Nontrivial solutions for an asymptotically linear $\Delta_\alpha$-Laplace equation

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Abstract. In this paper, we study a class of degenerate unperturbed problems. We first investigate some properties of eigenvalues and eigenfunctions for the strongly degenerate elliptic operator and then obtain two existence theorems of nontrivial solutions when the nonlinearity is a function with an asymptotically condition.

Keywords: asymptotically linear, saddle point theorem, strongly degenerate elliptic operator.

1 Introduction

Our aim in this paper is to study the following nonlinear elliptic equation:

$$\begin{align*}
-\Delta_\alpha u - \lambda u &= f(x, u), \quad x \in \Omega, \\
u &= 0, \quad x \in \partial \Omega,
\end{align*}$$

(1)

where $\Omega$ is a smooth bounded domain of $\mathbb{R}^N$ ($N > 2$), $\lambda$ is a parameter, and $\Delta_\alpha$ is a strongly degenerate elliptic operator of the form

$$\Delta_\alpha := \sum_{i=1}^{N} \partial_{x_i} \left( \alpha_i^2 \partial_{x_i} \right), \quad \alpha = (\alpha_1, \ldots, \alpha_N) : \mathbb{R}^N \to \mathbb{R}^N.$$
The strong degenerate elliptic operator $\Delta_\alpha$ was first introduced in [4], and the authors [5] remarked that $\Delta_\alpha$-Laplacian belong to the more general class of $X$-elliptic operators.

The $\Delta_\alpha$ operator contains many degenerate elliptic operators such as the Grushin-type operator $G_\alpha = \Delta_x + |x|^{2a} \Delta_y$, $a > 0$, where $(x, y)$ denotes the point of $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, $N_1 + N_2 = N$, and the operator of the form $P_{\alpha,b,c} = \Delta_x + |x|^{2a} \Delta_y + |x|^{2b} |y|^{2c} \Delta_z$, $(x, y, z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3}$, $N_1 + N_2 + N_3 = N$, where $a$, $b$ and $c$ are real positive constants. We refer readers to [1] for some important properties of this operator.

Many authors considered (1) with $\lambda = 0$, i.e.,

$$-\Delta_\alpha u = f(x, u), \quad x \in \Omega,$$

$$u = 0, \quad x \in \partial \Omega.$$  \hspace{1cm} (2)

In [1] the authors used the mountain pass theorem and the fountain theorem to study the existence and multiplicity of solutions for (2), where $f$ satisfies a superlinear growth condition, and in [17] the authors examined the case where (2) has a nontrivial solution using sub-super solutions and variational methods. In [6] the authors adopted the three critical point theorem to consider the case where (2) has at least two solutions, and in [3, 15] the authors investigated the existence of infinitely many solutions when $f$ satisfies a general superlinear growth condition. For more research to this kinds of equations, we also refer the readers to [7–14, 16, 19, 20] and the references therein.

Following [4], we denote by $W^{1,2}_\alpha(\Omega)$ the closure of $C^1_0(\Omega)$ with respect to the norm $\|u\|_{W^{1,2}_\alpha(\Omega)} = (\int_\Omega |\nabla_\alpha u|^2 \, dx)^{1/2}$, which is a Hilbert space with the inner product $\langle u, v \rangle = \int_\Omega \nabla_\alpha u \cdot \nabla_\alpha v \, dx$. Here $\nabla_\alpha = (\alpha_1 \partial_{x_1} u, \ldots, \alpha_N \partial_{x_N} u)$. For convenience, we abbreviate the norm $\|\cdot\|_{W^{1,2}_\alpha(\Omega)}$ as $\|\cdot\|$, and let $|\cdot|_q$ be the usual norm in the Lebesgue space $L^q(\Omega)$.

In order to study the asymptotically linear problem, we first present eigenvalues properties for $\Delta_\alpha$. We note that the author in [6] presented some properties for this operator, but the author did not provide proofs. For completeness, we first study the eigenvalue problem associated with (1),

$$-\Delta_\alpha u = \lambda u, \quad x \in \Omega,$$

$$u = 0, \quad x \in \partial \Omega,$$  \hspace{1cm} (3)

where $\lambda \in \mathbb{R}$ is the eigenvalue of the problem if there exists $u \in W^{1,2}_\alpha \setminus \{0\}$ such that (3) holds. Denote by $\sigma(-\Delta_\alpha)$ and $0 < \lambda_1 < \cdots < \lambda_k < \cdots$ the spectrum and the distinct eigenvalues of $-\Delta_\alpha$ in $W^{1,2}_\alpha(\Omega)$, respectively.

We note that problem (3) is equivalent to

$$\int_\Omega \nabla_\alpha u \cdot \nabla_\alpha v \, dx = \lambda \int_\Omega uv \, dx, \quad u, v \in W^{1,2}_\alpha.$$  \hspace{1cm} (4)

**Theorem 1.** Let $\Omega$ be an open bounded set of $\mathbb{R}^N$. Then the eigenvalues and eigenfunctions of $\Delta_\alpha$ have the following properties:
(i) Problem (4) has a positive eigenvalue \( \lambda_1 \), and its characteristic is

\[
\lambda_1 = \min_{\|u\|_2 = 1} \int_{\Omega} |\nabla_\alpha u|^2 \, dx
\]

or, equivalently,

\[
\lambda_1 = \min_{u \in W^{1,2}_\alpha} \frac{\int_\Omega |\nabla_\alpha u|^2 \, dx}{\int_\Omega |u|^2 \, dx}.
\]

(ii) There exists a positive function \( e_1 \in W^{1,2}_\alpha \), which is an eigenfunction corresponding to \( \lambda_1 \), attaining the minimum in (4), i.e., \( |e_1|_2 = 1 \) and

\[
\lambda_1 = \int_\Omega |\nabla_\alpha e_1|^2 \, dx.
\]

(iii) The first eigenvalue \( \lambda_1 \) is simple, i.e., if \( u \in W^{1,2}_\alpha \) is a solution of the following equation

\[
\int_\Omega \nabla_\alpha u \cdot \nabla_\alpha v \, dx = \lambda_1 \int_\Omega uv \, dx \quad \forall v \in W^{1,2}_\alpha,
\]

then \( u = \xi e_1 \) with \( \xi \in \mathbb{R} \).

(iv) The set of eigenvalues of (4) consists of a sequence \( \{\lambda_k\}_{k \in \mathbb{N}} \) with

\[
0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \lambda_{k+1} \leq \cdots
\]

and

\[
\lambda_k \to +\infty \quad \text{as} \quad k \to +\infty.
\]

Moreover, for any \( k \in \mathbb{N} \), the eigenvalues can be characterized as follows:

\[
\lambda_{k+1} = \min_{\|u\|_2 = 1, u \in \mathbb{P}_{k+1}} \int_{\Omega} |\nabla_\alpha u|^2 \, dx
\]

or, equivalently,

\[
\lambda_{k+1} = \min_{u \in \mathbb{P}_{k+1} \setminus \{0\}} \frac{\int_\Omega |\nabla_\alpha u|^2 \, dx}{\int_\Omega |u|^2 \, dx},
\]

where

\[
\mathbb{P}_{k+1} := \{ u \in W^{1,2}_\alpha \ s.t. \ \langle u, e_j \rangle = 0 \ \forall j = 1, \ldots, k \}.
\]

(v) For any \( k \in \mathbb{N} \), there exists a function \( e_{k+1} \in \mathbb{P}_{k+1} \), which is an eigenfunction corresponding to \( \lambda_{k+1} \), attaining the minimum in (9), i.e., \( |e_{k+1}|_2 = 1 \) and

\[
\lambda_{k+1} = \int_\Omega |\nabla_\alpha u|^2 \, dx.
\]
(vi) The sequence \( \{ e_k \} \) of eigenfunctions corresponding to \( \lambda_k \) is an orthonormal basis of \( L^2(\Omega) \) and orthogonal basis of \( W^{1,2}_0 \).

