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Global threshold analysis of an age-space structured disease model with relapse*

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Abstract. In this paper, an age-space structured disease model with age-dependent relapse rate is investigated. We first prove the well-posedness of the model including the existence and uniqueness of the solution, positivity, and boundedness. By performing the Laplace transformation to renewal equation, we derive the next generation operator, whose spectral radius is defined as the basic reproduction number. By checking the distribution of the roots of the characteristic equation, exploring the strong persistence property of the solution and designing the Lyapunov functionals, we establish the local and global dynamics of the model.

Keywords: age-space structured model, basic reproduction number, threshold dynamics, global asymptotic stability.

1 Introduction and derivation of the model

Relapse phenomenon of disease exists widely in animal and human diseases such as tuberculosis, human herpes virus infection [15, 22]. It directly threatens public health and increases the burdens of patients due to reactivation or reinfection. A clinical study in [9] estimates that approximately 7.5% of tuberculosis patients was had previous tuberculosis. On the other hand, it is also argued in [5] that tuberculosis patients infected with HIV are easier to relapse. Mathematical models can help us understand the long-time disease dynamics as investigation such models may provide guides and suggestions for disease control. At time t, let $S(t)$, $I(t)$, and $R(t)$ be, respectively, the numbers

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of susceptible, infectious, and recovered individuals. In an earlier and standard model [19] for herpes infections in human and animal populations formulated by a susceptible-infectious-recovered-infectious (SIRI) structure, it is assumed that recovered individuals return back to the infectious individuals due to the reactivation of latent infection. The authors established the threshold-type result that the basic reproduction number (BRN) is the key threshold value that determines whether the disease dies out or not. Further, the authors in [14] extended the model in [19] by incorporating more general incidence functions and obtained the same threshold result.

In order to investigate the different consequences of distinct settings on relapse period, van den Driessche and Zou [21] designed a step function distribution for the constant relapse period and formulated a delay differential equation (DDE) SIRI model but without considering the exposed class. It is also revealed in [21] that BRN is the key threshold value, and a constant relapse period is not the reason of sustained oscillations.

It should be mentioned in [21] that the proportion of recovered individuals coming from infectious individuals with recovery rate γ was formulated by

$$R(t) = \int_0^t \gamma I(\xi) \mathrm{e}^{-d(t-\xi)} P(t-\xi)\,\mathrm{d}\xi \quad \text{with } R(0) = 0,$$

where $P(t)$ stands for the fraction that after time t, recovered individuals are still remaining in the recovered class, and d represents the death rate of recovered individuals. Differentiating the above equation gives

$$\frac{\mathrm{d}R(t)}{\mathrm{d}t} = -\mathrm{d}R(t) + \gamma I(t) + \int_0^t \gamma I(\xi) \mathrm{e}^{-d(t-\xi)} d_t P(t-\xi)\,\mathrm{d}\xi.$$

With the different settings for $P(t)$, for example, a negative exponential, compact support, and a step function, the model in [21] will reduce to an ODE model, a DDE model with finite distributed delay, and a DDE model with a single delay, respectively. In the mean time, $P(t)$ was also used in [21] for representing the probability that an exposed individual still remains in the exposed class. In a setting for $P(t)$ with a step function, van den Driessche, Wang, and Zou [20] formulated a DDE model and studied the threshold-type results. Especially, the authors in [11] resolved the global stability problem of endemic equilibrium even for general nonlinear incidence function. The models in [20, 21] have been extended to be multigroup disease models with general exposed distribution and relapse or with general relapse distribution and latency involving heterogeneity (see, for example, [26]).

