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Stability of port-Hamiltonian systems with mixed time delays subject to input saturation*

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Abstract. In this paper, we investigate the stability of port-Hamiltonian systems with mixed time-varying delays as well as input saturation. Three types of time delays, including state delay, input delay, and output delay, are all assumed to be bounded. By introducing the output feedback control law and utilizing serval Lyapunov–Krasovskii functionals, we present three delay-dependent stability criteria in terms of the linear matrix inequality. Meanwhile, we use Wirtinger's inequality, constraint conditions, and Lyapunov–Krasovskii functionals of triple and quadruple integral form to obtain less conservative results. Some numerical examples demonstrate and support our results.

Keywords: port-Hamiltonian system, input saturation, time-varying delay, stability, Lyapunov–Krasovskii method.

1 Introduction

Port-Hamiltonian (PH) system is an important class of nonlinear systems. Since PH systems were proposed by Maschke and van der Schaft in [20], PH systems have been studied by many scholars owing to its nice structural properties [24, 30, 32]. As the total energy of systems, the Hamiltonian function of a PH system is often used as a good candidate of Lyapunov function to deal with the stability problem. At the same time, many effective control approaches have been established in the framework of PH systems, such as passivity-based control [34], interconnection and damping assignment passivity-

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based control [22], interconnection control [21], etc. Therefore, many practical problems can be converted into PH systems through modeling, and they can be further analyzed and controlled [6, 15, 23, 25].

In many practical control systems, time delays and saturation constraints are unavoidable factors, such as mechanical systems, control systems, communication systems, etc. Usually, the present of time delays and saturations will destroy the original structure of systems or make the stability of systems very difficult. Therefore, many scholars devote themselves to the stability design of systems. For time-delay systems, the Lyapunov–Krasovskii functional method [12,14] is a very important approach to determine the stability of systems. In addition, there are three methods to handle saturation terms, including sector nonlinearity models [36], diagonal matrix method [8], and convex combination-based method [39].

In recent years, many papers consider the stability of nonlinear systems with time delays and input saturation constraints in the framework of PH systems [5, 28, 33, 35]. Cao et al. proposed delay-dependent and delay-independent stability conditions for linear time-delay systems with input saturation by using the Lyapunov–Krasovskii functional method in [3]. Sun studied the stability of port-controlled Hamiltonian systems with state delay and input saturation; see [28]. Comparing with the criterion in [28], Aoues et al. in [1] proposed a more tractable criteria and used Wirtinger's inequality [13,26] to reduce the conservatism. In [38], Yang proposed stability conditions of input saturated nonlinear systems with state delay and input delay by constructing a new Lyapunov–Krasovskii functional. Then Sun et al. studied the global asymptotic stability of singular nonlinear Hamiltonian systems with input delay and output delay subject to input saturation, and they investigated H_∞ control problems with external disturbances in [29]. There are a number of literatures about time-delay nonlinear systems with input saturation [10, 33, 35]. However, to the best of our knowledge, there are few results on the stability of PH systems simultaneously with state delay, input delay, output delay, and input saturation.

In this paper, we study the asymptotic stability of the PH system with mixed delays subject to input saturation under the action of the output feedback controller. Three delay-dependent stability criteria are proposed as shown in three main theorems; see Theorems 1, 2, and 3 in Section 3. The contributions of this paper are as follows: (i) We consider three kinds of time delays, while considering input saturation, which generalized the related results in the previous literatures; (ii) Since the considering situation is more comprehensive and complex, we introduce Lyapunov–Krasovskii functionals in the forms of triple and quadruple integrals; (iii) In order to further reduce the conservatism of the system, we utilize Wirtinger's inequality and introduce constraint conditions.

The rest of paper is organized as follows. Section 2 is the problem formulation and preliminaries. In Section 3, we give three delay-dependent stability criteria based on different Lyapunov–Krasovskii functionals, and in Section 4, some numerical examples support our results. Finally, we give the conclusion in Section 5.

Notation. $\mathbb{R}$ is the set of real numbers; $\mathbb{R}^m$ denotes the m-dimensional Euclidean space, and $\mathbb{R}^{m\times n}$ is the set of $m\times n$ real matrices. I_m is a mth order identity matrix. For a matrix $M \in \mathbb{R}^{m\times m}$, we use M^{T}, M^{-1} to denote the transpose matrix and the inverse matrix of

M, respectively. The notation $M < 0$ (respectively, $M \leqslant 0$) represents that matrix M is negative definite (respectively, negative seminegative). For a mapping $H(x)$: $\mathbb{R}^m \to \mathbb{R}$ and a column vector $x \in \mathbb{R}^m$, we denote the gradient of $H(x)$ at x by $\nabla_x H(x)$. The Hessian matrix of $H(x)$ at x is denoted as $\mathrm{Hess}(H(x))$. $\dot{x}$ is the derivative of x with respect to t. The notation $*$ denotes the elements below the main diagonal of symmetric matrix. For a scalar a, $\mathrm{sign}(a)$ is the signum function of a, and $|a|$ denotes the absolute value of a.

2 Problem statement and preliminaries

Consider the following PH system with mixed time-varying delays subject to input saturation:

$$\begin{aligned} \dot{x} &= \big(J(x) - R(x)\big)\nabla_x H(x) + T(x)\nabla_{x_1} H(x_1) + g\,\mathrm{sat}\big(u\big(t - h_2(t)\big)\big), \\ y &= g^{\mathrm{T}} \nabla_{x_3} H(x_3), \end{aligned} \tag{1}$$

where $x = x(t) \in \mathbb{R}^n$ is the state vector, the terms $x_1 = x(t - h_1(t))$, $u(t - h_2(t))$, and $x_3 = x(t - h_3(t))$ are all time-varying delays. $u, y \in \mathbb{R}^m$ are the control input and the output, respectively. $J = J(x)$, $R = R(x) \in \mathbb{R}^{n\times n}$ are the interconnection matrix and damping matrix, which satisfy $J^{\mathrm{T}} = -J$, $R^{\mathrm{T}} = R > 0$. The Hamiltonian function $H(x) \in \mathbb{R}$ is the total energy of system (1), and it satisfies $H(x) \geqslant 0$, $H(0) = 0$. $T = T(x) \in \mathbb{R}^{n\times n}$ is a matrix depending on the state x, and $g \in \mathbb{R}^{n\times m}$ is a constant input matrix.

Input saturation $\mathrm{sat}(u) = [\mathrm{sat}(u_1), \mathrm{sat}(u_2), \dots, \mathrm{sat}(u_m)]^{\mathrm{T}}$ is the standard actuator saturation function with $\mathrm{sat}(u_i) = \mathrm{sign}(u_i)\min\{|u_i|, \rho_i\}$, $i = 1, 2, \dots, m$, and ρ_i is a positive scalar. The time-varying continuous differentiable functions $h_1(t)$, $h_2(t)$, and $h_3(t)$ are assumed to satisfy the following constraints:

$$\begin{aligned} &0 \leqslant h_1(t) \leqslant \alpha_1, \qquad &&0 \leqslant h_2(t) \leqslant \alpha_2, \qquad &&0 \leqslant h_3(t) \leqslant \alpha_3, \\ &\dot{h}_1(t) \leqslant \mu_1 < 1, &&\dot{h}_2(t) \leqslant \mu_2 < 1, &&\dot{h}_3(t) \leqslant \mu_3 < 1, \end{aligned}$$

where α_1, α_2, α_3, μ_1, μ_2, and μ_3 are all the positive scalars. The initial condition of system (1) is $x(s) = \phi(s)$, $s \in [-\alpha, 0]$, where $\alpha = \max\{\alpha_1, \alpha_2, \alpha_3\}$.

Remark 1. For the PH system (1), Sun et al. considered the stability with $h_1(t) = 0$, $h_3(t) = 0$ in [31], while Yang (see [38]) studied the stability in case of $h_1(t) = h_2(t) = c$, $h_3(t) = 0$, where c is a positive constant. Moreover, the stability of system (1) with $h_2(t) = h_3(t) = 0$ was investigated in [1, 28]. These cases are all the special situations of system (1).

Consider the output feedback controller

$$u\big(t - h_2(t)\big) = -Ky = -Kg^{\mathrm{T}}\nabla_{x_{23}} H(x_{23}), \tag{2}$$

where $x_{23} = x(t - h_2(t) - h_3(t - h_2(t)))$, $K \in \mathbb{R}^{m\times m}$ is a constant gain matrix to be determined later. This paper is aimed to find sufficient stability conditions for system (1) under the action of the above controller (2). To this end, we need the following preparations.