(vii) Each eigenvalue \( \lambda_k \) has finite multiplicity. More precisely, if \( \lambda_k \) satisfies

\[
\lambda_{k-1} < \lambda_k = \cdots = \lambda_{k+h} < \lambda_{k+h+1}
\]

for some \( h \in \mathbb{N}_0 \), then the set of all the eigenfunctions corresponding to \( \lambda_k \) agree with \( \text{span}\{e_k \cdots e_{k+h}\} \).

Now, on the basis of the above theorem, we study the existence and multiplicity of solutions for (1). For the nonlinear term \( f \) and \( \lambda \), we consider the following assumptions:

\begin{align*}
(f_1) & \ f \text{ is a Carathéodory function, and } \sup_{|t| \leq r} |f(\cdot, t)| \in L^\infty(\Omega) \text{ for all } r > 0. \\
(f_2) & \ \lim_{|t| \to +\infty} f(x, t)/t = 0 \text{ uniformly with respect to a.e. } x \in \Omega. \\
(f_3) & \ \lim_{t \to 0} f(x, t)/t = \lambda_0 \in \mathbb{R} \text{ uniformly with respect to a.e. } x \in \Omega. \\
(\lambda_1) & \ \lambda \notin \sigma(-\Delta_\alpha). \\
(\lambda_2) & \ (A_2) \text{ There exist } h, k \in \mathbb{N} \text{ with } k \geq h \text{ such that } \lambda_0 + \lambda < \lambda_h \leq \lambda_k \leq \lambda.
\end{align*}

We use the saddle point theorem and the pseudoindex theory introduced in [2] to discuss the existence and multiplicity of solutions for (1). Next, we state the main results:

**Theorem 2.** Assume that the nonlinearity \( f(x, u) \) satisfies \( (f_1) \), \( (f_2) \) and \( \lambda \) satisfies \( (\lambda_1) \). Then equation (1) has at least a nontrivial weak solution.

**Theorem 3.** Assume that the nonlinearity \( f(x, u) \) satisfies \( (f_1) \), \( (f_2) \) and \( (f_3) \) and \( \lambda \) satisfies \( (\lambda_1) \), \( (\lambda_2) \). Then equation (1) has at least \( k - h + 1 \) distinct pairs of nontrivial weak solutions.

**Remark 1.** Note that we were motivated partly by Theorem 3.1 in [2]. Here the nonlinearity is no longer superlinear, and we use the saddle point theorem to establish the existence of a solution. Also, we present eigenvalue properties of the operator \( \Delta_\alpha \).

## 2 Preliminaries

We recall the functional setting in [3, 4]. Consider the operator of the form \( \Delta_\alpha := \sum_{i=1}^N \partial_{x_i} (\alpha_i^2 \partial_{x_i}) \), where \( \partial_{x_i} = \partial/\partial x_i \), \( i = 1, \ldots, N \). Here the function \( \alpha_i : \mathbb{R}^N \to \mathbb{R} \) is continuous, strictly positive and of \( \mathcal{C}^1 \) outside the coordinate hyperplane, i.e., \( \alpha_i \geq 0 \), \( i = 1 \ldots N \), in \( \mathbb{R}^N \setminus \Pi \), where \( \Pi = \{(x_1, \ldots, x_N) \in \mathbb{R}^N : \Pi_{x_i=1} x_i = 0\} \). As in [4], we assume that \( \alpha_i \) satisfy the following properties:

\begin{enumerate}
  \item[(i)] \( \alpha_1(x) \equiv 1 \), \( \alpha_i(x) = \alpha_i(x_1, \ldots, x_{i-1}) \), \( i = 1, \ldots, N \).
  \item[(ii)] For every \( x \in \mathbb{R}^N \), \( \alpha_i(x) = \alpha_i(x^*) \), \( i = 1, \ldots, N \), where \( x^* = (|x_1|, \ldots, |x_N|) \) if \( x = (x_1, \ldots, x_N) \).
  \item[(iii)] There exists a constant \( \rho \geq 0 \) such that \( 0 \leq x_k \partial_{x_k} \alpha_i(x) \leq \rho \alpha_i(x) \) for all \( k \in \{1, \ldots, i-1\} \), \( i = 2, \ldots, N \), and for every \( x \in \mathbb{R}^N_+ := \{(x_1, \ldots, x_N) \in \mathbb{R}^N : x_i \geq 0 \ \forall i = 1, \ldots, N\} \).
\end{enumerate}
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(iv) There exists a group of dilations $\{\delta_t\}_{t>0}$, $\delta_t : \mathbb{R}^N \to \mathbb{R}^N$, $\delta_t(x) = \delta_t(x_1, \ldots, x_N) = (t^{\epsilon_1}x_1, \ldots, t^{\epsilon_N}x_N)$, where $1 \leq \epsilon_1 \leq \epsilon_2 \leq \cdots \leq \epsilon_N$ such that $\alpha_i$ is $\delta_t$-homogeneous of degree $\epsilon_i - 1$, i.e., $\alpha_i(\delta_t(x)) = t^{\epsilon_i-1}\alpha_i(x)$ for all $x \in \mathbb{R}^N$, $t > 0$, $i = 1, \ldots, N$. This implies that the operation $\Delta_\alpha$ is $\delta_t$-homogeneous of degree two, i.e., $\Delta_\alpha(u(\delta_t(x))) = t^2(\Delta_\alpha u)(\delta_t(x))$ for all $u \in C^{\infty}(\mathbb{R}^N)$.

We denote by $Q$ the homogeneous dimension of $\mathbb{R}^N$ with respect to the group of dilations $\{\delta_t\}_{t>0}$, i.e., $Q := \epsilon_1 + \cdots + \epsilon_N$. The homogeneous $Q$ plays a crucial role, both in the geometry and in the functional associated with the operator $\Delta_\alpha$.

**Proposition.** (See [4].) Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ ($N > 2$). Then the embedding $W^{1,2}_\alpha(\Omega) \hookrightarrow L^p(\Omega)$ is compact for every $p \in [1, 2_\alpha^*)$, where $2_\alpha^* = 2Q/(Q - 2)$.

**Remark 2.** For all $s \in [1, 2_\alpha^*)$, there exists a positive constant $C_s$ such that

$$\|u\|_s \leq C_s \|u\|.$$  \hfill (13)

**Theorem 4.** (See [18].) Let $E = V \oplus X$, where $E$ is a real Banach space, and $V \neq \{0\}$ is finite dimensional. Suppose that $I \in C^1(E, \mathbb{R})$ satisfies the (PS)-condition, and let

(i) there is a constant $\alpha$ and a bounded neighborhood $D$ of 0 in $V$ such that $I|_{\partial D} \leq \alpha$;

(ii) there is a constant $\beta > \alpha$ such that $I|_X \geq \beta$.

Then $I$ possess a critical value $c \geq \beta$. Moreover, $c$ can be characterized as $c = \inf_{h \in \Gamma} \max_{u \in \overline{D}} I(h(u))$, where $\Gamma = \{h \in C(\overline{D}, E) : h = id$ on $\partial D\}$.