Considering the fact that the relapse rate for recovery individuals varies from one to one, Liu et al. [10] introduced age-dependent relapse rate to model the waiting time for the risk of activation for tuberculosis and herpes virus infection. Denote by $R(t,a)$ the density of recovered individuals at time t with relapse age $a \geqslant 0$, then $M(t) = \int_0^\infty R(t,b)\,\mathrm{d}b$ represents the total number of recovered individuals. Assuming that the age-dependent relapse rate of recovered individuals is given by the function $r(a) \in L_+^\infty(0, +\infty)$, the rate of change of $R(t, a)$ in [10] is given by

$$\frac{\partial R(t,a)}{\partial t} + \frac{\partial R(t,a)}{\partial a} = -\big(\mu_R + r(a)\big)R(t,a) \quad \text{with } R(t,0) = kI(t),$$

where μ_R represents the natural death rate. In [10], the threshold-type results relying on the BRN were established by appealing to the integrated semigroup theory, persistence theory in infinite dimensional dynamical system, and Lyapunov functionals. A similar formulation for recovered individuals with age-dependent relapse rate can be found in [25]. It should be noted in [10, 25] that relapse phenomenon was described by partial differential equation (PDE) with age-dependent relapse rate instead of DDE with general relapse distribution.

Reaction–diffusion epidemic models have been widely adopted to model the spatial dynamics of infectious disease. It is widely accepted that reaction–diffusion equations for disease dynamics are meaningful and important for demonstrating the spatial heterogeneity in disease transmission, although more theoretic analysis tools are needed. Unlike in [10, 25], where the rate of change of $R(t, a)$ is dominated by a first-order PDE, here we allow reaction–diffusion equation of $R(t, a)$ in a bounded domain $\Omega \subset \mathbb{R}$ with smooth boundary $\partial\Omega$. We introduce the spatial variable $x \in \Omega$ and let $S := S(t, x)$ and $I := I(t, x)$ be, respectively, the spatial densities of susceptible and infectious individuals at location $x \in \Omega$ and time t, dispersing across habitat with diffusion coefficients d_S and d_I. Following from the standard argument on structured population and spatial diffusion [13], the density of recovered individuals at time t and $x \in \Omega$ with relapse age $a \geqslant 0$, denoted by $R := R(t, a, x)$, fulfills

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right)R = d_R \Delta R - \big(\mu_R + r(a)\big)R, \quad t > 0,\ x \in \Omega, \\
&R(t,0,x) = kI, \quad t > 0,\ x \in \Omega,
\end{aligned}$$

where d_R is the diffusion coefficient. To make things not to be complicated, we use the simple growth term for susceptible individuals with the recruitment rate λ and death rate μ_S, and the interactions between susceptible and infectious individuals fulfill the mass action infection mechanism with disease transmission rate $\beta > 0$. For the biologically significant, we denote by

$$S(0,x) = \phi_1(x), \quad I(0,x) = \phi_2(x), \quad R(0,a,x) = \phi_3(a,x), \quad a \geqslant 0,\ x \in \Omega,$$

the initial data for susceptible, infectious, and recovered individuals at time $t = 0$, and $r^+ := \operatorname{ess\,sup}_{a\geqslant 0} r(a) < +\infty$, the essential upper bounds of $r(a) \in L_+^\infty(0, +\infty)$. Let n be the outward normal on $\partial\Omega$. We impose the following no flux condition on the boundary:

$$\frac{\partial \mathcal{W}}{\partial n} = 0, \quad \mathcal{W} = S, I, R, \quad t > 0,\ x \in \partial\Omega, \tag{1}$$

where means that no populations across the boundary of the domain. With these preparations, we shall investigate the following model:

$$\begin{aligned}
&\frac{\partial S}{\partial t} = d_S \Delta S + \lambda - \mu_S S - \beta SI,\\
&\frac{\partial I}{\partial t} = d_I \Delta I + \beta SI - (\mu_I + k)I + \int_0^\infty r(a) R \,\mathrm{d}a,\\
&\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) R = d_R \Delta R - \big(\mu_R + r(a)\big) R,\\
&R(t, 0, x) = kI
\end{aligned} \tag{2}$$

for $t > 0$, $x \in \Omega$ and boundary condition (1).