Firstly, according to the expression of (2), we define $h_{23}(t) = h_2(t) + h_3(t - h_2(t))$. It is obviously that $h_{23}(t) \leqslant \alpha_2 + \alpha_3 = \alpha_{23}$. Meanwhile, $h_{23}(t)$ satisfies the following lemma.

Lemma 1. *There exists a unique $t^* > 0$ such that*

$$\begin{aligned} t &< h_{23}(t), \quad t < t^*, \\ t &\geqslant h_{23}(t), \quad t \geqslant t^*. \end{aligned}$$

Proof. Let $f(t) = t - h_{23}(t)$. Taking the derivative of $f(t)$ with respect to t, we obtain

$$\begin{aligned} \dot{f}(t) &= 1 - \dot{h}_{23}(t) \\ &= 1 - \dot{h}_2(t) - \dot{h}_3\big(t - h_2(t)\big)\big(1 - \dot{h}_2(t)\big) \\ &= \big(1 - \dot{h}_2(t)\big)\big(1 - \dot{h}_3\big(t - h_2(t)\big)\big) > 0. \end{aligned}$$

It means that $f(t)$ is an increasing function in the domain of definition. Since $f(0) = -h_{23}(0) = -h_2(0) - h_3(-h_2(0)) \leqslant 0$ and $f(+\infty) = +\infty$, there exists a unique t^* such that $f(t^*) = 0$, that is, $t^* = h_{23}(t^*)$. Therefore, when $t < t^*$, $t < h_{23}(t)$; when $t \geqslant t^*$, $t \geqslant h_{23}(t)$. $\square$

It is obviously that α_{23} is the upper bound of t^*. Since system (1) cannot receive any information from the controller when $t < h_{23}(t)$, so we only focus on $t \geqslant h_{23}(t)$. According to Lemma 1, $\dot{h}_{23} \leqslant \mu_{23} = 1 - (1 - \mu_2)(1 - \mu_3) < 1$, where μ_{23} is also a positive scalar.

Remark 2. In [16], Liu and Fridman assumed that there exists a unique t^* such that $t - \tau(t) < 0$, $t < t^*$, and $t - \tau(t) \geqslant 0$, $t \geqslant t^*$. This assumption also was introduced by Sun and his collaborators; see [29, Ass. 2.1]. However, according to the definition of $h(t)$, the assumption holds naturally by using the idea of Lemma 1 in this paper.

In order to deal with the input saturation term, we introduce a transformation

$$v = \mathrm{sat}(u) - u. \tag{3}$$

In the meantime, in order to obtain the stability conditions, the following lemmas will be needed.

Lemma 2. (*See* [36].) *If there exists the nonlinear transformation* (3)*, then the following inequality holds*:

$$v^{\mathrm{T}} v \leqslant u^{\mathrm{T}} u.$$

Lemma 3. *For any vectors $a, b \in \mathbb{R}^m$, then the following inequality is true*:

$$2a^{\mathrm{T}} b \leqslant a^{\mathrm{T}} a + b^{\mathrm{T}} b.$$

Lemma 4 [Schur complement]. (*See* [2].) *The linear matrix inequality*

$$\begin{pmatrix} A(x) & B(x) \\ * & C(x) \end{pmatrix} > 0$$

is equivalent to $A(x) - B(x)C^{-1}(x)B^{\mathrm{T}}(x) > 0$ *and* $C(x) > 0$*, where* $A(x) = A^{\mathrm{T}}(x)$*,* $C(x) = C^{\mathrm{T}}(x)$*, and* $B(x)$ *depend affinely on* x*.*

Lemma 5. *For any positive-definite matrix* $D \in \mathbb{R}^{m\times m}$*, positive scalar* τ*, and vector function* $f : [-\tau, +\infty) \to \mathbb{R}^m$ *such that the following integrations are well defined, the inequalities*

$$-\tau \int_{t-\tau}^{t} f^{\mathrm{T}}(s) D f(s)\,\mathrm{d}s \leqslant -\left(\int_{t-\tau}^{t} f(s)\,\mathrm{d}s\right)^{\mathrm{T}} D \left(\int_{t-\tau}^{t} f(s)\,\mathrm{d}s\right), \tag{4}$$

$$\begin{aligned} &-\frac{\tau^2}{2}\int_{-\tau}^{0}\int_{t+\theta}^{t} f^{\mathrm{T}}(s) D f(s)\,\mathrm{d}s\,\mathrm{d}\theta \\ &\quad\leqslant -\left(\int_{-\tau}^{0}\int_{t+\theta}^{t} f(s)\,\mathrm{d}s\,\mathrm{d}\theta\right)^{\mathrm{T}} D \left(\int_{-\tau}^{0}\int_{t+\theta}^{t} f(s)\,\mathrm{d}s\,\mathrm{d}\theta\right), \end{aligned} \tag{5}$$

and

$$\begin{aligned} &-\frac{\tau^3}{6}\int_{-\tau}^{0}\int_{\beta}^{0}\int_{t+\theta}^{t} f^{\mathrm{T}}(s) D f(s)\,\mathrm{d}s\,\mathrm{d}\beta\,\mathrm{d}\theta \\ &\quad\leqslant -\left(\int_{-\tau}^{0}\int_{\beta}^{0}\int_{t+\theta}^{t} f(s)\,\mathrm{d}s\,\mathrm{d}\beta\,\mathrm{d}\theta\right)^{\mathrm{T}} D \left(\int_{-\tau}^{0}\int_{\beta}^{0}\int_{t+\theta}^{t} f(s)\,\mathrm{d}s\,\mathrm{d}\beta\,\mathrm{d}\theta\right) \end{aligned} \tag{6}$$

hold.

Proof. The first inequality (4) was proposed in [11], and the second one (5) has been proved in [27]. The third inequality is used in [7, 37] with no proof. For the convenience of readers, we give the proof of the third inequality in the following.

Notice that

$$\begin{pmatrix} f^{\mathrm{T}}(s) D f(s) & f^{\mathrm{T}}(s) \\ f(s) & D^{-1} \end{pmatrix} \geqslant 0. \tag{7}$$

Integrate three times of (7) on the intervals $[t+\theta,\, t]$, $[\beta, 0]$, and $[-\tau, 0]$ in turn, we have

$$\begin{pmatrix} \int_{-\tau}^{0}\int_{\beta}^{0}\int_{t+\theta}^{t} f^{\mathrm{T}}(s) D f(s)\,\mathrm{d}s\,\mathrm{d}\beta\,\mathrm{d}\theta & \int_{-\tau}^{0}\int_{\beta}^{0}\int_{t+\theta}^{t} f^{\mathrm{T}}(s)\,\mathrm{d}s\,\mathrm{d}\beta\,\mathrm{d}\theta \\ \int_{-\tau}^{0}\int_{\beta}^{0}\int_{t+\theta}^{t} f(s)\,\mathrm{d}s\,\mathrm{d}\beta\,\mathrm{d}\theta & \frac{\tau^3}{6}D^{-1} \end{pmatrix} \geqslant 0, \tag{8}$$

where $\theta \in [-\beta, 0]$, $\beta \in [-\tau, 0]$. Here $\int_{-\tau}^{0}\int_{\beta}^{0}\int_{t+\theta}^{t} D^{-1}\,\mathrm{d}s\,\mathrm{d}\beta\,\mathrm{d}\theta = (\tau^3/6)D^{-1}$. By using Schur complement (see Lemma 4), (8) is equivalent to inequality (6). Hence, we complete the proof. □

Lemma 6. (*See* [26].) *For a given positive-definite matrix $W \in \mathbb{R}^{n\times n}$ and for all continuous functions $f : [a,b] \to \mathbb{R}^n$, the following inequality holds*:

$$\int_a^b \dot{f}^{\mathrm{T}}(s) W \dot{f}(s)\,\mathrm{d}s \geqslant \frac{1}{b-a}\begin{pmatrix}\Omega_0\\ \Omega_1\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}W & 0\\ * & 3W\end{pmatrix}\begin{pmatrix}\Omega_0\\ \Omega_1\end{pmatrix}, \tag{9}$$

where $\Omega_0 = f(b) - f(a)$, $\Omega_1 = f(b) + f(a) - (2/(b-a))\int_a^b f(s)\,\mathrm{d}s$.

Remark 3. As mentioned in [26], the first term of the right-hand side of inequality (9) is Jensen's inequality. Since the second term $(3/(b-a))\Omega_1^{\mathrm{T}} W \Omega_1$ is nonnegative definite. Therefore, Wirtinger's inequality encompasses Jensen's inequality. In another word, Wirtinger's inequality is less conservative than Jensen's inequality.

