}}(x, y) := \frac{\mathcal{F}(x-y) - \mathcal{F}(x)}{\mathcal{F}(x)}, \quad \Delta_{\mathcal{G}}(x, y) := \frac{\mathcal{G}(x-y) - \mathcal{G}(x)}{\mathcal{G}(x)}.
$$

**Proof of (17).** According to definitions of $\mathcal{F}(x)$ and $\mathcal{G}(x)$,

$$
\mu_{\mathcal{F}} = \int_{0}^{\infty} \frac{\mathcal{F}(y)}{y} \, dy \leq 6! + 6! \sum_{n=3}^{\infty} \frac{1}{(n!)^2} ((n+1)! - n!) < 1238,
$$

**Proof of (18).**

**Proof of (19).**

**Proof of (20).**

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\[ \mu_G = \int_0^\infty \mathcal{G}(y) \, dy \leq 8! + 8! \sum_{n=3}^{\infty} \frac{1}{((2^n)!)^2}((2^n + 1)! - (2^n)!) \]

\[ + 8! \sum_{n=3}^{\infty} \left( \frac{1}{(2^n + 1)!} - \frac{1}{(2^{n+1})!} \right) < 98243, \]

implying (17).

\[ \square \]

**Proof of (18).** Suppose that \( n \) is sufficiently large and let

\[(n + 1)! - b_n d_n \leq x < (n + 1)!. \]  \hspace{1cm} (21)

For such \( x \), we have

\[ \Delta \mathcal{F}(x, y) = \begin{cases} \frac{y}{d_n} & \text{if } 0 \leq y \leq x - ((n + 1)! - b_n d_n), \\ \frac{x - ((n + 1)! - b_n d_n)}{d_n + (n + 1)! - x} & \text{if } x - ((n + 1)! - b_n d_n) < y \leq \frac{x}{2}. \end{cases} \]

Therefore, for \( x \) in (21), we have

\[ J \mathcal{F}(x) := \int_0^{x/2} \Delta \mathcal{F}(x, y) \mathcal{F}(y) \, dy \]

\[ \leq \frac{1}{d_n} \int_0^{b_n d_n} y \mathcal{F}(y) \, dy + b_n \int_{b_n d_n}^{\infty} \mathcal{F}(y) \, dy =: K_1 + K_2. \]  \hspace{1cm} (22)

Define \( k_n := \max\{k: (k + 1)! \leq b_n d_n \} \) and write

\[ K_1 \leq \frac{(6!)^2}{2d_n} \left\{ 1 + \sum_{k=6}^{k_n+1} \frac{1}{(k!)^2} \left( ((k + 1)! - b_k d_k)^2 - (k!)^2 \right) \\ + \sum_{k=6}^{k_n+1} \frac{1}{((k + 1)!)^2} \left( 1 + \frac{(k + 1)!}{d_k} \right) \left( ((k + 1)!)^2 - ((k + 1)! - b_k d_k)^2 \right) \right\} \]

\[ \leq \frac{(6!)^2}{2d_n} \left( 1 + 3 \sum_{k=6}^{k_n+1} b_k \right) = \frac{(6!)^2}{2d_n} \left( 1 + 3 \sum_{k=6}^{k_n+1} (k^2 + 2k) \right) \]  \hspace{1cm} (23)

because

\[ \frac{1}{(k!)^2} \left( ((k + 1)! - b_k d_k)^2 - (k!)^2 \right) \leq (k + 1)^2 - 1 = b_k \]

and

\[ \left( 1 + \frac{(k + 1)!}{d_k} \right) \left( ((k + 1)!)^2 - ((k + 1)! - b_k d_k)^2 \right) \leq 2((k + 1)!)^2 b_k. \]
Thus, by (23) and Lemma 4,
\[ K_1 \leq (6!)^2 k_n^3 d_n \leq (6!)^2 \left( \frac{2 \log b_n}{\log \log b_n} \right)^3 \frac{1}{(\log b_n)^3} = \frac{8(6!)^2}{(\log \log b_n)^3}. \] (24)