For convenience, we let $\Pi(a) = \mathrm{e}^{-\int_0^a [\mu_R + r(\sigma)]\mathrm{d}\sigma}$, and let $\Gamma_3(a, x, y)$ be the Green function of $d_R\Delta$ subject to (1). An application of the standard argument of characteristics to solve the third equation in (2) leads to

$$R = \begin{cases} \Pi(a) \int_\Omega \Gamma_3(a, x, y) kI(t-a,\, y)\,\mathrm{d}y, & t - a > 0,\ x \in \Omega,\\ \frac{\Pi(a)}{\Pi(a-t)} \int_\Omega \Gamma_3(t, x, y)\phi_3(a-t,\, y)\,\mathrm{d}y, & a - t \geqslant 0,\ x \in \Omega. \end{cases} \tag{3}$$

Let $(d_S, d_I, d_R) = (d_1, d_2, d_3)$. With the help of (3), we substitute (3) into the I equation of (2), which results in the following coupled system:

$$\begin{aligned}
&\frac{\partial S}{\partial t} = d_1 \Delta S + \lambda - \mu_S S - \beta SI,\\
&\frac{\partial I}{\partial t} = d_2 \Delta I + \beta SI - (\mu_I + k)I + \mathfrak{F}_1 + \mathfrak{F}_2,\\
&\mathfrak{F}_1 = \int_0^t r(a)\Pi(a) \int_\Omega \Gamma_3(a, x, y) kI(t-a, y)\,\mathrm{d}y\,\mathrm{d}a,\\
&\mathfrak{F}_2 = \int_t^{+\infty} r(a) \frac{\Pi(a)}{\Pi(a-t)} \int_\Omega \Gamma_3(t, x, y)\phi_3(a-t, y)\,\mathrm{d}y\,\mathrm{d}a
\end{aligned} \tag{4}$$

for $t > 0$ and $x \in \Omega$.

Our main motivation of this paper is to resolve the question that whether the threshold-type results as those in [10, 25] can be preserved in the reaction–diffusion model with age-dependent relapse rate. To this end, we give a detailed analysis of the well-posedness of the model in Section 2. In Section 3, the BRN is derived through seeking the next generation operator. By checking the distribution of the roots of the characteristic equation, we will investigate the local dynamics of the model in Section 4. Section 5 is spent on studying the strong persistence property of the model. By designing the Lyapunov functionals, the global attractivity of the equilibria is obtained in Section 6. Section 7 ends the paper with a brief conclusion and some discussions.

2 Well-posedness of the model

This section is spent on investigating the positivity, existence and uniqueness, and boundedness of the solution of reformulated system (4). Before going into details, we define the appropriate phase space for (4). Let $\mathbb{X} := C(\bar{\Omega}, \mathbb{R})$ be the space of continuous functions with usual norm $\|\varphi\|_{\mathbb{X}} = \max\{|\varphi|\}$, $\varphi \in \mathbb{X}$, and positive cone $\mathbb{X}^+$. Let $\mathbb{Y} := L^1(\mathbb{R}_+, \mathbb{X})$ be the Lebesgue measure space with the norm $\|\varphi\|_{\mathbb{Y}} := \int_0^{+\infty} \|\varphi(a)\|_{\mathbb{X}}\,\mathrm{d}a$, $\varphi \in \mathbb{Y}$, and positive cone $\mathbb{Y}^+$. Denote by Γ_i $(i = 1, 2)$ the Green functions of $d_i\Delta$ $(i = 1, 2)$ subject to (1). By a standard argument as in [16, Thm. 1.5] and [17, Cor. 7.2.3], the Laplace operator $d_i\Delta$ $(i = 1, 2)$ subject to (1) generates a strongly positive and compact semigroup on $\mathbb{X}^+$:

$$\big(T_i(t)[\phi]\big)(x) = \int\limits_{\Omega} \Gamma_i(t, x, y)\phi(y)\,\mathrm{d}y, \quad i = 1, 2.$$