By introducing the output feedback controller (2) and combining the transformation (3), system (1) becomes the closed-loop system

$$\dot{x} = (J-R)\nabla_x H(x) + T\nabla_{x_1} H(x_1) - gKg^{\mathrm{T}}\nabla_{x_{23}} H(x_{23}) + gv\big(t - h_2(t)\big). \tag{10}$$

Therefore, our object is to study the asymptotic stability condition of system (10), which is equivalent to the asymptotic stability condition of system (1) under the action of the controller (2).

3 Main results

In this section, we establish three types of stability sufficient conditions by constructing three different Lyapunov–Krasovskii functionals.

Theorem 1. *Consider system* (10). *If there exist positive-definite symmetric matrices Q, P and a matrix K with proper dimensions such that $\Psi < 0$, then system* (10) *is asymptotically stable, where*

$$\Psi = \begin{pmatrix} -2R + Q + P + gg^{\mathrm{T}} & -gKg^{\mathrm{T}} & T & 0\\ * & -(1-\mu_{23})Q & 0 & gK^{\mathrm{T}}\\ * & * & -(1-\mu_1)P & 0\\ * & * & * & -I_{m\times m}\end{pmatrix}.$$

Proof. Choose a Lyapunov–Krasovskii functional as

$$V = V_1 + V_2, \tag{11}$$

where

$$\begin{aligned} V_1 &= 2H(x),\\ V_2 &= \int_{t-h_{23}(t)}^{t} \big(\nabla_x H\big(x(s)\big)\big)^{\mathrm{T}} Q \nabla_x H\big(x(s)\big)\,\mathrm{d}s\\ &\quad + \int_{t-h_1(t)}^{t} \big(\nabla_x H\big(x(s)\big)\big)^{\mathrm{T}} P \nabla_x H\big(x(s)\big)\,\mathrm{d}s, \end{aligned}$$

and Q, P are positive-definite symmetric matrices, which to be determined later. Calculating the derivatives of V_i, $i = 1, 2$, along the trajectory of system (10), we obtain

$$\dot{V}_1 = 2\big(\nabla_x H(x)\big)^{\mathrm{T}}(J-R)\nabla_x H(x) + 2\big(\nabla_x H(x)\big)^{\mathrm{T}}T(x)\nabla_{x_1}H(x_1) \\ - 2\big(\nabla_x H(x)\big)^{\mathrm{T}}gKg^{\mathrm{T}}\nabla_{x_{23}}H(x_{23}) + 2\big(\nabla_x H(x)\big)^{\mathrm{T}}gv\big(t-h_2(t)\big)$$

and

$$\dot{V}_2 = \big(\nabla_x H(x)\big)^{\mathrm{T}}Q\nabla_x H(x) - \big(1-\dot{h}_{23}(t)\big)\big(\nabla_{x_{23}}H(x_{23})\big)^{\mathrm{T}}Q\nabla_{x_{23}}H(x_{23}) \\ + \big(\nabla_x H(x)\big)^{\mathrm{T}}P\nabla_x H(x) - \big(1-\dot{h}_1(t)\big)\big(\nabla_{x_1}H(x_1)\big)^{\mathrm{T}}P\nabla_{x_1}H(x_1).$$

According to Lemmas 2 and 3, the last term of $\dot{V}_1$ satisfies

$$\begin{aligned} &2\big(\nabla_x H(x)\big)^{\mathrm{T}}gv\big(t-h_2(t)\big) \\ &\quad\leqslant \big(\nabla_x H(x)\big)^{\mathrm{T}}gg^{\mathrm{T}}\nabla_x H(x) + v\big(t-h_2(t)\big)^{\mathrm{T}}v\big(t-h_2(t)\big) \\ &\quad= \big(\nabla_x H(x)\big)^{\mathrm{T}}gg^{\mathrm{T}}\nabla_x H(x) + \big(\nabla_{x_{23}}H(x_{23})\big)^{\mathrm{T}}gK^{\mathrm{T}}Kg^{\mathrm{T}}\nabla_{x_{23}}H(x_{23}). \end{aligned}$$

Combining the above equations, we obtain

$$\begin{aligned} \dot{V} \leqslant{} & \big(\nabla_x H(x)\big)^{\mathrm{T}}\big\{-2R+P+Q+gg^{\mathrm{T}}\big\}\nabla_x H(x) \\ & + 2\big(\nabla_x H(x)\big)^{\mathrm{T}}T\nabla_{x_1}H(x_1) - 2\big(\nabla_x H(x)\big)^{\mathrm{T}}gKg^{\mathrm{T}}\nabla_{x_{23}}H(x_{23}) \\ & + \big(\nabla_{x_{23}}H(x_{23})\big)^{\mathrm{T}}\big\{gK^{\mathrm{T}}Kg^{\mathrm{T}} - \big(1-\dot{h}_{23}(t)\big)Q\big\}\nabla_{x_{23}}H(x_{23}) \\ & - \big(1-\dot{h}_1(t)\big)\big(\nabla_{x_1}H(x_1)\big)^{\mathrm{T}}P\nabla_{x_1}H(x_1) \\ \leqslant{} & \eta^{\mathrm{T}}\Phi\eta, \end{aligned}$$

where

$$\eta = \begin{pmatrix} \nabla_x H(x) \\ \nabla_{x_{23}}H(x_{23}) \\ \nabla_{x_1}H(x_1) \end{pmatrix},$$

$$\Phi = \begin{pmatrix} -2R+Q+P+gg^{\mathrm{T}} & -gKg^{\mathrm{T}} & T \\ * & gK^{\mathrm{T}}Kg^{\mathrm{T}} - (1-\mu_{23})Q & 0 \\ * & * & -(1-\mu_1)P \end{pmatrix}.$$

Since Φ is a nonlinear matrix, we get a linear matrix inequality $\Psi < 0$, which is equivalent to inequality $\Phi < 0$ by utilizing the Schur complement.

Therefore, we have $\dot{V} < 0$. By the Lyapunov–Krasovskii functional stability theorem, system (1) is asymptotically stable under the feedback controller (2). □

Remark 4. For Theorem 1, we can easily determine the values of Q, P, K, which satisfy $\Psi < 0$. However, in order to reduce conservatism, we introduce constraint conditions and other types of Lyapunov–Krasovskii functionals.

Remark 5. When $h_1(t)$, $h_2(t)$, $h_3(t)$ are all positive scalars, the above Theorem 1 is a delay-independent stability criterion. In [38], the author investigated the stabilization problem for a class of nonlinear time-delay Hamiltonian systems with actuator saturation and constant input delay. Sun [28] considered the stabilization of a class of Hamiltonian systems with time-varying state delay and input saturation. Theorem 1 extends the results in these papers and solves more complex cases involving input and output delays.

Next, we give a less conservative stability criterion by introducing constraint conditions and other types of Lyapunov–Krasovskii functionals, and we use Wirtinger's inequality [26] to deal with the derivative terms.

Theorem 2. *Consider system* (10). *If there exist positive-definite symmetric matrices Q, P, W, M, free weighted matrices N_1, N_2, and a proper dimensions matrix K such that $\Theta < 0$, then system* (10) *is asymptotically stable, where*

$$\Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & 6W & 6M & \Theta_{16} & N_1 g & 0 & 0 \\ * & \Theta_{22} & 0 & 6W & 0 & 0 & -gK^{\mathrm{T}} & -gK^{\mathrm{T}} & -gK^{\mathrm{T}} \\ * & * & \Theta_{33} & 0 & 6M & T^{\mathrm{T}} N_2^{\mathrm{T}} & 0 & 0 & 0 \\ * & * & * & -12W & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -12M & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Theta_{66} & 0 & N_2 g & 0 \\ * & * & * & * & * & * & -I_{m\times m} & 0 & 0 \\ * & * & * & * & * & * & * & -I_{m\times m} & 0 \\ * & * & * & * & * & * & * & * & -I_{m\times m} \end{pmatrix},$$

where

$$\begin{aligned} \Theta_{11} &= -2R + Q + P - 4W - 4M + gg^{\mathrm{T}} + N_1(J-R) + (J-R)^{\mathrm{T}} N_1^{\mathrm{T}}, \\ \Theta_{12} &= -2W - gKg^{\mathrm{T}}, \qquad \Theta_{13} = -2M + T + N_1 T, \\ \Theta_{16} &= -N_1 + (J-R)^{\mathrm{T}} N_2^{\mathrm{T}}, \qquad \Theta_{22} = -(1-\mu_{23})Q - 4W, \\ \Theta_{33} &= -(1-\mu_1)P - 4M, \qquad \Theta_{66} = \alpha_{23}^2 \beta_1 + \alpha_1^2 \beta_2 - N_2 - N_2^{\mathrm{T}}, \end{aligned}$$

and $\beta_1 = (\mathrm{Hess}(H(x)))W(\mathrm{Hess}(H(x)))^{\mathrm{T}}$, $\beta_2 = (\mathrm{Hess}(H(x)))M(\mathrm{Hess}(H(x)))^{\mathrm{T}}$.