For the second integral in (22), we have
\[ K_2 \leq b_n (6! - b_n d_n)^+ + b_n \sum_{k=k_n+1}^{\infty} \frac{6!}{(k!)^2} ((k+1)! - k!) \]
\[ = b_n (6! - b_n d_n)^+ + b_n 6! \sum_{k=k_n}^{\infty} \frac{1}{k!} \]
\[ \leq b_n (6! - b_n d_n)^+ + \frac{24(6!)}{(\log \log b_n)^2 \log b_n} \] (25)
because of the following estimate:
\[ b_n \sum_{k=k_n}^{\infty} \frac{1}{k!} \leq \frac{e \cdot b_n d_n}{k_n! d_n} \leq \frac{e \cdot (k_n + 2)!}{k_n! d_n} \]
\[ \leq \frac{2e k_n^2}{d_n} \leq \frac{6 \left( \frac{2 \log b_n}{\log \log b_n} \right)^2}{(\log b_n)^3} \frac{1}{(\log \log b_n)^3} \]
\[ = \frac{24 (6!)}{(\log \log b_n)^2 \log b_n}. \]

Here we have used that, by definition of \( k_n \), \( b_n d_n \leq (k_n + 2)! \leq 2k_n! \) and then applied Lemma 4. Substituting estimates (24)–(25) into (22), we get that for \( x \) from (21), it holds
\[ J_{\mathcal{F}}(x) \leq \frac{c_1}{(\log \log b_n)^3} \] (26)
for some positive constant \( c_1 \).

Now, consider \( x \) satisfying
\[ (n + 1)! \leq x < (n + 2)! - b_{n+1} d_{n+1}. \] (27)

We split this interval into three subintervals
\[ (n + 1)! \leq x < 2((n + 1)! - b_n d_n), \] (28)
\[ 2((n + 1)! - b_n d_n) \leq x < 2(n + 1)!, \] (29)
\[ 2(n + 1)! \leq x < (n + 2)! - b_{n+1} d_{n+1} \] (30)
and estimate \( J_{\mathcal{F}}(x) \) in each case separately.

In case (28), we have
\[ \Delta_{\mathcal{F}}(x, y) = \begin{cases} \frac{x}{d_n} & \text{if } 0 \leq y \leq x - (n + 1)!, \\
\frac{(n+1)! - x + y}{d_n} & \text{if } x - (n + 1)! < y \leq x - ((n + 1)! - b_n d_n), \\
b_n & \text{if } x - ((n + 1)! - b_n d_n) < y \leq \frac{x}{2}. \end{cases} \]
Since
\[
\frac{(n+1)! - x + y}{d_n} \leq \min \left\{ \frac{y}{d_n}, b_n \right\},
\]
for the \(x\) from (28) and for \(x - (n+1)! < y \leq x - ((n+1)! - b_n d_n)\), we get that \(J_F(x) \leq K_1 + K_2\) and estimate (26) holds again.

Consider now case (29). We have
\[
\Delta_F(x, y) = \begin{cases} 
0 & \text{if } 0 \leq y \leq x - (n+1)!, \\
\frac{(n+1)! - x + y}{d_n} & \text{if } x - (n+1)! < y \leq \frac{x}{2}.
\end{cases}
\]
Thus,
\[
J_F(x) = \int_{x-(n+1)!}^{x/2} \frac{(n+1)! - x + y}{d_n} F(y) \, dy \\
\leq b_n \int_{b_n d_n}^{\infty} F(y) \, dy = K_2 \leq \frac{c_2}{\log b_n (\log \log b_n)^2},
\]
according to estimate (25), where \(c_2\) is some positive constant.

Finally, in case (30), \(\Delta_F(x, y) = 0\) for all \(0 \leq y \leq x/2\), implying \(J_F(x) = 0\).