It then follows from the properties of Γ_i that

$$\big\|T_i(t)\phi\big\|_{\mathbb{X}} \leqslant \int\limits_{\Omega} \Gamma_i(t, x, y)\,\mathrm{d}y\,\|\phi\|_{\mathbb{X}} = \|\phi\|_{\mathbb{X}}. \tag{5}$$

Further, $T(t) = (T_1(t), T_2(t)) : \mathbb{X}^+ \times \mathbb{X}^+ \to \mathbb{X}^+ \times \mathbb{X}^+$, $t \geqslant 0$, forms a strongly continuous semigroup.

The well-posedness result of (4) is stated as follows.

Theorem 1. *For each $\phi \in \mathbb{X}^+ \times \mathbb{X}^+$, system* (4) *has a unique global nonnegative classical solution (S, I), which is defined on $[0, +\infty) \times \overline{\Omega}$. Further, the semiflow generated by* (4)

$$\Phi[\phi](t) = \big(S(t, \cdot), I(t, \cdot)\big), \quad t \geqslant 0,$$

admits a global compact attractor in $\mathbb{X}^+ \times \mathbb{X}^+$.

The assertions in Theorem 1 will be verified by the following lemmas.

Lemma 1. *Let $\phi \in \mathbb{X}^+ \times \mathbb{X}^+$. (S, I) is the unique solution of system* (4) *on $[0, T) \times \bar{\Omega}$ with $T > 0$.*

Proof. Directly solving the S and I equation of (4) gives

$$\begin{aligned} S &= F_S + \int\limits_0^t \mathrm{e}^{-\mu_S(t-s)} \int\limits_{\Omega} \Gamma_1(t-s,\, x, y)\big(\lambda - \beta S(s, y)I(s, y)\big)\,\mathrm{d}y\,\mathrm{d}s \\ &=: \mathcal{F}_1(S, I)(t, x), \\ I &= F_I + \int\limits_0^t \mathrm{e}^{-(\mu_I+k)(t-s)} \int\limits_{\Omega} \Gamma_2(t-s,\, x, y)\big[\mathfrak{B}(S, I)(s, y)\big]\,\mathrm{d}y\,\mathrm{d}s \\ &=: \mathcal{F}_2(S, I)(t, x), \end{aligned}$$

where $F_S := F_S(t, x) = \mathrm{e}^{-\mu_S t} \int_{\Omega} \Gamma_1(t, x, y)\phi_1(y)\,\mathrm{d}y$, $F_I := F_I(t, x) = \mathrm{e}^{-(\mu_I+k)t} \int_{\Omega} \Gamma_2(t, x, y)\phi_2(y)\,\mathrm{d}y$, and $\mathfrak{B}(S, I)(s, y) = \beta S(s, y)I(s, y) + \mathfrak{F}_1(s, y) + \mathfrak{F}_2(s, y)$.

For $T > 0$, we set $\mathbb{X}_T := C([0,T],\mathbb{X})$ with $\|\psi\|_{\mathbb{X}_T} := \sup_{0\leqslant t\leqslant T}\|\psi(t,\cdot)\|_{\mathbb{X}}$, $\psi \in \mathbb{X}_T$, and $\mathbb{W}_T := \mathbb{X}_T \times \mathbb{X}_T$ with $\|(\psi_1,\psi_2)\|_{\mathbb{W}_T} := \|\psi_1\|_{\mathbb{X}_T} + \|\psi_2\|_{\mathbb{X}_T}$, $(\psi_1,\psi_2) \in \mathbb{W}_T$. Let

$$\mathcal{F}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} := \begin{pmatrix}\mathcal{F}_1(\psi_1,\psi_2)\\ \mathcal{F}_2(\psi_1,\psi_2)\end{pmatrix}, \quad \psi_1,\psi_2 \in \mathbb{W}_T. \tag{6}$$