Proof. Choose the Lyapunov–Krasovskii functional with

$$V = V_1 + V_2 + V_3.$$

Here V_1, V_2 are functions described as in (11), and

$$\begin{aligned} V_3 = {} & \alpha_{23} \int_{-\alpha_{23}}^{0} \int_{t+s}^{t} \bigl(\nabla_x \dot{H}\bigl(x(v)\bigr)\bigr)^{\mathrm{T}} W \bigl(\nabla_x \dot{H}\bigl(x(v)\bigr)\bigr) \mathrm{d}v \, \mathrm{d}s \\ & + \alpha_1 \int_{-\alpha_1}^{0} \int_{t+s}^{t} \bigl(\nabla_x \dot{H}\bigl(x(v)\bigr)\bigr)^{\mathrm{T}} M \bigl(\nabla_x \dot{H}\bigl(x(v)\bigr)\bigr) \mathrm{d}v \, \mathrm{d}s. \end{aligned}$$

Taking the derivatives of V_i, $i = 1, 2, 3$, along the trajectory of system (10), we obtain

$$\begin{aligned}\dot{V}_1 &= 2\big(\nabla_x H(x)\big)^{\mathrm{T}}(J-R)\nabla_x H(x) + 2\big(\nabla_x H(x)\big)^{\mathrm{T}} T \nabla_{x_1} H(x_1) \\ &\quad - 2\big(\nabla_x H(x)\big)^{\mathrm{T}} g K g^{\mathrm{T}} \nabla_{x_{23}} H(x_{23}) + 2\big(\nabla_x H(x)\big)^{\mathrm{T}} g v\big(t - h_2(t)\big), \\ \dot{V}_2 &= \big(\nabla_x H(x)\big)^{\mathrm{T}} Q \nabla_x H(x) - \big(1 - \dot{h}_{23}(t)\big)\big(\nabla_{x_{23}} H(x_{23})\big)^{\mathrm{T}} Q \nabla_{x_{23}} H(x_{23}) \\ &\quad + \big(\nabla_x H(x)\big)^{\mathrm{T}} P \nabla_x H(x) - (1 - \dot{h}_1(t))\big(\nabla_{x_1} H(x_1)\big)^{\mathrm{T}} P \nabla_{x_1} H(x_1),\end{aligned}$$

and

$$\begin{aligned}\dot{V}_3 &= \alpha_{23}^2 (\dot{x})^{\mathrm{T}} \beta_1 \dot{x} - \alpha_{23} \int_{t-\alpha_{23}}^{t} \big(\nabla_x \dot{H}\big(x(s)\big)\big)^{\mathrm{T}} W \big(\nabla_x \dot{H}\big(x(s)\big)\big)\,\mathrm{d}s \\ &\quad + \alpha_1^2 (\dot{x})^{\mathrm{T}} \beta_2 \dot{x} - \alpha_1 \int_{t-\alpha_1}^{t} \big(\nabla_x \dot{H}\big(x(s)\big)\big)^{\mathrm{T}} M \big(\nabla_x \dot{H}\big(x(s)\big)\big)\,\mathrm{d}s,\end{aligned}$$

where $\beta_1 = (\mathrm{Hess}(H(x)))W(\mathrm{Hess}(H(x)))^{\mathrm{T}}$, $\beta_2 = (\mathrm{Hess}(H(x)))M(\mathrm{Hess}(H(x)))^{\mathrm{T}}$.

With the advantage of Wirtinger's inequality, as showed in Lemma 6, we know the second and the last terms in V_3 satisfying

$$-\alpha_{23} \int_{t-\alpha_{23}}^{t} \big(\nabla_x \dot{H}\big(x(s)\big)\big)^{\mathrm{T}} W \big(\nabla_x \dot{H}\big(x(s)\big)\big)\,\mathrm{d}s \leqslant -\xi_1^{\mathrm{T}}(t) \begin{pmatrix} W & 0 \\ * & 3W \end{pmatrix} \xi_1(t)$$

and

$$-\alpha_1 \int_{t-\alpha_1}^{t} \big(\nabla_x \dot{H}\big(x(s)\big)\big)^{\mathrm{T}} M \big(\nabla_x \dot{H}\big(x(s)\big)\big)\,\mathrm{d}s \leqslant -\xi_2^{\mathrm{T}}(t) \begin{pmatrix} M & 0 \\ * & 3M \end{pmatrix} \xi_2(t),$$

respectively, where

$$\xi_1(t) = \begin{pmatrix} \nabla_x H(x) - \nabla_{x_{23}} H(x_{23}) \\ \nabla_x H(x) + \nabla_{x_{23}} H(x_{23}) - \frac{2}{h_{23}(t)} \int_{t-h_{23}(t)}^{t} (\nabla_x H(x(s)))^{\mathrm{T}} W (\nabla_x H(x(s)))\,\mathrm{d}s \end{pmatrix},$$

and

$$\xi_2(t) = \begin{pmatrix} \nabla_x H(x) - \nabla_{x_1} H(x_1) \\ \nabla_x H(x) + \nabla_{x_1} H(x_1) - \frac{2}{h_1(t)} \int_{t-h_1(t)}^{t} (\nabla_x H(x(s)))^{\mathrm{T}} M (\nabla_x H(x(s)))\,\mathrm{d}s \end{pmatrix}.$$

From the expression of system (10) let $z = \dot{x}$. We have

$$\begin{aligned}&(J-R)\nabla_x H(x) + T \nabla_{x_1} H(x_1) - g K g^{\mathrm{T}} \nabla_{x_{23}} H(x_{23}) \\ &\quad + g v\big(t - h_2(t)\big) - z = 0.\end{aligned} \tag{12}$$

Next, we introduce the following constraint conditions:

$$\begin{aligned}&\big(\nabla_x H(x)\big)^{\mathrm{T}} N_1\big\{(J-R)\nabla_x H(x) + T\nabla_{x_1} H(x_1) - gKg^{\mathrm{T}}\nabla_{x_{23}} H(x_{23})\\ &\quad + gv\big(t-h_2(t)\big) - z\big\} = 0\end{aligned} \tag{13}$$

and

$$\begin{aligned}&z^{\mathrm{T}} N_2\big\{(J-R)\nabla_x H(x) + T\nabla_{x_1} H(x_1) - gKg^{\mathrm{T}}\nabla_{x_{23}} H(x_{23})\\ &\quad + gv\big(t-h_2(t)\big) - z\big\} = 0,\end{aligned} \tag{14}$$

where N_1 and N_2 are free weighted matrices with proper dimensions. Since $z = \dot{x}$, $\dot{V}_3$ becomes

$$\begin{aligned}\dot{V}_3 &= \alpha_{23}^2 z^{\mathrm{T}}\beta_1 z - \alpha_{23}\int\limits_{t-\alpha_{23}}^{t} \big(\nabla_x \dot{H}\big(x(s)\big)\big)^{\mathrm{T}} W\big(\nabla_x \dot{H}\big(x(s)\big)\big)\,\mathrm{d}s\\ &\quad + \alpha_1^2 z^{\mathrm{T}}\beta_2 z - \alpha_1\int\limits_{t-\alpha_1}^{t} \big(\nabla_x \dot{H}\big(x(s)\big)\big)^{\mathrm{T}} M\big(\nabla_x \dot{H}\big(x(s)\big)\big)\,\mathrm{d}s.\end{aligned}$$