Summarizing, estimate (26) holds for all \(x\) in (27) and for all sufficiently large \(n\). This implies relation (18).

\[\square\]

Proof of (19). Suppose that \(n\) is sufficiently large and split the interval
\[
(2^n)! \leq x < (2^{n+1})!
\]
into following subintervals:
\[
(2^n)! \leq x < 2(2^n)!, \quad (2^n + 1)! - \hat{b}_n \hat{d}_n \leq x < (2^n + 1)!, \quad 2((2^n + 1)! - \hat{b}_n \hat{d}_n) \leq x < 2(2^n + 1)!, \quad 2(2^n + 1)! \leq x < (2^{n+1})!.
\]
As in the case of \(F\), for each subset above, we will obtain the exact expressions for \(\Delta_G(x, y)\) and then, the upper bounds for \(J_G(x)\).

In case (31),
\[
\Delta_G(x, y) = \begin{cases} 
0 & \text{if } 0 \leq y \leq x - (2^n)!, \\
\frac{((2^n)!)^2 - (x-y)^2}{(x-y)^2} & \text{if } x - (2^n)! < y \leq \frac{x}{2},
\end{cases}
\]

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and, consequently,

\[ J_G(x) \leq \int_0^{x/2} \left( \left( \frac{x}{x-y} \right)^2 - 1 \right) \overline{G}(y) \, dy. \]

Since \((x/(x-y))^2 \leq 4\), by the dominated convergence theorem, we have that

\[
\sup_{(2^n)! \leq x < 2(2^n)!} J_G(x) = \epsilon_1(n) \to 0, \quad n \to \infty.
\] (37)

In case (32), \(\Delta_G(x, y) = 0\) for \(0 \leq y \leq x/2\), implying

\[
\sup_{2(2^n)! \leq x < (2^n+1)!-\hat{b}_n \hat{d}_n} J_G(x) = 0.
\] (38)

In case (33), we have

\[
\Delta_G(x, y) = \begin{cases} 
\frac{y}{(2^n+1)!+\hat{d}_n-x} & \text{if } 0 \leq y < x - ((2^n+1)! - \hat{b}_n \hat{d}_n), \\
\frac{x-((2^n+1)!-\hat{b}_n \hat{d}_n)}{(2^n+1)!+\hat{d}_n-x} & \text{if } x - ((2^n+1)! - \hat{b}_n \hat{d}_n) \leq y < \frac{x}{2},
\end{cases}
\]

implying that

\[
J_G(x) \leq \int_0^{x/2} \min \left\{ \frac{y}{\hat{d}_n}, \hat{b}_n \right\} \overline{G}(y) \, dy \leq \frac{1}{\hat{d}_n} \int_0^{\hat{d}_n \hat{b}_n} y \overline{G}(y) \, dy + \hat{b}_n \int_{\hat{d}_n \hat{b}_n}^{\infty} \overline{G}(y) \, dy
\]

\[=: L_1 + L_2.\]

Analogously to \(k_n\), define \(\hat{k}_n := \max\{k: (2^k+1)! \leq \hat{b}_n \hat{d}_n\}\). We get

\[
L_1 \leq \frac{1}{\hat{d}_n} \int_0^{\hat{d}_n \hat{b}_n} y \left\{ 1_{(-\infty,8!]}(y) + (8!)^2 \sum_{k=3}^{\infty} \frac{1}{((2^k+1)!)^2} \left[ (2^k+1)!-(2^k+1)!\hat{b}_k \hat{d}_k \right](y) \right. \\
+ \left. (8!)^2 \sum_{k=3}^{\infty} \frac{1}{(2^k+1)!} \left[ (2^k+1)!\hat{b}_k \hat{d}_k - (2^k+1)!\right]\right\} \, dy \\
\leq \frac{(8!)^2}{2\hat{d}_n} + \frac{(8!)^2 \hat{k}_n+1}{\hat{d}_n} \sum_{k=3}^{\infty} \frac{1}{(2^k+1)!(2^k)!} \int_{(2^k)!}^{(2^k+1)!} y \, dy + \frac{(8!)^2}{\hat{d}_n} \int_0^{9!} \frac{1}{y} \, dy \\
\leq \frac{(8!)^2}{2\hat{d}_n} + \frac{(8!)^2 \hat{k}_n+1}{\hat{d}_n} \log(\hat{b}_n \hat{d}_n) + \frac{(8!)^2}{2\hat{d}_n} \sum_{k=3}^{\hat{k}_n+1} \hat{b}_k = \epsilon_2(n) \to 0, \quad n \to \infty,
\] (39)
because for sufficiently large $n$,
\[
\sum_{k=3}^{\hat{k}_n+1} \hat{b}_k \leq \frac{16}{3} 2^{2\hat{k}_n} + 8 \cdot 2^{2\hat{k}_n} \leq 6 \cdot 2^{2\hat{k}_n} \leq \frac{24(\log \hat{b}_n)^2}{(\log \log \hat{b}_n)^2}
\]
due to Lemma 4.