Next, we show that (4) has a unique solution on $[0,T]\times\overline{\Omega}$ through verifying that $\mathcal{F} : \mathbb{W}_T \to \mathbb{W}_T$ has a fixed point. For any $(S',I'), (S'',I'') \in \mathbb{W}_T$, we have

$$\|\beta S'I' - \beta S''I''\|_{\mathbb{X}_T} \leqslant \beta\big(\|I'\|_{\mathbb{X}_T}\|S'-S''\|_{\mathbb{X}_T} + \|S''\|_{\mathbb{X}_T}\|I'-I''\|_{\mathbb{X}_T}\big).$$

By (5), we obtain

$$\begin{aligned}
&\big\|\mathcal{F}_1(S',I') - \mathcal{F}_1(S'',I'')\big\|_{\mathbb{X}_T}\\
&\quad\leqslant \int_0^t \mathrm{e}^{-\mu_S(t-s)}\,\mathrm{d}s\,\|\beta S'I' - \beta S''I''\|_{\mathbb{X}_T}\\
&\quad\leqslant \frac{\beta(1-\mathrm{e}^{-\mu_S t})}{\mu_S}\big(\|I'\|_{\mathbb{X}_T}\|S'-S''\|_{\mathbb{X}_T} + \|S''\|_{\mathbb{X}_T}\|I'-I''\|_{\mathbb{X}_T}\big)\\
&\quad\leqslant g_1(T)\left\|\begin{pmatrix}S'\\ I'\end{pmatrix} - \begin{pmatrix}S''\\ I''\end{pmatrix}\right\|_{\mathbb{W}_T},
\end{aligned}$$

where

$$g_1(T) := \frac{\beta(1-\mathrm{e}^{-\mu_S T})}{\mu_S}\max\big(\|I'\|_{\mathbb{X}_T}, \|S''\|_{\mathbb{X}_T}\big).$$

Note that for any $0 < T_* < T$, we can regard (S',I'), (S'',I'') as functions in $\mathbb{W}_{T_*}$, and

$$\begin{aligned}
g_1(T_*) &= \frac{\beta(1-\mathrm{e}^{-\mu_S T_*})}{\mu_S}\max\big(\|I'\|_{\mathbb{X}_{T_*}}, \|S''\|_{\mathbb{X}_{T_*}}\big)\\
&\leqslant \frac{\beta(1-\mathrm{e}^{-\mu_S T_*})}{\mu_S}\max\big(\|I'\|_{\mathbb{X}_T}, \|S''\|_{\mathbb{X}_T}\big) = \frac{1-\mathrm{e}^{-\mu_S T_*}}{1-\mathrm{e}^{-\mu_S T}}g_1(T),
\end{aligned}$$

and thus, $g_1(T_*) \to 0$ as $T_* \to +0$. Hence, without loss of generality, we select $T > 0$ small enough as a new initial time such that $g_1(T) < 1$ (regarding T_* such that $g(T_*) < 1$ as a new T). In a similar manner,

$$\begin{aligned}
&\big\|\mathfrak{B}(S',I') - \mathfrak{B}(S'',I'')\big\|_{\mathbb{X}_T}\\
&\quad\leqslant \sup_{0\leqslant s\leqslant T}\left\{\|\beta S'I' - \beta S''I''\|_{\mathbb{X}_T} + \int_0^s kr(a)\int_\Omega \varGamma_3(a,x,y)\|I'-I''\|_{\mathbb{X}_T}\varPi(a)\,\mathrm{d}y\,\mathrm{d}a\right\}\\
&\quad\leqslant \beta\big(\|I'\|_{\mathbb{X}_T}\|S'-S''\|_{\mathbb{X}_T} + \|S''\|_{\mathbb{X}_T}\|I'-I''\|_{\mathbb{X}_T}\big) + \mathfrak{M}\|I'-I''\|_{\mathbb{X}_T},
\end{aligned}$$