Substituting $\dot{V}_1$, $\dot{V}_2$, $\dot{V}_3$, (13), and (14) into $\dot{V}$, we obtain

$$\dot{V} \leqslant \eta^{\mathrm{T}}\varTheta_0\eta,$$

where

$$\eta = \begin{pmatrix} \nabla_x H(x)\\ \nabla_{x_{23}} H(x_{23})\\ \nabla_{x_1} H(x_1)\\ \frac{1}{h_{23}(t)}\int_{t-h_{23}(t)}^{t} \nabla_x H\big(x(s)\big)\,\mathrm{d}s\\ \frac{1}{h_1(t)}\int_{t-h_1(t)}^{t} \nabla_x H\big(x(s)\big)\,\mathrm{d}s \end{pmatrix},$$

$$\varTheta_0 = \begin{pmatrix} \varTheta_{11}^0 & \varTheta_{12}^0 & \varTheta_{13} & 6W & 6M & \varTheta_{16}\\ * & \varTheta_{22}^0 & 0 & 6W & 0 & -gK^{\mathrm{T}}g^{\mathrm{T}}N_2^{\mathrm{T}}\\ * & * & \varTheta_{33} & 0 & 6M & T^{\mathrm{T}}N_2^{\mathrm{T}}\\ * & * & * & -12W & 0 & 0\\ * & * & * & * & -12M & 0\\ * & * & * & * & * & \varTheta_{66}^0 \end{pmatrix}$$

with $\varTheta_{11}^0 = \varTheta_{11} + N_1 gg^{\mathrm{T}}N_1^{\mathrm{T}}$, $\varTheta_{12}^0 = \varTheta_{12} - N_1 gKg^{\mathrm{T}}$, $\varTheta_{22}^0 = \varTheta_{22} + 3gK^{\mathrm{T}}Kg^{\mathrm{T}}$, $\varTheta_{66}^0 = \varTheta_{66} + N_2 gg^{\mathrm{T}}N_2^{\mathrm{T}}$. Terms $2(\nabla_x H(x))^{\mathrm{T}}N_1 gv(t-h_2(t))$ and $2z^{\mathrm{T}}N_2 gv(t-h_2(t))$ can be handled in the same way as the last term of $\dot{V}_1$ in Theorem 1.

Since $\varTheta_0$ is a nonlinear matrix, we obtain a linear matrix inequality $\varTheta < 0$, which is equivalent to inequality $\varTheta_0 < 0$ by using Lemma 4. Therefore, we have $\dot{V} < 0$. By utilizing the Lyapunov–Krasovskii functional stability theorem, system (1) is also asymptotically stable under the feedback controller (2). □

Remark 6. In fact, the construction of equation (12) is a common technique, which is called the descriptor system method; see [9] for instance. The free weighted matrices N_1, N_2 are introduced to reduce the conservatism. Since Theorem 2 is derived using the descriptor system method and Wirtinger-based integral inequality, it exhibits less conservatism than the results presented in [28, 38].

Next, we give another criterion by using triple and quadruple integral form Lyapunov–Krasovskii functionals. We have the following theorem.

Theorem 3. *Consider system* (10). *If there exist positive-definite symmetric matrices Q, P, W, M, R_1, R_2, Z_1, Z_2, free weighted matrices N_1, N_2, and matrix K with proper dimensions such that $\Upsilon < 0$, then system* (10) *is asymptotically stable, where*

$$\Upsilon = \begin{pmatrix} \Upsilon_{11} & \Upsilon_{12} & \Upsilon_{13} & 6W & 6M & \Upsilon_{16} & \Upsilon_{17} & \Upsilon_{18} & \Upsilon_{19} & \Upsilon_{110} & N_1 g & 0 & 0 \\ * & \Upsilon_{22} & 0 & 6W & 0 & 0 & 0 & 0 & 0 & 0 & -gK^{\mathrm{T}} & -gK^{\mathrm{T}} & -gK^{\mathrm{T}} \\ * & * & \Upsilon_{33} & 0 & 6M & T^{\mathrm{T}} N_2^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -12W & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -12M & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Upsilon_{66} & 0 & 0 & 0 & 0 & 0 & N_2 g & 0 \\ * & * & * & * & * & * & -R_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -R_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & -Z_1 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & -Z_2 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & -I_m & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & -I_m & 0 \\ * & * & * & * & * & * & * & * & * & * & * & * & -I_m \end{pmatrix}$$

with

$$\begin{aligned} \Upsilon_{11} &= -2R + Q + P - 4W - 4M + gg^{\mathrm{T}} + N_1(J-R) + (J-R)^{\mathrm{T}} N_1^{\mathrm{T}} \\ &\quad - \alpha_{23}^2 R_1 - \alpha_1^2 R_2 - \frac{\alpha_{23}^4}{4} Z_1 - \frac{\alpha_1^4}{4} Z_2, \\ \Upsilon_{12} &= -2W - gKg^{\mathrm{T}}, \qquad \Upsilon_{13} = -2M + T + N_1 T, \\ \Upsilon_{16} &= -N_1 + (J-R)^{\mathrm{T}} N_2^{\mathrm{T}}, \qquad \Upsilon_{17} = \alpha_{23} R_1, \\ \Upsilon_{18} &= \alpha_1 R_2, \qquad \Upsilon_{19} = \frac{\alpha_{23}^2}{2} Z_1, \qquad \Upsilon_{110} = \frac{\alpha_1^2}{2} Z_2, \\ \Upsilon_{22} &= -(1-\mu_{23})Q - 4W, \qquad \Upsilon_{33} = -(1-\mu_1)P - 4M, \\ \Upsilon_{66} &= \alpha_{23}^2 \beta_1 + \alpha_1^2 \beta_2 - N_2 - N_2^{\mathrm{T}} + \frac{\alpha_{23}^4}{4} \beta_3 + \frac{\alpha_1^4}{4} \beta_4 + \frac{\alpha_{23}^6}{36} \beta_5 + \frac{\alpha_1^6}{36} \beta_6, \end{aligned}$$

and $\beta_i = (\mathrm{Hess}(H(x))) S_i (\mathrm{Hess}(H(x)))^{\mathrm{T}}$, $i = 1, \dots, 6$, $S = \{W, M, R_1, R_2, Z_1, Z_2\}$.

Proof. Let us construct a Lyapunov–Krasovskii functional with

$$V = V_1 + V_2 + V_3 + V_4 + V_5,$$

where

$$V_1 = 2H(x),$$

$$V_2 = \int_{t-h_{23}(t)}^{t} \big(\nabla_x H\big(x(s)\big)\big)^{\mathrm{T}} Q \nabla_x H\big(x(s)\big)\,\mathrm{d}s + \int_{t-h_1(t)}^{t} \big(\nabla_x H\big(x(s)\big)\big)^{\mathrm{T}} P \nabla_x H\big(x(s)\big)\,\mathrm{d}s,$$

$$V_3 = \alpha_{23} \int_{-\alpha_{23}}^{0} \int_{t+\theta}^{t} \big(\nabla_x \dot H\big(x(s)\big)\big)^{\mathrm{T}} W \big(\nabla_x \dot H\big(x(s)\big)\big)\,\mathrm{d}s\,\mathrm{d}\theta + \alpha_1 \int_{-\alpha_1}^{0} \int_{t+\theta}^{t} \big(\nabla_x \dot H\big(x(s)\big)\big)^{\mathrm{T}} M \big(\nabla_x \dot H\big(x(s)\big)\big)\,\mathrm{d}s\,\mathrm{d}\theta,$$

$$V_4 = \frac{\alpha_{23}^2}{2} \int_{-\alpha_{23}}^{0} \int_{\theta}^{0} \int_{t+\lambda}^{t} \big(\nabla_x \dot H\big(x(s)\big)\big)^{\mathrm{T}} R_1 \big(\nabla_x \dot H\big(x(s)\big)\big)\,\mathrm{d}s\,\mathrm{d}\lambda\,\mathrm{d}\theta + \frac{\alpha_1^2}{2} \int_{-\alpha_1}^{0} \int_{\theta}^{0} \int_{t+\lambda}^{t} \big(\nabla_x \dot H\big(x(s)\big)\big)^{\mathrm{T}} R_2 \big(\nabla_x \dot H\big(x(s)\big)\big)\,\mathrm{d}s\,\mathrm{d}\lambda\,\mathrm{d}\theta,$$

and

$$V_5 = \frac{\alpha_{23}^3}{6} \int_{-\alpha_{23}}^{0} \int_{\phi}^{0} \int_{\theta}^{0} \int_{t+\lambda}^{t} \big(\nabla_x \dot H\big(x(s)\big)\big)^{\mathrm{T}} Z_1 \big(\nabla_x \dot H\big(x(s)\big)\big)\,\mathrm{d}s\,\mathrm{d}\lambda\,\mathrm{d}\theta\,\mathrm{d}\phi + \frac{\alpha_1^3}{6} \int_{-\alpha_1}^{0} \int_{\phi}^{0} \int_{\theta}^{0} \int_{t+\lambda}^{t} \big(\nabla_x \dot H\big(x(s)\big)\big)^{\mathrm{T}} Z_2 \big(\nabla_x \dot H\big(x(s)\big)\big)\,\mathrm{d}s\,\mathrm{d}\lambda\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