For the integral $L_2$, we obtain
\[
L_2 \leq \hat{b}_n \left( (8! - \hat{b}_n \hat{d}_n)^+ + (8!)^2 \sum_{k=\hat{k}_n+1}^{\infty} \frac{(2^k + 1)! - (2^k)!}{(2^k - 1)!} + (8!)^2 \int_{\hat{b}_n \hat{d}_n}^\infty \frac{dy}{y^2} \right)
\]
\[
\leq \hat{b}_n \left( (8! - \hat{b}_n \hat{d}_n)^+ + (8!)^2 \sum_{k=\hat{k}_n+1}^{\infty} \frac{1}{(2^k - 1)!} + \frac{(8!)^2}{\hat{b}_n \hat{d}_n} \right)
\]
\[
\leq \hat{b}_n (8! - \hat{b}_n \hat{d}_n)^+ + \frac{(8!)^2 \hat{b}_n}{(2^{\hat{k}_n+1} - 1)!} + \frac{(8!)^2}{\hat{d}_n} = \epsilon_3(n) \to 0, \quad n \to \infty,
\]
(40)
because $(2^{\hat{k}_n+1} - 1)! > \hat{b}_n \hat{d}_n$ for large $n$, according to definition of the sequence $\hat{k}_n$.

Relations (39)–(40) imply that
\[
\sup_{2(2^n)! \leq x < (2^n+1)! - b_n \hat{d}_n} J_G(x) \leq \epsilon_2(n) + \epsilon_3(n) \to 0, \quad n \to \infty.
\]
(41)

In case (34), we obtain
\[
\Delta_G(x, y) = \begin{cases} 
\frac{y(2x-y)}{(x-y)^2} & \text{if } 0 \leq y \leq x - (2^n+1)!, \\
\frac{x^2(\hat{d}_n+(2^n+1)!-y+x+y)}{\hat{d}_n((2^n+1)!)^2} & \text{if } x - (2^n+1)! < y \leq x - ((2^n+1)! - \hat{b}_n \hat{d}_n), \\
\left( \frac{x}{(2^n)!} \right)^2 - 1 & \text{if } x - ((2^n+1)! - \hat{b}_n \hat{d}_n) < y \leq \frac{x}{2}.
\end{cases}
\]
(42)

Hence,
\[
J_G(x) = \int_0^{x-(2^n+1)!} \frac{y(2x-y)}{(x-y)^2} \mathcal{G}(y) \, dy + \int_{x-(2^n+1)!}^{x-(2^n+1)! - \hat{b}_n \hat{d}_n} \left( \frac{x^2(\hat{d}_n+(2^n+1)!-x+y)}{\hat{d}_n((2^n+1)!)^2} - 1 \right) \mathcal{G}(y) \, dy
\]
\[
+ \int_{x-(2^n+1)!}^{x/2} \left( \left( \frac{x}{(2^n)!} \right)^2 - 1 \right) \mathcal{G}(y) \, dy.
\]
(43)
In particular case of (34), when \((2^n + 1)! \leq x < (2^n + 1)! + \hat{b}_n \hat{d}_n\), by estimating the above integrals separately, we get