where $\mathfrak{M} = kr^+T$, and hence,

$$\left\|\mathcal{F}_2(S', I') - \mathcal{F}_2(S'', I'')\right\|_{\mathbb{X}_T} \leqslant g_2(T)\left\|\begin{pmatrix} S' \\ I' \end{pmatrix} - \begin{pmatrix} S'' \\ I'' \end{pmatrix}\right\|_{\mathbb{W}_T},$$

$$g_2(T) := \frac{(1 - \mathrm{e}^{-(\mu_I + k)T})}{(\mu_I + k)} \max\bigl\{\beta\|I'\|_{\mathbb{X}_T}, \beta\|S''\|_{\mathbb{X}_T} + \mathfrak{M}\bigr\}.$$

Similar to the case of g_1, we can set sufficiently small initial time T ensuring that $g_2(T) < 1$. Consequently, we obtain

$$\left\|\begin{pmatrix} S' \\ I' \end{pmatrix} - \mathcal{F}\begin{pmatrix} S'' \\ I'' \end{pmatrix}\right\|_{\mathbb{W}_T} \leqslant \max\bigl\{g_1(T), g_2(T)\bigr\}\left\|\begin{pmatrix} S' \\ I' \end{pmatrix} - \begin{pmatrix} S'' \\ I'' \end{pmatrix}\right\|_{\mathbb{W}_T}.$$

As $\max\{g_1(T), g_2(T)\} < 1$, the operator $\mathcal{F}$ admits a unique fixed point in $\mathbb{W}_T$ in the sense that $\mathcal{F}$ is a strict contraction map in $\mathbb{W}_T$. Thus, the local existence of solution (S, I) of system (4) directly follows. □

The following result indicates that the solution of (4) is positive.

Lemma 2. *For each $\phi \in \mathbb{X}^+ \times \mathbb{X}^+$, the solution of* (4) *is positive for* $(t, x) \in (0, T) \times \Omega$, *that is,* $S(t, x) > 0$, $I(t, x) \geqslant 0$ *for all* $(0, T) \times \Omega$.

Proof. We first show that $S(t, x) > 0$ for $(t, x) \in (0, T) \times \Omega$. Denote $Q_T = (0, T] \times \Omega$ and $S_T = (0, T] \times \partial\Omega$. We proceed it indirectly and suppose that $S(t, x)$ is a negative solution. Hence, there exists $(t_1, x_*) \in Q_T$ such that $S(t_1, x_*) = 0$ and $S'(t_1, x_*) \leqslant 0$. However, from the S equation of (4) and strong maximum principle we get $S'(t_1, x_*) = d_S \Delta S(t_1, x_*) + \lambda > 0$, a contradiction. Meanwhile, if $S(t_1', x_*') = 0$ for some $(t_1', x_*') \in S_T$, then by the Hopf boundary lemma, we can get $\partial S(t_1', x_*')/\partial n < 0$, a contradiction.

We next verify the positivity of I on $(t, x) \in [0, T) \times \Omega$ by appealing to the arguments on Picard sequences. We first set

$$\boldsymbol{I}_0 = F_I + \int_0^t \mathrm{e}^{-(\mu_I + k)(t-s)} \int_\Omega \Gamma_2(t - s,\ x, y)\mathfrak{F}_2(s, y)\,\mathrm{d}y\,\mathrm{d}s > 0.$$