Let $X$ be a Banach space,

$$\Sigma = \Sigma(X) = \{A \subset X : A \text{ closed and symmetric w.r.t. the origin,}
\quad \text{i.e., } -u \in A \text{ if } u \in A\}$$

and $\mathcal{H} = \{h \in C(X, X) : h \text{ odd}\}$. Taking $A \in \Sigma$, $A \neq \emptyset$, the genus of $A$ is $\gamma(A) = \inf\{k \in \mathbb{N}^* : \exists \psi(\psi(-u)) = -\psi(u) \forall u \in A\}$ if such an infimum exists, otherwise, $\gamma(A) = +\infty$. Assume that $\gamma(\emptyset) = 0$.

**Theorem 5.** (See [2].) Let $H$ be a real Hilbert space, $J \in C^1(H, \mathbb{R})$ an even functional, $(\Sigma, \mathcal{H}, \gamma)$ an index theory on $H$. Let $S \in \Sigma$ and consider the pseudoindex theory $(S, \mathcal{H}^*, \gamma^*)$, where $\mathcal{H}^* = \{h \in \mathcal{H} : h \text{ bounded homeomorphism s.t. } h(u) = u \text{ if } u \notin J^{-1}(0, +\infty]\}$, and $\gamma^* = \min_{h \in \mathcal{H}^*} \gamma(h(A) \cap S)$ for all $A \in \Sigma$. Taking $a, b, c_0, c_\infty \in \overline{R}$, $-\infty \leq a < c_0 < c_\infty < b \leq +\infty$, we assume that:

(i) the functional $J$ satisfies $(PS)$-condition in $(a, b)$;

(ii) $S \subset J^{-1}([c_0, +\infty[$);

(iii) there exist an integer $\overline{k} \geq 1$ and $\overline{A} \in \Sigma$ such that $\overline{A} \subset J^{-\infty}$ and $\gamma^*(\overline{A}) \geq \overline{k}$.

Then the numbers $c_i = \inf_{A \in \Sigma_i} \sup_{u \in A} J(u)$, $i \in \{1, \ldots, \overline{k}\}$, with $\Sigma_i = \{A \in \Sigma : \gamma^* \geq i\}$ are critical values for $J$ and $c_0 \leq c_1 \leq \cdots \leq c_{\overline{k}} \leq c_\infty$. Furthermore, if $c = c_i = \cdots = c_{i+r}$ with $i \geq 1$ and $i + r \leq \overline{k}$, then $\gamma(K_c) \geq r + 1$. 

3 Proof of the theorems

Before proving Theorem 1, we define the functional $I : W^{1,2}_\alpha \to \mathbb{R}$ as follows:

$$I(u) = \frac{1}{2} \int_\Omega |\nabla_\alpha u|^2 \, dx$$

and

$$\langle I'(u), v \rangle = \int_\Omega \nabla_\alpha u \cdot \nabla_\alpha v \, dx = \langle u, v \rangle.$$

In order to obtain Theorem 1, we prove the following lemmas.

**Lemma 1.** If $\mathcal{A} \neq \emptyset$ is a weakly closed subspace of $W^{1,2}_\alpha$ and $\mathcal{M} := \{ u \in \mathcal{A} : |u|_2 = 1 \}$, then there exists $u_0 \in \mathcal{M}$ such that

$$\min_{u \in \mathcal{M}} I(u) = I(u_0)$$

and

$$\langle u_0, v \rangle = \int_\Omega \nabla_\alpha u_0 \cdot \nabla_\alpha v \, dx = \lambda_0 \int_\Omega u_0 v \, dx \quad \forall v \in \mathcal{A},$$

where $\lambda_0 := 2I(u_0) > 0$.

**Proof.** Let $\{u_j\}$ be the minimization sequence of $I$ on $\mathcal{M}$, i.e., a sequence $u_j \in \mathcal{M}$ is such that

$$I(u_j) \to \inf_{u \in \mathcal{M}} I(u) \geq 0 > -\infty$$

as $j \to +\infty$. Then $I(u_j)$ is bounded in $\mathbb{R}$. From the definition of $I$ we have that $\|u_j\|$ is bounded.

Note that $W^{1,2}_\alpha$ is a reflexive Banach space, and we have a subsequence still denoted as $u_j$ and $u_j \rightharpoonup u_0$ in $W^{1,2}_\alpha$ for some $u_0 \in \mathcal{A}$. Thus,

$$\int_\Omega \nabla_\alpha u_j \cdot \nabla_\alpha v \, dx \to \int_\Omega \nabla_\alpha u_0 \cdot \nabla_\alpha v \, dx \quad \forall v \in W^{1,2}_\alpha; \quad j \to +\infty.$$

From $\|u_j\|$ bounded and the embedding theorem we have $u_j \to u_0$ in $L^2(\Omega)$ as $j \to \infty$. Then $\|u_0\|_2 = 1$, $u_0 \in \mathcal{M}$. By the weak lower semicontinuity we have

$$\lim_{j \to +\infty} I(u_j) = \frac{1}{2} \lim_{j \to +\infty} \int_\Omega |\nabla_\alpha u_j|^2 \, dx \geq \frac{1}{2} \int_\Omega |\nabla_\alpha u_0|^2 \, dx$$

$$= I(u_0) \geq \inf_{u \in \mathcal{M}} I(u).$$

Therefore, from (16) $I(u_0) = \inf_{u \in \mathcal{M}} I(u)$. Hence, (14) is established.
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Next, we let $\varepsilon \in (-1, 1)$, $v \in \mathcal{M}$, $c_\varepsilon := |u_0 + \varepsilon v|_2$ and $u_\varepsilon := (u_0 + \varepsilon v)/c_\varepsilon$. We have that $u_\varepsilon \in \mathcal{M}$, and according to $|u_0|_2 = 1$, we get

$$2J(u_\varepsilon) = \int_{\Omega} |\nabla_{\alpha} u_\varepsilon|^2 \, dx = \int_{\Omega} \left| \nabla_{\alpha} \frac{u_0 + \varepsilon v}{c_\varepsilon} \right|^2 \, dx = \frac{|u_0 + \varepsilon v|^2}{c_\varepsilon^2}$$

$$= \frac{\langle u_0 + \varepsilon v, u_0 + \varepsilon v \rangle}{(\int_{\Omega} |u_0 + \varepsilon v|^2 \, dx)^2} = \frac{|u_0|^2 + 2\varepsilon \langle u_0, v \rangle + o(\varepsilon)}{|u_0|^2 + 2\varepsilon \int_{\Omega} u_0(x)v(x) \, dx + o(\varepsilon)}$$

$$= (2I(u_0) + 2\varepsilon \langle u_0, v \rangle + o(\varepsilon)) \left( 1 - 2 \int_{\Omega} u_0(x)v(x) \, dx + o(\varepsilon) \right)$$

$$= 2I(u_0) + 2\varepsilon \left( \langle u_0, v \rangle - 2I(u_0) \int_{\Omega} u_0(x)v(x) \, dx \right) + o(\varepsilon).$$

Note the minimum value of $u_0$, and we have (15). The proof is complete.