\[
J_G(x) \leq \int_0^\infty \left( \frac{x}{x-y} \right)^2 1_{[0,x/2]}(y) \mathcal{G}(y) \, dy - \int_0^\infty \mathcal{G}(y) \, dy + \epsilon_4(n)
\]

\[
+ \frac{2}{d_n} \int_0^\infty y \mathcal{G}(y) \, dy + 2\hat{b}_n \int \mathcal{G}(y) \, dy + \hat{b}_n \int \mathcal{G}(y) \, dy
\]

for some vanishing function \(\epsilon_4(n)\). Thus, for large \(n\) and for all \(x \in [(2^n + 1)!, (2^n + 1)! + \hat{b}_n \hat{d}_n)\), we have that

\[
J_G(x) \leq \epsilon_1(n) + \epsilon_4(n) + 3(\epsilon_2(n) + \epsilon_3(n)) \rightarrow 0. \tag{44}
\]

For the remaining subinterval of (34), where \((2^n + 1)! + \hat{b}_n \hat{d}_n \leq x < 2((2^n + 1)! - \hat{b}_n \hat{d}_n)\), using expressions (42) and (43), we obtain

\[
J_G(x) \leq \int_0^\infty \left( \frac{x}{x-y} \right)^2 1_{[0,x/2]}(y) \mathcal{G}(y) \, dy - \int_0^\infty \mathcal{G}(y) \, dy + 3\hat{b}_n \int \mathcal{G}(y) \, dy
\]

\[
\leq \epsilon_1(n) + 3 \epsilon_3(n) \rightarrow 0. \tag{45}
\]

Relations (44) and (45) imply that

\[
\sup_{2(2^n + 1)! \leq x < 2((2^n + 1)! - \hat{b}_n \hat{d}_n)} J_G(x) \leq \epsilon_5(n) \tag{46}
\]

with some vanishing function \(\epsilon_5\).

Consider now case (35). For such \(x\),

\[
\Delta_G(x, y) = \begin{cases} 
\frac{y(2x-y)}{(x-y)^2} & \text{if } 0 \leq y \leq x - (2^n + 1)!, \\
\frac{x^2(\hat{d}_n + (2^n + 1)! - x + y)}{d_n((2^n + 1)!)^2} - 1 & \text{if } x - (2^n + 1)! < y \leq \frac{x}{2},
\end{cases}
\]

implying that

\[
J_G(x) = \int_0^{x-(2^n + 1)!} \frac{y(2x-y)}{(x-y)^2} \mathcal{G}(y) \, dy
\]

\[
+ \int_{x-(2^n + 1)!}^{x/2} \left( \frac{x^2(\hat{d}_n + (2^n + 1)! - x + y)}{\hat{d}_n((2^n + 1)!)^2} - 1 \right) \mathcal{G}(y) \, dy.
\]
In the case under consideration, we have that \( x - (2^n + 1)! \geq \hat{b}_n \hat{d}_n \) and

\[
\frac{x^2(\hat{d}_n + (2^n + 1)! - x + y)}{\hat{d}_n((2^n + 1)!)^2} - 1 \leq \frac{\hat{d}_n}{\hat{d}_n + b_n d_n} \leq \frac{\hat{d}_n + b_n d_n}{\hat{d}_n} \leq 5\hat{b}_n.
\]

The derived estimates yield

\[
J_{G}(x) \leq \int_{0}^{\infty} \left( \frac{x}{x - y} \right)^2 1_{[0,x/2]}(y) \mathcal{G}(y) \, dy - \int_{0}^{\infty} \mathcal{G}(y) \, dy + 5L_2,
\]

which implies that

\[
\sup_{2((2^n + 1)! - \hat{b}_n \hat{d}_n) \leq x < 2(2^n + 1)!} J_{G}(x) \leq \epsilon_1(n) + 5\epsilon_3(n). \tag{47}
\]

Finally, consider case (36). For these \( x \) and for all \( 0 \leq y \leq x/2 \),

\[
\Delta_{G}(x, y) = \frac{y(2x - y)}{(x - y)^2}.
\]

Thus,

\[
\sup_{2(2^n + 1)! \leq x < (2^n + 1)!} J_{G}(x) \leq \epsilon_1(n). \tag{48}
\]

The derived estimates (37), (38), (41), (46), (47), and (48) imply that

\[
\lim_{n \to \infty} \sup_{(2^n)! \leq x < (2^n + 1)!} J_{G}(x) \to 0,
\]

showing the validity of (19).