The derivative of terms V_1, V_2, V_3 have been calculated in the proof of Theorem 2. Hence, we only consider V_4 and V_5. Taking the derivatives of V_i, $i = 4, 5$, along the trajectory of system (10), we get

$$\dot V_4 = \frac{\alpha_{23}^4}{4} (\dot x)^{\mathrm{T}} \beta_3 \dot x - \frac{\alpha_{23}^2}{2} \int_{-\alpha_{23}}^{0} \int_{t+\theta}^{t} \big(\nabla_x \dot H\big(x(s)\big)\big)^{\mathrm{T}} R_1 \big(\nabla_x \dot H\big(x(s)\big)\big)\,\mathrm{d}s\,\mathrm{d}\theta + \frac{\alpha_1^4}{4} (\dot x)^{\mathrm{T}} \beta_4 \dot x - \frac{\alpha_1^2}{2} \int_{-\alpha_1}^{0} \int_{t+\theta}^{t} \big(\nabla_x \dot H\big(x(s)\big)\big)^{\mathrm{T}} R_2 \big(\nabla_x \dot H\big(x(s)\big)\big)\,\mathrm{d}s\,\mathrm{d}\theta$$

and

$$\dot{V}_5 = \frac{\alpha_{23}^6}{36}(\dot{x})^{\mathrm{T}}\beta_5\dot{x} - \frac{\alpha_{23}^3}{6}\int_{-\alpha_{23}}^{0}\int_{\phi}^{0}\int_{t+\theta}^{t}\big(\nabla_x\dot{H}\big(x(s)\big)\big)^{\mathrm{T}}Z_1\big(\nabla_x\dot{H}\big(x(s)\big)\big)\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}\phi$$
$$+ \frac{\alpha_1^6}{36}(\dot{x})^{\mathrm{T}}\beta_6\dot{x} - \frac{\alpha_1^3}{6}\int_{-\alpha_1}^{0}\int_{\phi}^{0}\int_{t+\theta}^{t}\big(\nabla_x\dot{H}\big(x(s)\big)\big)^{\mathrm{T}}Z_2\big(\nabla_x\dot{H}\big(x(s)\big)\big)\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

For the second and the fourth terms of V_4, utilizing inequality (5) in Lemma 5, we have

$$-\frac{\alpha_{23}^2}{2}\int_{-\alpha_{23}}^{0}\int_{t+\theta}^{t}\big(\nabla_x\dot{H}\big(x(s)\big)\big)^{\mathrm{T}}R_1\big(\nabla_x\dot{H}\big(x(s)\big)\big)\,\mathrm{d}s\,\mathrm{d}\theta$$
$$\leqslant -\left(\alpha_{23}\nabla_x H(x) - \int_{t-\alpha_{23}}^{t}\nabla_x H\big(x(s)\big)\,\mathrm{d}s\right)^{\mathrm{T}}R_1$$
$$\times\left(\alpha_{23}\nabla_x H(x) - \int_{t-\alpha_{23}}^{t}\nabla_x H\big(x(s)\big)\,\mathrm{d}s\right)$$

and

$$-\frac{\alpha_1^2}{2}\int_{-\alpha_1}^{0}\int_{t+\theta}^{t}\big(\nabla_x\dot{H}\big(x(s)\big)\big)^{\mathrm{T}}R_2\big(\nabla_x\dot{H}\big(x(s)\big)\big)\,\mathrm{d}s\,\mathrm{d}\theta$$
$$\leqslant -\left(\alpha_1\nabla_x H(x) - \int_{t-\alpha_1}^{t}\nabla_x H\big(x(s)\big)\,\mathrm{d}s\right)^{\mathrm{T}}R_2$$
$$\times\left(\alpha_1\nabla_x H(x) - \int_{t-\alpha_1}^{t}\nabla_x H\big(x(s)\big)\,\mathrm{d}s\right).$$

For the second and the fourth terms of V_5, using inequality (6) in Lemma 5, we also obtain

$$-\frac{\alpha_{23}^3}{6}\int_{-\alpha_{23}}^{0}\int_{\phi}^{0}\int_{t+\theta}^{t}\big(\nabla_x\dot{H}\big(x(s)\big)\big)^{\mathrm{T}}Z_1\big(\nabla_x\dot{H}\big(x(s)\big)\big)\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}\phi$$
$$\leqslant -\left(\frac{\alpha_{23}^2}{2}\nabla_x H(x) - \int_{-\alpha_{23}}^{0}\int_{t+\phi}^{t}\nabla_x H\big(x(s)\big)\,\mathrm{d}s\,\mathrm{d}\phi\right)^{\mathrm{T}}Z_1$$
$$\times\left(\frac{\alpha_{23}^2}{2}\nabla_x H(x) - \int_{-\alpha_{23}}^{0}\int_{t+\phi}^{t}\nabla_x H\big(x(s)\big)\,\mathrm{d}s\,\mathrm{d}\phi\right)$$

and

$$-\frac{\alpha_1^3}{6}\int\limits_{-\alpha_1}^{0}\int\limits_{\phi}^{0}\int\limits_{t+\theta}^{t}\big(\nabla_x\dot{H}\big(x(s)\big)\big)^{\mathrm{T}}Z_2\big(\nabla_x\dot{H}\big(x(s)\big)\big)\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}\phi$$
$$\leqslant -\Bigg(\frac{\alpha_1^2}{2}\nabla_x H(x)-\int\limits_{-\alpha_1}^{0}\int\limits_{t+\phi}^{t}\nabla_x H\big(x(s)\big)\,\mathrm{d}s\,\mathrm{d}\phi\Bigg)^{\mathrm{T}}Z_2$$
$$\times\Bigg(\frac{\alpha_1^2}{2}\nabla_x H(x)-\int\limits_{-\alpha_1}^{0}\int\limits_{t+\phi}^{t}\nabla_x H\big(x(s)\big)\,\mathrm{d}s\,\mathrm{d}\phi\Bigg).$$

Here we also add constraints conditions (13) and (14). Taking $z=\dot{x}$, $\dot{V}_4$ and $\dot{V}_5$ become

$$\dot{V}_4=\frac{\alpha_{23}^4}{4}z^{\mathrm{T}}\beta_3 z-\frac{\alpha_{23}^2}{2}\int\limits_{-\alpha_{23}}^{0}\int\limits_{t+\theta}^{t}\big(\nabla_x\dot{H}\big(x(s)\big)\big)^{\mathrm{T}}R_1\big(\nabla_x\dot{H}\big(x(s)\big)\big)\,\mathrm{d}s\,\mathrm{d}\theta$$
$$+\frac{\alpha_1^4}{4}z^{\mathrm{T}}\beta_4 z-\frac{\alpha_1^2}{2}\int\limits_{-\alpha_1}^{0}\int\limits_{t+\theta}^{t}\big(\nabla_x\dot{H}\big(x(s)\big)\big)^{\mathrm{T}}R_2\big(\nabla_x\dot{H}\big(x(s)\big)\big)\,\mathrm{d}s\,\mathrm{d}\theta$$

and

$$\dot{V}_5=\frac{\alpha_{23}^6}{36}z^{\mathrm{T}}\beta_5 z-\frac{\alpha_{23}^3}{6}\int\limits_{-\alpha_{23}}^{0}\int\limits_{\phi}^{0}\int\limits_{t+\theta}^{t}\big(\nabla_x\dot{H}\big(x(s)\big)\big)^{\mathrm{T}}Z_1\big(\nabla_x\dot{H}\big(x(s)\big)\big)\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}\phi$$
$$+\frac{\alpha_1^6}{36}z^{\mathrm{T}}\beta_6 z-\frac{\alpha_1^3}{6}\int\limits_{-\alpha_1}^{0}\int\limits_{\phi}^{0}\int\limits_{t+\theta}^{t}\big(\nabla_x\dot{H}\big(x(s)\big)\big)^{\mathrm{T}}Z_2(\nabla_x\dot{H}(x(s)))\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