**Lemma 2.** Let $\lambda \neq \bar{\lambda}$ be different eigenvalues of problem (4) with eigenfunctions $e$ and $\bar{e} \in W^{1,2}_\alpha$. Then

$$\langle e, \bar{e} \rangle = 0 = \int_{\Omega} e(x)\bar{e}(x) \, dx.$$

**Proof.** If $e = 0$ or $\bar{e} = 0$, then the proof is complete. Now we consider the case when $e \neq 0$ and $\bar{e} \neq 0$. First, consider the characteristic function $f := e/|e|_2$ and $\bar{f} := \bar{e}/|\bar{e}|_2$. Substitute $f$, $\bar{f}$ into (4), and we have

$$\lambda \int_{\Omega} f(x)\bar{f}(x) \, dx = \int_{\Omega} |\nabla_{\alpha} f| \cdot |\nabla_{\alpha} \bar{f}| \, dx = \bar{\lambda} \int_{\Omega} f(x)\bar{f}(x) \, dx,$$

and then

$$(\lambda - \bar{\lambda}) \int_{\Omega} f(x)\bar{f}(x) \, dx = 0.$$

Note that $\lambda \neq \bar{\lambda}$, we obtain

$$\int_{\Omega} f(x)\bar{f}(x) \, dx = 0.$$  \hspace{1cm} (18)

Combine (17) and (18), and we get

$$\left\langle \frac{e}{|e|_2}, \frac{\bar{e}}{|\bar{e}|_2} \right\rangle = \langle f, \bar{f} \rangle = \int_{\Omega} |\nabla_{\alpha} f| \cdot |\nabla_{\alpha} \bar{f}| \, dx = 0.$$

Thus, $\langle e, \bar{e} \rangle = 0$ is established. The proof is complete.

**Lemma 3.** If $e$ is an eigenfunction of (4), the corresponding eigenvalue is $\lambda$, then

$$\int_{\Omega} |\nabla_{\alpha} e|^2 \, dx = \lambda |e|_2^2.$$
Proof. In (4), replacing \( v \) by \( e \), we obtain
\[
\int_{\Omega} |\nabla_\alpha e|^2 \, dx = \lambda \int_{\Omega} e^2 \, dx.
\]
The proof is complete. \( \square \)

Proof of Theorem 1. (i) According to Lemma 1 (choosing \( A := W^{1,2}_\alpha \)), we obtain that there is a \( \lambda_1 \), i.e.,
\[
\lambda_1 = \min_{|u|_2 = 1} \int_{\Omega} |\nabla_\alpha u|^2 \, dx.
\]
Moreover, it is an eigenvalue.

(ii) For this, we get that \( e_1 \) is an eigenfunction corresponding to \( \lambda_1 \) by (15). Hence (with \( A := W^{1,2}_\alpha \) in Lemma 1), (6) is established. It can be seen from (14) that the minimum \( \lambda_1 \) is attained at some \( e_1 \in W^{1,2}_\alpha \), where \( |e_1|_2 = 1 \). To complete the proof of (ii), we first show that if \( e \) is an eigenfunction corresponding to \( \lambda_1 \), with \( |e|_2 = 1 \), then both \( e \) and \( |e| \) attain the minimum in (5), also, either \( e \geq 0 \) or \( e \leq 0 \) a.e. in \( \Omega \). From Lemma 3 and (6) we obtain
\[
2I(e) = \int_{\Omega} |\nabla_\alpha e|^2 \, dx = \lambda_1 = 2I(e_1).
\]
Also, we get \( I(|e|) = I(e) = I(e_1) \), where \( |e| \in W^{1,2}_\alpha \) and \( \|e\|^2 = \lambda_1 \), and either \( \{e > 0\} \) or \( \{e < 0\} \) has zero measure. Hence, by replacing \( e \) with \( e_1 \), we obtain that \( e_1 \geq 0 \). Thus, there exists a function \( e_1 \in W^{1,2}_\alpha \) with \( e_1 \geq 0 \) and is an eigenfunction relative to \( \lambda_1 \), attaining the minimum in (4).

(iii) Assume that \( \lambda_1 \) also corresponds to another eigenfunction \( u \) in \( W^{1,2}_\alpha \) with \( 0 \neq u \) and \( u \neq e_1 \). It follows from the proof of (ii) that \( u \geq 0 \) or \( u \leq 0 \) a.e. in \( \Omega \). First, consider the case \( u \geq 0 \) a.e. in \( \Omega \). We set
\[
g := \frac{u}{|u|_2}, \quad g_1 := e_1 - g.
\]
Next, we prove that
\[
g_1(x) = 0 \quad \text{a.e.} \ x \in \Omega.
\]
Suppose that \( g_1(x) \neq 0 \) a.e. \( x \in \Omega \), and we can conclude that \( g_1 \) is an eigenfunction corresponding to \( \lambda_1 \). Using the proof of (ii) again, we get that \( g_1 \geq 0 \) or \( g_1 \leq 0 \) a.e. in \( \Omega \). Thus, either \( e_1 \geq g \) or \( e_1 \leq g \). From \( e_1 \geq 0 \) we have one of the following:
\[
e_1^2 \geq g^2 \quad \text{or} \quad e_1^2 \leq g^2 \quad \text{a.e. in} \ \Omega.
\]
Also,
\[
\int_{\Omega} (e_1^2(x) - g^2(x)) \, dx = |e_1|_2^2 - |g|_2^2 = 0.
\]
According to the above, we get $e_1^2 - g^2 = 0$, so $e_1 = g$. Hence, $g_1 = 0$ a.e. in $\Omega$. That is a contradiction, so (19) is established. Therefore, $f_1$ is proportional to $e_1$, i.e. $u = \xi e_1$, $\xi \in \mathbb{R}$. The situation when $u \leq 0$ a.e. in $\Omega$ is similar.

(iv) By Lemma 1 (choosing $A := \mathbb{P}_{k+1}$) we see that there exists $\lambda_{k+1}$ such that (9) holds, and it is attained at some $e_{k+1} \in \mathbb{P}_{k+1}$. Also, from $\mathbb{P}_{k+1} \subseteq \mathbb{P}_{k} \subseteq W_{\alpha}^{1,2}$ we have

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \lambda_{k+1}.$$ 

First, we prove (7). In fact, we show $\lambda_1 \neq \lambda_2$. Indeed, if $\lambda_1 = \lambda_2$ and $e_2 \in \mathbb{P}_2$ also is an eigenfunction relative to $\lambda_1$, from (iii) we get that $e_2 = \xi e_1$ with $\xi \in \mathbb{R}$ and $\xi \neq 0$ so $e_2 \neq 0$. From $e_2 \in \mathbb{P}_2$ we have

$$0 = \langle e_1, e_2 \rangle = \langle e_1, \xi e_1 \rangle = \xi \| e_1 \|^2.$$ 

We conclude that $e_1 = 0$, which is a contradiction.