It remains to prove inequality (20). Integral from this inequality is bounded from below by

\[
J_{F,G}(x) := \int_{0}^{x/2} \frac{\mathcal{G}(x - y) - \mathcal{G}(x)}{\mathcal{G}(x)} \mathcal{F}(y) \, dy.
\]

Take \( x_n := (2^n + 1)! \). Then \( \mathcal{F}(x_n) = \mathcal{G}(x_n) = 1/x_n^2 \), implying that

\[
J_{F,G}(x_n) = \frac{1}{2} \int_{0}^{x_n/2} \frac{\mathcal{G}(x_n - y) - \mathcal{G}(x_n)}{\mathcal{G}(x_n)} \mathcal{F}(y) \, dy
\]

\[
\geq \frac{1}{2} \int_{0}^{\Delta_{G}(x_n, y)} \mathcal{F}(y) \, dy
\]

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for large $n$. According to (42),

$$\Delta G(x_n, y) = \frac{\hat{d}_n}{d_n} + \frac{y}{d_n} \geq \frac{y}{d_n}.$$  

Consequently, denoting $\tilde{k}_n = \max\{k: (k + 1)! \leq \hat{b}_n \hat{d}_n\}$, we get

$$J_{F, G}(x_n) \geq \frac{1}{2\hat{d}_n} \int_0^\infty yF(y) \, dy \geq \frac{1}{2\hat{d}_n} \int_0^\infty y\left\{1_{(-\infty,6!]}(y) + (6!)^2 \sum_{k=6}^\infty \frac{1}{(k!)^2} 1_{[k!(k+1)!-b_k d_k]}(y) + (6!)^2 \sum_{k=6}^\infty \frac{1}{(k+1)!} \left(1 + \frac{(k+1)! - y}{d_k}\right) 1_{[(k+1)!-b_k d_k, (k+1)!]}(y)\right\} \, dy \geq \frac{(6!)^2}{4\hat{d}_n} \sum_{k=6}^{\tilde{k}_n} \frac{(k+1)! - b_k d_k}{(k!)^2} - (k!)^2$$

$$= \frac{(6!)^2}{4\hat{d}_n} \sum_{k=6}^{\tilde{k}_n} \left(\frac{k+1}{k!} - \frac{b_k d_k}{k!}\right)^2 - 1 \geq \frac{(6!)^2}{4\hat{d}_n} \left(\sum_{k=6}^{\tilde{k}_n} k^2 - 2 \sum_{k=6}^{\tilde{k}_n} \frac{k+1}{k!} b_k d_k\right).$$

Since the series

$$\sum_{k=6}^\infty \frac{k+1}{k!} b_k d_k$$

converges, we have that

$$J_{F, G}(x_n) \geq c_2 \left(\frac{\tilde{k}_n^3}{\hat{d}_n} - c_3\right)$$

for large $n$ with some positive constants $c_2$ and $c_3$. As $\hat{b}_n \hat{d}_n = \hat{b}_n (\log \hat{b}_n)^2$, applying Lemma 4 with $a_n = \hat{b}_n$ and $\beta = 2$ to the sequence $\tilde{k}_n$, we get

$$\frac{\tilde{k}_n^3}{\hat{d}_n} \geq \frac{\log \hat{b}_n}{(\log \log \hat{b}_n)^3} \to \infty, \quad n \to \infty.$$

Therefore,

$$\lim_{n \to \infty} J_{F, G}(x_n) = \infty,$$

and the desired inequality (20) follows. Theorem 2 is proved.
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References