Combining the above discussion, constraint conditions (12), (13), and using $\dot{V}_1$, $\dot{V}_2$, and $\dot{V}_3$ in Theorem 2, we get

$$\dot{V}\leqslant\zeta^{\mathrm{T}}\Upsilon_0\zeta,$$

where

$$\zeta=\Bigg(\big(\nabla_x H(x)\big)^{\mathrm{T}},\ \big(\nabla_{x_{23}}H(x_{23})\big)^{\mathrm{T}},\ \big(\nabla_{x_1}H(x_1)\big)^{\mathrm{T}},\ \frac{1}{h_{23}(t)}\int\limits_{t-h_{23}(t)}^{t}\big(\nabla_x H\big(x(s)\big)\big)^{\mathrm{T}}\mathrm{d}s,$$
$$\frac{1}{h_1(t)}\int\limits_{t-h_1(t)}^{t}\big(\nabla_x H\big(x(s)\big)\big)^{\mathrm{T}}\mathrm{d}s,\ z,\ \int\limits_{t-\alpha_{23}}^{t}\big(\nabla_x H\big(x(s)\big)\big)^{\mathrm{T}}\mathrm{d}s,\ \int\limits_{t-\alpha_1}^{t}\big(\nabla_x H\big(x(s)\big)\big)^{\mathrm{T}}\mathrm{d}s,$$
$$\int\limits_{-\alpha_{23}}^{0}\int\limits_{t+\phi}^{t}\big(\nabla_x H\big(x(s)\big)\big)^{\mathrm{T}}\mathrm{d}s\,\mathrm{d}\phi,\ \int\limits_{-\alpha_1}^{0}\int\limits_{t+\phi}^{t}\big(\nabla_x H\big(x(s)\big)\big)^{\mathrm{T}}\mathrm{d}s\,\mathrm{d}\phi\Bigg)^{\mathrm{T}},$$

$$\varUpsilon_0 = \begin{pmatrix} \varUpsilon_{11}^0 & \varUpsilon_{12}^0 & \varUpsilon_{13} & 6W & 6M & \varUpsilon_{16} & \varUpsilon_{17} & \varUpsilon_{18} & \varUpsilon_{19} & \varUpsilon_{110} \\ * & \varUpsilon_{22}^0 & 0 & 6W & 0 & \varUpsilon_{26}^0 & 0 & 0 & 0 & 0 \\ * & * & \varUpsilon_{33} & 0 & 6M & T^{\mathrm{T}}N_2^{\mathrm{T}} & 0 & 0 & 0 & 0 \\ * & * & * & -12W & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -12M & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \varUpsilon_{66}^0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -R_1 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -R_2 & 0 & 0 \\ * & * & * & * & * & * & * & * & -Z_1 & 0 \\ * & * & * & * & * & * & * & * & * & -Z_2 \end{pmatrix}$$

with $\varUpsilon_{11}^0 = \varUpsilon_{11} + N_1 g g^{\mathrm{T}} N_1^{\mathrm{T}}$, $\varUpsilon_{12}^0 = \varUpsilon_{12} - N_1 g K g^{\mathrm{T}}$, $\varUpsilon_{22}^0 = \varUpsilon_{22} + 3gK^{\mathrm{T}}Kg$, $\varUpsilon_{26}^0 = -gK^{\mathrm{T}}g^{\mathrm{T}}N_2^{\mathrm{T}}$, and $\varUpsilon_{66}^0 = \varUpsilon_{66} + N_2 g g^{\mathrm{T}} N_2^{\mathrm{T}}$, where terms $2(\nabla_x H(x))^{\mathrm{T}} N_1 g v(t - h_2(t))$ and $2z^{\mathrm{T}} N_2 g v(t - h_2(t))$ are handled in the same way as the last term of $\dot{V}_1$ in Theorem 1.

Since $\varUpsilon_0$ is a nonlinear matrix, we obtain a linear matrix inequality $\varUpsilon < 0$, which is equivalent to inequality $\varUpsilon_0 < 0$, by utilizing the Schur complement. Hence, we have $\dot{V} < 0$. By the Lyapunov–Krasovskii functional stability theorem, system (1) is asymptotically stable under the feedback controller (2). □

Remark 7. Here we construct the Lyapunov–Krasovskii functionals with the form of triple and quadruple integrals. It is the first time to introduce these Lyapunov–Krasovskii functionals in the framework of the PH system. As mentioned in [7, 27, 37], they play key roles in the further reduction of conservativeness.

Remark 8. In [1], Aoues et al. studied state-delay PH systems subject to input saturation by using the Wirtinger-based integral inequality. By employing Wirtinger-based integral inequality method, Sun presented an improved stability analysis method for Hamiltonian systems with input saturation and time-varying delay. Compared to the results presented in [1, 28, 31, 38], Theorem 3 has less conservative with the aid of triple and quadruple integral form Lyapunov–Krasovskii functionals.

4 Numerical examples

In this section, we give numerical examples to verify the validity of our main results. Specifically, consider a PH system with mixed time-varying delays subject to input saturation:

$$\begin{aligned} \dot{x} &= (J - R)\nabla_x H(x) + T\nabla_{x_1} H(x_1) + g\,\mathrm{sat}\big(u\big(t - h_2(t)\big)\big), \\ y &= g^{\mathrm{T}} \nabla_{x_3} H(x_3), \end{aligned} \tag{15}$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \qquad T = \begin{pmatrix} -1 & -0.7 \\ 0.3 & -0.5 \end{pmatrix}, \qquad g = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Here the Hamiltonian function is $H(x) = 0.5x_1^2 + \sin^2 x_2$, and

$$\operatorname{sat}(u_i) = \operatorname{sign}\min\bigl\{|u_i|, 0.03\bigr\}, \quad i = 1, 2.$$

State-delay, input-delay, and out-delay functions $h_1(t)$, $h_2(t)$, $h_3(t)$ are given by

$$h_1(t) = \frac{\pi + 2\arctan t}{6}, \qquad h_2(t) = 0.2\sin^2 t, \qquad h_3(t) = \frac{\pi + 2\arctan t}{4},$$

respectively. It is obviously that the delay functions satisfy

$$\begin{aligned} &0 \leqslant h_1(t) \leqslant \frac{\pi}{3}, \qquad &&0 \leqslant h_2(t) \leqslant 0.2, \qquad &&0 \leqslant h_3(t) \leqslant \frac{\pi}{2}, \\ &\dot{h}_1(t) \leqslant \frac{1}{3} < 1, \qquad &&\dot{h}_2(t) \leqslant 0.2 < 1, \qquad &&\dot{h}_3(t) \leqslant \frac{1}{2} < 1. \end{aligned}$$

Therefore, $\alpha_1 = \pi/3$, $\alpha_2 = 0.2$, $\alpha_3 = \pi/2$, $\mu_1 = 1/3$, $\mu_2 = 0.2$, and $\mu_3 = 1/2$. According to the expression of $h_{23}(t) = h_2(t) + h_3(t - h_2(t))$, $\dot{h}_{23} = \dot{h}_2 + \dot{h}_3(t - h_2(t)) \times (1 - \dot{h}_2)$, we take $\mu_{23} = 0.6$, $\alpha_{23} = 0.2 + \pi/2$.

Since the matrix $g = (0, 1)^{\mathrm{T}}$, it is obviously that the first component of $g\operatorname{sat}(u(t - h_2(t)))$ is 0. Hence, we only consider the saturation control $\operatorname{sat}(u_2)$. In addition, when $t < h_{23}(t)$, system (15) cannot receive any information from the controller. In other words, $u = 0$ for $t < h_{23}(t)$.

Firstly, we verify Theorem 1. By applying MATLAB tool box, we obtain $\Psi < 0$ with the following parameters:

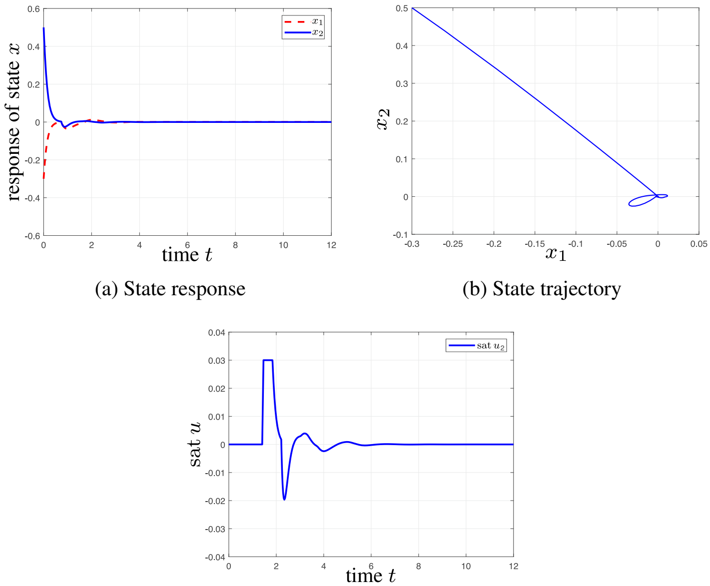
$$Q = \begin{pmatrix} 2.8745 & 0 \\ 0 & 2.5370 \end{pmatrix}, \qquad P = \begin{pmatrix} 2.7520 & 0 \\ 0 & 2.4810 \end{pmatrix}, \qquad K = -0.3853.$$

Hence, we introduce the output feedback controller $u = 0.3853y = 0.3853 \times \sin(2x_2(t - h_{23}(t)))$. The simulation with the initial condition $x_0 = (-0.3, 0.5)$ is given in Fig. 1. Figure 1(a) depicts the response of the system state, Fig. 1(b) shows the motion trajectory of the state variable x of the system, and Fig. 1(c) represents the control saturation input of the system. These numerical results all show that system (15) is asymptotically stable under the action of the controller $u = 0.3853y$. This is consistent with the result of our Theorem 1.