Now apply (15) with $A = \mathbb{P}_{k+1}$, and we have

$$\int \nabla_{\alpha} e_{k+1} \cdot \nabla_{\alpha} v \, dx = \lambda_{k+1} \int e_{k+1}(x)v(x) \, dx \quad \forall v \in \mathbb{P}_{k+1}. \quad (20)$$

In order to show that $\lambda_{k+1}$ is an eigenvalue with eigenfunction $e_{k+1}$, we need to show that the above formula holds for any $v \in W_{\alpha}^{1,2}$ not only in $\mathbb{P}_{k+1}$. We define

$$X_1 = \text{span}\{e_1, \ldots, e_k\}, \quad X_2 = X_1^\perp = \mathbb{P}_{k+1}, \quad W_{\alpha}^{1,2} = X_1 \oplus X_2.$$ 

Hence, for any $v \in W_{\alpha}^{1,2}$, $v := v_1 + v_2$, where $v_2 \in \mathbb{P}_{k+1}$, and $v_1 = \sum_{j=1}^k c_j e_j$ for some $c_1, \ldots, c_k \in \mathbb{R}$. Put $v_2 = v - v_1$ into (20), and with the definition of $v_1$ we deduce

$$\int \nabla_{\alpha} e_{k+1} \cdot \nabla_{\alpha} v \, dx - \lambda_{k+1} \int e_{k+1}(x)v(x) \, dx \quad \Omega$$

$$= \int \nabla_{\alpha} e_{k+1} \cdot \nabla_{\alpha} v_1 \, dx - \lambda_{k+1} \int e_{k+1}(x)v_1(x) \, dx \quad \Omega$$

$$= \sum_{j=1}^k c_j \left[ \int \nabla_{\alpha} e_{k+1} \cdot \nabla_{\alpha} e_j \, dx - \lambda_{k+1} \int e_{k+1}(x)e_j(x) \, dx \right]. \quad (21)$$

Test the eigenvalue equation (4) for $e_j$ against $e_{k+1}$ for $j = 1, \ldots, k$, furthermore, $e_{k+1} \in \mathbb{P}_{k+1}$, and we have

$$0 = \int \nabla_{\alpha} e_{k+1} \cdot \nabla_{\alpha} e_j \, dx = \lambda_j \int e_{k+1}(x)e_j(x) \, dx,$$

so

$$0 = \int \nabla_{\alpha} e_{k+1} \cdot \nabla_{\alpha} e_j \, dx = \int e_{k+1}(x)e_j(x) \, dx \quad \forall j = 1, \ldots, k.$$
Putting the above formula into (21), we deduce that (20) is established for any \( v \in W^{1,2}_\alpha \).

Hence \( \lambda_{k+1} \) is an eigenvalue with eigenfunction \( e_{k+1} \).

Next, in order to obtain (8), we prove that if \( k, h \in \mathbb{N} \), \( k \neq h \), then \( \langle e_k, e_h \rangle = 0 = \int_\Omega e_k(x) e_h(x) \, dx \). In fact, let \( k > h \), then \( k - 1 > h \), and

\[
e_k \in \mathbb{P}_k = \left( \text{span}\{e_1, \ldots, e_{k-1}\} \right)^\perp \subseteq \left( \text{span}\{e_h\} \right)^\perp.
\]

Then \( \langle e_k(x), e_h(x) \rangle = 0 \). However, \( e_k \) is an eigenfunction. We substitute \( e_k \) into (4) and replace \( v \) with \( e_h \), and we have

\[
\int_\Omega \nabla_\alpha e_k \cdot \nabla_\alpha e_h \, dx = \lambda_k \int_\Omega e_k e_h \, dx,
\]

so \( \langle e_k, e_h \rangle = 0 = \int_\Omega e_k(x) e_h(x) \, dx \). To prove \( \lambda_k \to +\infty \), we assume \( \lambda_k \to c, k \to +\infty \) for some \( c \in \mathbb{R} \), so \( \lambda_k \) is bounded in \( \mathbb{R} \). By Lemma 3 we get \( \|e_k\|^2 = \lambda_k \), and there exists a subsequence \( \{e_{k_j}\} \) and some \( e_\infty \in L^2(\Omega) \) with

\[
e_{k_j} \to e_\infty \text{ in } L^2(\Omega) \text{ as } k_j \to \infty.
\]

According to the previous analysis, we see that \( e_{k_j} \) and \( e_{k_i} \) are orthogonal in \( L^2(\Omega) \), and we get

\[
|e_{k_j} - e_{k_i}|^2 = |e_{k_j}|^2 + |e_{k_i}|^2 = 2.
\]

We have a contradiction since \( e_{k_j} \) is a Cauchy sequence in \( L^2(\Omega) \). Thus, (8) is established.

Finally, we show (9). Suppose that there exists an eigenvalue \( \lambda \notin \{\lambda_k\}_{k \in \mathbb{N}} \), and let \( e \in W^{1,2}_\alpha \) be an eigenfunction corresponding to \( \lambda \), so \( |e|_2 = 1 \) is obtained by normalization. According to Lemma 3, we get

\[
2I(e) = \int_\Omega |\nabla_\alpha e|^2 \, dx = \lambda. \tag{22}
\]

Also, by (5) and (6) we have

\[
\lambda = 2I(e) \geq 2I(e_1) = \lambda_1.
\]

From \( \lambda \notin \{\lambda_k\}_{k \in \mathbb{N}} \) and (8) we see that there exists \( k \in \mathbb{N} \) such that

\[
\lambda_k < \lambda < \lambda_{k+1}.
\]

Assume that \( e \in \mathbb{P}_{k+1} \) and (22) and (9) imply that \( \lambda = 2I(e) \geq \lambda_{k+1} \), which is a contradiction. Thus, we have \( e \notin \mathbb{P}_{k+1} \), and there exists \( j \in \{1, \ldots, k\} \) such that \( \langle e, e_j \rangle \neq 0 \), so this contradicts Lemma 2. This completes the proof of (iv).

(v) Apply Lemma 1, let \( A \) be replaced by \( \mathbb{P}_{k+1} \), and the minimum defining \( \lambda_{k+1} \) is attained for some \( e_{k+1} \in \mathbb{P}_{k+1} \). By Lemma 1 we have (11). According to the proof of (iv), we see that (21) holds for any \( v \in W^{1,2}_\alpha \), so we can conclude that \( e_{k+1} \) is an eigenfunction relative to \( \lambda_{k+1} \). This completes the proof of (v).
(vi) From the proof of (iv) we see that the sequence \( \{e_k\}_{k \in \mathbb{N}} \) of eigenfunctions corresponding to \( \lambda_k \) is an orthonormal basis. Next, to complete the proof of (vi), we claim that the \( \{e_k\}_{k \in \mathbb{N}} \) is a basis for both \( W^{1,2}_\alpha \) and \( L^2(\Omega) \).

First, we prove that it is a basis of \( W^{1,2}_\alpha \). We prove that if \( v \in W^{1,2}_\alpha \) is such that for all \( k \in \mathbb{N} \), \( \langle v, e_k \rangle = 0 \), then \( v \equiv 0 \). Assume that \( v \neq 0 \) and there exists a nontrivial \( v \in W^{1,2}_\alpha \) such that for all \( k \in \mathbb{N} \), \( \langle v, e_k \rangle = 0 \), and by normalization we assume that \( |v|_2 = 1 \). Therefore, from (8) there exists a \( k \in \mathbb{N} \) such that

\[
2I(v) < \lambda_{k+1} = \min_{\|u\|_2 = 1} \int_{\Omega} |\nabla u|^2 \, dx.
\]

We see \( v \notin \mathbb{P}_{k+1} \) and there exists \( j \in \mathbb{N} \) such that \( \langle v, e_j \rangle \neq 0 \). This contradicts the assumption. Now we define \( E_i := e_i/\|e_i\| \), and let \( g \in W^{1,2}_\alpha \), \( g_j := \sum_{i=1}^j \langle g, E_i \rangle E_i \). Also, \( g_j \in \text{span}\{e_1, \ldots, e_j\} \) for all \( j \in \mathbb{N} \). Define \( G_j := g - g_j \), and by the orthogonality of \( \{e_k\}_{k \in \mathbb{N}} \) in \( W^{1,2}_\alpha \)

\[
0 \leq \|G_j\|^2 = \langle G_j, G_j \rangle = \langle g - g_j, -g_j \rangle
\]

\[
= \|g\|^2 + \|g_j\|^2 - 2\langle g, g_j \rangle = \|g\|^2 + \langle g, g_j \rangle - 2\left( g, \sum_{i=1}^j \langle g, E_i \rangle E_i \right)
\]

\[
= \|g\|^2 - 2 \sum_{i=1}^j \langle g, E_i \rangle^2.
\]