For Theorem 2, we also use Matlab tool box to get the inequality $\Theta < 0$ with

$$\begin{aligned} &Q = \begin{pmatrix} 1.7127 & 0.0664 \\ 0.0664 & 2.6009 \end{pmatrix}, \qquad &&P = \begin{pmatrix} 1.8850 & 0.1311 \\ 0.1311 & 1.3575 \end{pmatrix}, \\ &W = \begin{pmatrix} 0.0314 & 0.0008 \\ 0.0008 & 0.0035 \end{pmatrix}, \qquad &&M = \begin{pmatrix} 0.0938 & 0.0020 \\ 0.0020 & 0.0134 \end{pmatrix}, \\ &N_1 = \begin{pmatrix} 0.4761 & 0.1096 \\ 0.0434 & 0.1213 \end{pmatrix}, \qquad &&N_2 = \begin{pmatrix} 0.3438 & 0.0236 \\ 0.0489 & 0.1697 \end{pmatrix}, \qquad K = 0.4672. \end{aligned}$$

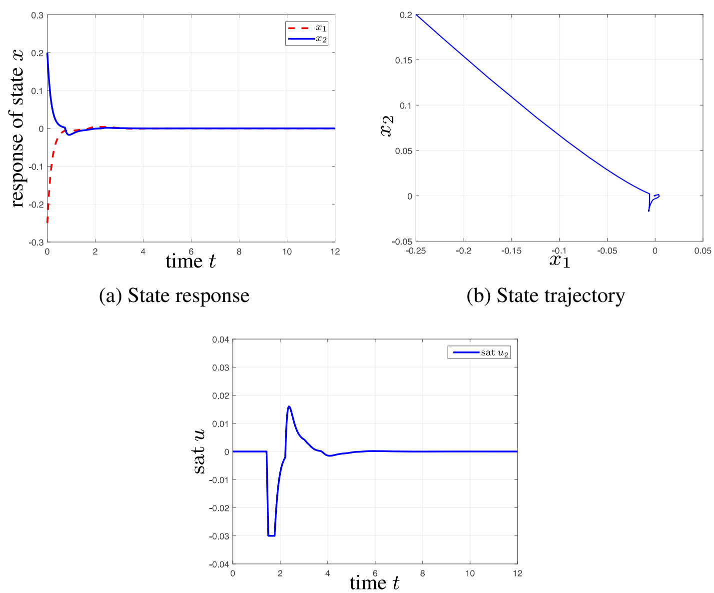
Given controller $u = -0.4672y$ with initial condition $x_0 = (-0.25, 0.2)$, there are also three figure in Fig. 2, which represent the response of the system state, the motion



(a) State response (b) State trajectory

(c) Input saturation

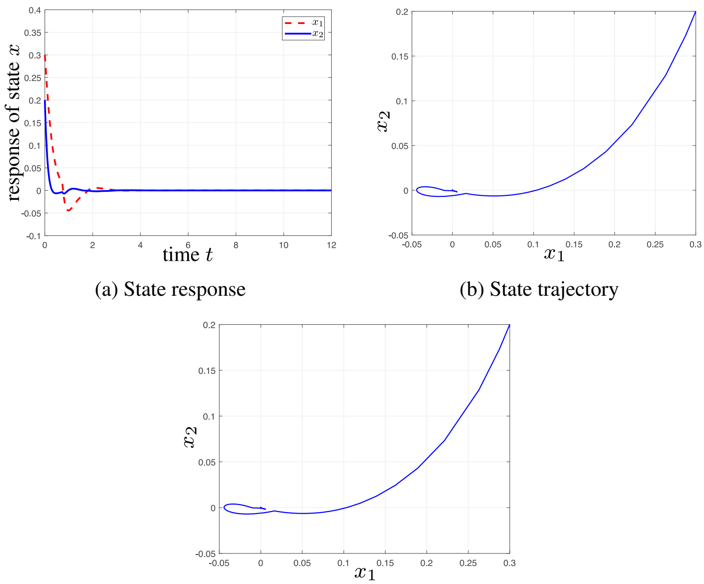
Figure 1. The simulations of Theorem 1 with $x_0 = (-0.3, 0.5)$, $u = 0.3853y$.



(a) State response (b) State trajectory

(c) Input saturation

Figure 2. The simulations of Theorem 2 with $x_0 = (-0.25, 0.2)$, $u = -0.4672y$.



(a) State response

(b) State trajectory

(c) Input saturation

Figure 3. The simulations of Theorem 3 with $x_0 = (0.3, 0.2)$, $u = -0.3217y$.

trajectory of the state, and the control saturation input of the system. The results show that system (15) is asymptotically stable under the action of the controller $u = -0.4672y$. This also verifies Theorem 2.

Similarly, calculating $\Upsilon < 0$ in Theorem 3, we have the following parameters:

$$Q = \begin{pmatrix} 1.0487 & 0.0700 \\ 0.0700 & 1.5373 \end{pmatrix}, \qquad P = \begin{pmatrix} 1.2552 & 0.2694 \\ 0.2694 & 1.2091 \end{pmatrix},$$
$$W = \begin{pmatrix} 0.0164 & 0 \\ 0 & 0.0022 \end{pmatrix}, \qquad M = \begin{pmatrix} 0.0529 & 0.0001 \\ 0.0001 & 0.0081 \end{pmatrix},$$
$$R_1 = \begin{pmatrix} 0.0200 & 0.0002 \\ 0.0002 & 0.0024 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 0.1661 & 0.0018 \\ 0.0018 & 0.0277 \end{pmatrix},$$
$$Z_1 = \begin{pmatrix} 0.0596 & 0.0004 \\ 0.0004 & 0.0105 \end{pmatrix}, \qquad Z_2 = \begin{pmatrix} 1.4453 & 0.0054 \\ 0.0054 & 0.2457 \end{pmatrix},$$
$$N_1 = \begin{pmatrix} 0.0114 & 0.04935 \\ 0.3387 & -0.0283 \end{pmatrix}, \qquad N_2 = \begin{pmatrix} 0.3093 & 0.0089 \\ 0.0672 & 0.1994 \end{pmatrix}, \qquad K = 0.3217.$$

Using MATLAB tool box, three figures representing the state response, the state trajectory, and the input saturation can be obtained as shown in Fig. 3 with initial condition $x_0 = (0.3, 0.2)$. It can be found from these figures that under the action of the controller $u = -0.3217y$, system (15) is also asymptotically stable. This is also consistent with the result of our Theorem 3.

5 Conclusion

In this paper, we propose three delay-dependent stability criteria in terms of linear matrix inequality by utilizing three different form Lyapunov–Krasovskii functionals. Using Wirtinger's inequality, constraint conditions, and triple and quadruple integral form Lyapunov–Krasovskii functionals, we prove that system (1) is asymptotically stable under the action of controller (2). Simulations show that the results obtained in this paper are very effective in analyzing the stabilization of some PH systems with mixed time delays subject to input saturation.

Recently, as in the case of nonlinear equations, there are studies on nonlocal equations, involving the reflection points, for example, nonlocal nonlinear Schrödinger-type equations [18, 19]. Such nonlocal differential equations, involving all three reflection points $(-x, t)$, $(x, -t)$, and $(-x, -t)$, are integrable, namely, they possess infinitely many symmetries and conservation laws. It would be particularly interesting and helpful to remark on whether the presented methodology could be applied to establishing stability of soliton solutions to such nonlocal equations or other stability [4, 17].

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