Then \( 2 \sum_{i=1}^j \langle g, E_i \rangle^2 \leq \|g\|^2 \) for all \( j \in \mathbb{N} \). We deduce that \( \sum_{i=1}^{+\infty} \langle g, E_i \rangle^2 \) is a convergent series. Now we assume that \( \omega_j := \sum_{i=1}^j \langle g, E_i \rangle^2 \), and since \( \omega_j \) is a convergent series, it is a Cauchy sequence in \( \mathbb{R} \). Also, from the orthogonality of \( \{e_k\}_{k \in \mathbb{N}} \) in \( W^{1,2}_\alpha \) we get

\[
\|G_j - G_j \|^2 = \|g_j - g_j \|^2 = \left\| \sum_{i=1}^j \langle g, E_i \rangle E_i \right\|^2
\]

\[
= \sum_{i=1}^j \langle g, E_i \rangle^2 = \omega_j - \omega_j \quad \text{if } J > j.
\]

We obtain that \( G_j \) is a Cauchy sequence in \( W^{1,2}_\alpha \), and by the completeness of \( W^{1,2}_\alpha \) there exists a \( G \in W^{1,2}_\alpha \) such that

\[
G_j \to G \quad \text{in } W^{1,2}_\alpha, \quad j \to \infty.
\]  

Moreover,

\[
\langle G_j, E_k \rangle = \langle g, E_k \rangle - \langle g_j, E_k \rangle = \langle g, E_k \rangle - \langle g, E_k \rangle = 0.
\]

From (23), for any \( k \in \mathbb{N} \), we have \( \langle G, E_k \rangle = 0 \), that is, \( G = 0 \). Hence,

\[
g_j \to g \quad \text{as } j \to \infty \text{ in } W^{1,2}_\alpha.
\]
Finally, we prove that \( \{e_k\}_{k \in \mathbb{N}} \) is a basis for \( L^2(\Omega) \). Let \( v \in L^2(\Omega) \) and \( v_j \in C^1_0(\Omega) \) such that \( |v_j - v|_2 \leq 1/j \). From \( C^1_0(\Omega) \subseteq W^{1,2}_\alpha \) it follows that \( v_j \in W^{1,2}_\alpha \). Since \( \{e_k\}_{k \in \mathbb{N}} \) is a basis for \( W^{1,2}_\alpha \), hence there exists \( k_j \in \mathbb{N} \) and function \( \mu_j \), and \( \mu_j \in \text{span}\{e_1, \ldots, e_{k_j}\} \) such that \( \|v_j - \mu_j\| \leq 1/j \). Thus,

\[
|v_j - \mu_j|_2 \leq C\|v_j - \mu_j\| \leq \frac{C}{j}
\]

and

\[
|v - \mu_j|_2 \leq |v - v_j|_2 + |v_j - \mu_j|_2 \leq \frac{C + 1}{j}.
\]

Therefore, the sequence \( \{e_k\}_{k \in \mathbb{N}} \) of eigenfunctions of (4) is a basis in \( L^2(\Omega) \).

(vii) Consider some \( h \in \mathbb{N}_0 \) such that \( \lambda_{k-1} < \lambda_k = \cdots = \lambda_{k+h} < \lambda_{k+h+1} \). We let

\[
W^{1,2}_\alpha = \text{span}\{e_k, \ldots, e_{k+h}\} \oplus (\text{span}\{e_k, \ldots, e_{k+h}\})^\perp,
\]

and \( \phi = \phi_1 + \phi_2 \), where \( \phi_1 \in \text{span}\{e_k, \ldots, e_{k+h}\} \), and \( \phi_2 \in (\text{span}\{e_k, \ldots, e_{k+h}\})^\perp \). We have

\[
\langle \phi_1, \phi_2 \rangle = 0.
\]

(24)

Since \( \phi \) is an eigenfunction relative to \( \lambda_k \), substituting \( \phi \) into (4), from (24) we have

\[
\|\phi_1\|^2 + \|\phi_2\|^2 = \|\phi\|^2 = \int_\Omega |\nabla_\alpha \phi|^2 \, dx = \lambda_k \int_\Omega \phi^2 \, dx.
\]

(25)

According to (v), we have that \( e_k, \ldots, e_{k+h} \) are eigenfunctions relative to \( \lambda_k = \cdots = \lambda_{k+h} \) and \( \phi \) is also an eigenfunction corresponding to \( \lambda_k \). Hence, substituting both \( \phi_1 \) and \( \phi_2 \) into (4), we have

\[
\lambda_k \int_\Omega \phi_1 \phi_2 \, dx = \int_\Omega |\nabla_\alpha \phi_1| \cdot |\nabla_\alpha \phi_2| \, dx = \langle \phi_1, \phi_2 \rangle = 0 \implies \int_\Omega \phi_1 \phi_2 \, dx = 0
\]

and

\[
|\phi_1|_2^2 = |\phi_1 + \phi_2|_2^2 = |\phi_1|_2^2 + |\phi_2|_2^2.
\]

(26)

Let

\[
\phi_1 = \sum_{i=k}^{k+h} c_j e_j, \quad c_j \in \mathbb{R}.
\]

By (v) and the orthogonality in (vi) we have

\[
\|\phi_1\|^2 = \langle \phi_1, \phi_1 \rangle = \sum_{j=k}^{k+h} \langle c_j e_j, c_j e_j \rangle = \sum_{j=k}^{k+h} c_j^2 \|e_j\|^2 = \lambda_k \sum_{j=k}^{k+h} c_j^2 = \lambda_k |\phi_1|_2^2.
\]

(27)
Since $\phi_1$ and $\phi$ are eigenfunctions corresponding to $\lambda_k$, hence we deduce that $\phi_2$ is also an eigenfunction corresponding to $\lambda_k$. From Lemma 2 and (12) we get
\[
\langle \phi_2, e_1 \rangle = \langle \phi_2, e_2 \rangle = \cdots = \langle \phi_2, e_{k-1} \rangle = 0
\]
and
\[
\phi_2 \in \left( \text{span}\{e_k, \ldots, e_{k+h}\} \right)^\perp = \mathbb{P}_{k+h+1}.
\]
Now we prove that $\phi_2 = 0$ via contradiction. If $\phi_2 \neq 0$, from (10)
\[
\lambda_k < \lambda_{k+h+1} = \min_{u \in \mathbb{P}_{k+h+1} \setminus \{0\}} \frac{\int_\Omega \left| \nabla \alpha u \right|^2 \, dx}{\int_\Omega |u|^2 \, dx} = \frac{\int_\Omega \left| \nabla \alpha \phi_2 \right|^2 \, dx}{\int_\Omega |\phi_2|^2 \, dx} \geq \frac{\|\phi_2\|^2}{\|\phi_2\|^2},
\]
(28)
Also, from (25)–(28) we have
\[
\lambda_k |\phi_2|^2 = \|\phi_1\|^2 + \|\phi_2\|^2 > \lambda_k |\phi_1|^2 + \lambda_k |\phi_2|^2 = \lambda_k |\phi|^2.
\]
This is a contradiction. Therefore, we deduce that $\phi = \phi_1 \in \text{span}\{e_k, \ldots, e_{k+h}\}$. The proof is complete.

Now we define the following energy functional:
\[
J_\lambda(u) = \frac{1}{2} \int _\Omega \left| \nabla \alpha u \right|^2 \, dx - \frac{\lambda}{2} \int _\Omega u^2 \, dx - \int _\Omega F(x, u) \, dx,
\]
where $F(x, u) = \int_0^u f(x, t) \, dt$. From the hypotheses on $f$ we observe that $J_\lambda$ is well defined on $W^{1,2}_\alpha(\Omega)$ and $J_\lambda \in C^1(W^{1,2}_\alpha(\Omega), \mathbb{R})$ with
\[
\langle J'_\lambda(u), v \rangle = \int _\Omega \nabla \alpha u \nabla \alpha v \, dx - \lambda \int _\Omega uv \, dx - \int _\Omega f(x, u)v \, dx \quad \forall v \in W^{1,2}_\alpha.
\]
(29)
To establish Theorems 2 and 3, we first provide the following lemmas, which shows that the (PS)-condition is satisfied.

**Lemma 4.** Let $(f_1)$, $(f_2)$ be satisfied. Then any (PS)-sequence $\{u_j\}$ of $J_\lambda$ is bounded in $W^{1,2}_\alpha$.

**Proof.** Let $\{u_j\} \subset W^{1,2}_\alpha$ be a (PS)-sequence such that
\[
J_\lambda(u_j) \leq c, \quad J'_\lambda(u_j) \to 0,
\]
(30)
and then we have
\[
\langle u_j, \varphi \rangle - \lambda \int _\Omega u_j \varphi \, dx - \int _\Omega f(x, u_j) \varphi \, dx = o(1) \quad \forall \varphi \in W^{1,2}_\alpha.
\]
(31)
From $(f_1)$ and $(f_2)$, for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that
\[
|f(x, t)| \leq \varepsilon |t| + C_\varepsilon \quad \text{a.e. } x \in \Omega \quad \forall t \in \mathbb{R}.
\]
(32)
Assume that \(\|u_j\| \to \infty\) as \(j \to \infty\). Set \(v_j := u_j/\|u_j\|\), and then \(\|v_j\| = 1\). Thus, we have \(v_j \rightharpoonup v\) in \(W^{1,2}_\alpha\), \(v_j \to v\) in \(L^p\) for \(1 \leq p < 2\alpha\). In (31), \(\varphi\) is replaced by \(v_j - v\), and dividing by \(\|u_j\|\), one has

\[
\langle v_j, v_j - v \rangle = \lambda \int_\Omega v_j(v_j - v) \, dx + \int_\Omega \frac{f(x, u_j)}{\|u_j\|} (v_j - v) \, dx + o(1).
\]

From \(v_j \rightharpoonup v\) in \(L^p\) for \(1 \leq p < 2\alpha\) and Hölder’s inequality we have

\[
\left| \int_\Omega v_j(v_j - v) \, dx \right| \leq |v_j|^2|v_j - v|^2 = o(1).
\]  

(33)

Moreover, from (32) we have that

\[
\left| \int_\Omega \frac{f(x, u_j)}{\|u_j\|} (v_j - v) \, dx \right| \leq \left| \int_\Omega \left( \varepsilon v_j + \frac{C_\varepsilon}{\|u_j\|} \right) (v_j - v) \, dx \right|
\]

\[
\leq \varepsilon |v_j|^2|v_j - v|^2 + C_\varepsilon \frac{|v_j - v|_1}{\|u_j\|}
\]

\[
= o(1).
\]  

(34)

Hence, from (33), (34), we get \(\langle v_j, v_j - v \rangle = o(1)\). Thus, \(v_j \to v\) strongly in \(W^{1,2}_\alpha\). If \(v = 0\), we obtain \(\|v_j\| \to 0\), a contradiction. Hence, \(v \neq 0\). Now, dividing (31) by \(\|u_j\|\), we obtain

\[
\langle v_j, \varphi \rangle - \lambda \int_\Omega v_j \varphi \, dx - \int_\Omega \frac{f(x, u_j)\varphi}{\|u_j\|} \, dx = o(1).
\]  

(35)

From (f2) we have

\[
\lim_{j \to +\infty} \int_\Omega \frac{f(x, u_j)}{\|u_j\|} \varphi \, dx = \lim_{j \to +\infty} \int_\Omega \frac{f(x, u_j)}{u_j} v_j \varphi \, dx = 0.
\]

Passing to the limit in (35), from \(v_j \to v\) strongly in \(W^{1,2}_\alpha\) we have that \(\langle v, \varphi \rangle = \lambda \int_\Omega v \varphi \, dx\) for all \(\varphi \in W^{1,2}_\alpha\). This implies that \(\lambda \in \sigma(-\Delta_\alpha)\), which contradicts \((\lambda_1)\). Therefore, \(\{u_j\}\) is a bounded sequence.

\(\square\)

**Lemma 5.** Let \((f_1), (f_2)\) be satisfied. Then any \((\text{PS})\)-sequence \(\{u_j\}\) has a convergent subsequence.

**Proof.** We see that \(\{u_j\}\) is bounded in \(W^{1,2}_\alpha\), and therefore, we can assume that there is a subsequence, still denoted by \(\{u_j\}\) and there exists \(u_1 \in W^{1,2}_\alpha\) such that \(u_j \rightharpoonup u_1\) in \(W^{1,2}_\alpha\) and \(u_j \to u_1\) in \(L^p\) for all \(p \in [1, 2^*_\alpha]\). From (29) and (30) we get

\[
\langle J'_\lambda(u_j), u_j - u_1 \rangle \to 0, \quad j \to +\infty.
\]  

(36)
Moreover, from (32) and the Hölder inequality we have
\[
\int_\Omega |f(x, u_j)| |u_j - u_1| \, dx \leq \int_\Omega (\varepsilon |u_j| + C_\varepsilon) |u_j - u_1| \, dx \\
\leq \varepsilon |u_j| |u_j - u_1| + C_\varepsilon |u_j - u_1|_2 \rightarrow 0, \quad j \rightarrow +\infty.
\]
Therefore, from (29) and (36) it follows that
\[
\langle J'_\lambda(u_j - u_1), u_j - u_1 \rangle = \int_\Omega |\nabla_\alpha (u_j - u_1)|^2 \, dx - \lambda \int_\Omega |u_j - u_1|^2 \, dx \\
- \int_\Omega f(x, u_j - u_1) |u_j - u_1| \, dx \rightarrow 0.
\]
Thus, \(\|u_j - u_1\|^2 \rightarrow 0\). Hence \(u_j \rightarrow u_1\) in \(W^{1,2}_\alpha\). The proof is complete. \(\square\)

Proof of Theorem 2. Let \(\{\varepsilon_k\}_k\) the eigenfunctions corresponding to \(\lambda_k\) be the orthonormal basis of \(W^{1,2}_\alpha\). According to the proof of Theorem 1, we have
\[W^{1,2}_\alpha = X_1 \oplus X_2.\]
Consider \(\lambda > \lambda_1\), and \(\lambda \notin \sigma(\Delta_\alpha)\). By the definition of the eigenvalues, we get
\[
\|u\|^2 \leq \lambda_k |u|_2^2 \quad \forall u \in X_1 \quad \text{and} \quad \|u\|^2 \geq \lambda_{k+1} |u|_2^2 \quad \forall u \in X_2. \tag{37}
\]
According to (32), we have \(|F(x, t)| \leq C_1 (1 + t^2)\) for a.e. \(x \in \Omega\), for all \(t \in \mathbb{R}\). For every \(u \in X_1\), we have
\[
J_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla_\alpha u|^2 \, dx - \lambda \int_\Omega u^2 \, dx - \int_\Omega F(x, u) \, dx \\
\leq \frac{\lambda_k}{2} |u|_2^2 - \frac{\lambda}{2} |u|_2^2 + \frac{\varepsilon}{2} \int_\Omega u^2 \, dx + C_\varepsilon \int_\Omega u \, dx \\
\leq \frac{1}{2} (\lambda_k - \lambda + \varepsilon) |u|_2^2 + C_2 |u|_2. \tag{38}
\]
Let \(\lambda_k < \lambda\) be such that \(\lambda_k + \varepsilon < \lambda\). Then, since \(X_1\) is a finite dimensional subspace and \(J_\lambda \rightarrow -\infty\) as \(\|u\|\) diverges in \(X_1\), there exists a positive constant \(C_3\) such that \(J_\lambda(u) \leq -C_3\) for all \(u \in X_1\). On the other hand, from (37), for every \(u \in X_2\), we have
\[
J_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla_\alpha u|^2 \, dx - \lambda \int_\Omega u^2 \, dx - \int_\Omega F(x, u) \, dx \\
\geq \frac{1}{2} \|u\|^2 - \frac{\lambda}{2 \lambda_{k+1}} \|u\|^2 - \int_\Omega C_1 (1 + u^2) \, dx \\
\geq \frac{1}{2} \left(1 - \frac{\lambda + C_4}{\lambda_{k+1}}\right) \|u\|^2 - C_5.
\]
We obtain that $J_\lambda(u) \geq C_6$ for all $u \in X_2$. Therefore, let $E = W^{1,2}_\alpha$, $V = X_1$ and $X = X_2$, and it follows from Lemmas 4 and 5 that all the conditions of Theorem 4 are satisfied.

If $\lambda < \lambda_1$, the functional $J_\lambda$ is coercive and can be shown to a global minimum using the method of the Weierstrass Theorem. The proof is complete.

\textbf{Lemma 6.} Let $(f_1)$–$(f_3)$, $(\lambda_1)$, $(\lambda_2)$ be satisfied. Then there exist $\rho > 0$ and $c_0 > 0$ such that $J_\lambda(u) \geq c_0$ for all $u \in S_\rho \cap X_2$, $S_\rho := \{u \in W^{1,2}_\alpha : \|u\| = \rho\}$.

\textbf{Proof.} According to $(A_2)$, we have that $\lambda_0 < 0$. Therefore, it follows from $(f_2)$ that for every $\varepsilon > 0$, there exist $\delta_1 \geq 1$ and $\delta_2 \geq 0$ such that

$$\left| F(x, t) \right| \leq \frac{\varepsilon t^2}{2} \text{ if } |t| > \delta_1 \text{ for a.e. } x \in \Omega,$$

and

$$\left| F(x, t) - \frac{\lambda_0}{2} t^2 \right| \leq \frac{\varepsilon t^2}{2} \text{ if } |t| < \delta_2 \text{ for a.e. } x \in \Omega.$$  

From $(f_1)$, choosing any constant $p \in [0, 4Q/(Q - 2))$, there exists $\epsilon > 0$ such that

$$\left| F(x, t) \right| \leq \epsilon |t|^{p+2} \text{ if } \delta_2 \leq |t| \leq \delta_1 \text{ for a.e. } x \in \Omega.$$  

Hence, it follows from (39)–(41) that there exists $\epsilon_1$ such that $|F(x, t)| \leq (\lambda_0 + \varepsilon)t^2/2 + \epsilon_1|t|^{p+2}$, $t \in \mathbb{R}$, for a.e. $x \in \Omega$. Integrate both sides of the above formula, and we have that

$$\int_{\Omega} |F(x, u)| \, dx \leq \frac{\lambda_0 + \varepsilon}{2} |u|^2 + \epsilon_1 |u|^{p+2} \quad \forall u \in W^{1,2}_\alpha.$$  

Then from (13), (37) and (42) we have

$$J_\lambda(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \frac{\lambda}{2} \int_{\Omega} u^2 \, dx - \int_{\Omega} F(x, u) \, dx \\
\geq \frac{1}{2} \|u\|^2 - \frac{\lambda}{2} |u|^2 - \frac{\lambda_0 + \varepsilon}{2} |u|^2 - \epsilon_1 |u|^{p+2} \\
\geq \frac{1}{2} \|u\|^2 - \frac{\lambda}{2\lambda_{k+1}} \|u\|^2 - \frac{\lambda_0 + \varepsilon}{2\lambda_{k+1}} \|u\|^2 - \epsilon_1 \|u\|^{p+2} \\
\geq \frac{1}{2} \left( 1 - \frac{\lambda + \lambda_0 + \varepsilon}{\lambda_{h}} \right) \|u\|^2 - \epsilon_1 \|u\|^{p+2}. $$

Based on $(\lambda_2)$, note that $\varepsilon$ can be small enough, there exists a constant $\alpha$ such that $J_\lambda(u) \geq \alpha \|u\|^2 - \epsilon' \|u\|^{p+2}$. If $\rho$ is small enough, there exist $c_0 > 0$ such that $J_\lambda(u) \geq c_0$. 

\textbf{Lemma 7.} Let $(f_1)$, $(f_3)$ and $(\lambda_2)$ be satisfied. Then there exist $c_\infty > c_0$ such that $J_\lambda(u) \leq c_\infty$ for all $u \in X_1$. 

https://www.journals.vu.lt/nonlinear-analysis
**Proof.** From (38), for every \( u \in X_1 \), we have that \( J_\lambda(u) \leq (1/2)(1 - (\lambda + C_2)/\lambda_k)\|u\|^2 \) as \( X_1 \) is a finite dimensional subspace, and \( J_\lambda \to -\infty \) as \( \|u\| \) diverges in \( X_1 \). We see that there exists \( c_\infty = c_\infty(\varepsilon)(c_\infty > c_0) \) such that \( J_\lambda(u) \leq c_\infty \).

**Proof of Theorem 3.** From Lemmas 4 and 5 we see that \( J_\lambda \) satisfies the (PS)-condition. Also, by Lemmas 6 and 7 we consider the pseudoindex theory \( (S_\rho \cap X_2, H^*, \gamma^*) \) related to the genus, \( S_\rho \cap X_2 \) and \( J_\lambda \). By Theorem A.2 in [18], with \( V = X_1, \partial B = S_\rho \) and \( W = X_2 \), we get \( \gamma(X_1 \cap h(S_\rho \cap X_2)) \geq \dim X_1 - \text{codim } X_2 \) for all \( h \in X_1 \), which implies that \( \gamma^*(X_1) \geq k - h - 1 \). Hence, with \( \overline{A} = X_1, S = S_\rho \cap X_2 \), all the conditions of Theorem 5 are satisfied. Thus, \( J_\lambda \) has at least \( k - h - 1 \) distinct pairs of critical points corresponding to at most \( k - h - 1 \) distinct critical values \( c_i \). The proof is complete.

**References**