Assume that $I_n > 0$, $n \in \mathbb{N}$. Due to the positivity of β, k, $\Pi(\cdot)$, and Γ_i ($i = 2, 3$), it is obvious that

$$\begin{aligned}
\boldsymbol{I}_{n+1} = \boldsymbol{I}_0 &+ \int_0^t \mathrm{e}^{-(\mu_I + k)(t-s)} \int_\Omega \Gamma_2(t - s,\ x, y)\beta S(s, y)\boldsymbol{I}_n(s, y)\,\mathrm{d}y\,\mathrm{d}s \\
&+ \int_0^t \mathrm{e}^{-(\mu_I + k)(t-s)} \int_\Omega \Gamma_2(t - s,\ x, y) \int_0^s r(b)\Pi(b) \\
&\quad \times \int_\Omega \Gamma_3(b, y, z)k\boldsymbol{I}_n(s - b,\ z)\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}s \\
&> 0.
\end{aligned}$$

It remains to investigate the convergence of the sequence $\{\boldsymbol{I}_n\}_0^\infty$ as $n \to \infty$ by setting

$$\tilde{\boldsymbol{I}}_n = \mathrm{e}^{-\rho t}\boldsymbol{I}_n, \quad \rho \in \mathbb{R}_+.$$

It is readily seen that

$$\begin{aligned}
\tilde{\boldsymbol{I}}_{n+1} &= \mathrm{e}^{-\rho t}\boldsymbol{I}_0 + \beta\int_0^t \mathrm{e}^{-(\mu_I+k)s}\int_\Omega \Gamma_2(s,x,y)\mathrm{e}^{-\rho t}S(t-s,\,y)\boldsymbol{I}_n(t-s,\,y)\,\mathrm{d}y\,\mathrm{d}s \\
&\quad + k\int_0^t \mathrm{e}^{-(\mu_I+k)s}\int_\Omega \Gamma_2(s,x,y)\int_0^{t-s} r(b)\Pi(b) \\
&\qquad \times \int_\Omega \Gamma_3(b,y,z)\mathrm{e}^{-\rho t}\boldsymbol{I}_n(t-s-b,\,z)\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}s \\
&= \mathrm{e}^{-\rho t}\boldsymbol{I}_0 + \beta\int_0^t \mathrm{e}^{-(\mu_I+k)s}\int_\Omega \Gamma_2(s,x,y)\mathrm{e}^{-\rho s}S(t-s,\,y)\tilde{\boldsymbol{I}}_n(t-s,\,y)\,\mathrm{d}y\,\mathrm{d}s \\
&\quad + k\int_0^t \mathrm{e}^{-(\mu_I+k)s}\int_\Omega \Gamma_2(s,x,y)\int_0^{t-s} r(b)\Pi(b) \\
&\qquad \times \int_\Omega \Gamma_3(b,y,z)\mathrm{e}^{-\rho(s+b)}\tilde{\boldsymbol{I}}_n(t-s-b,\,z)\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}s.
\end{aligned}$$

Let

$$I_n^\sharp = \max_{(t,x)\in[0,T]\times\Omega} \tilde{\boldsymbol{I}}_n(t,x), \quad n \in \mathbb{N}.$$

By elementary calculation, then we have

$$\begin{aligned}
\big\|\boldsymbol{I}_{n+1}^\sharp - \boldsymbol{I}_n^\sharp\big\|_\infty &\leqslant \beta S^+\int_0^t \mathrm{e}^{-(\mu_I+k)s}\int_\Omega \Gamma_2(s,x,y)\mathrm{e}^{-\rho s}\,\mathrm{d}y\,\mathrm{d}s\,\big\|\boldsymbol{I}_n^\sharp - \boldsymbol{I}_{n-1}^\sharp\big\|_\infty \\
&\quad + k\int_0^t \mathrm{e}^{-(\mu_I+k)s}\int_\Omega \Gamma_2(s,x,y)\int_0^{t-s} r(b)\Pi(b) \\
&\qquad \times \int_\Omega \Gamma_3(b,y,z)\mathrm{e}^{-\rho(s+b)}\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}s\big\|\boldsymbol{I}_n^\sharp - \boldsymbol{I}_{n-1}^\sharp\big\|_\infty \\
&\leqslant \frac{\beta S^+\rho + kr^+}{\rho(\mu_I+k+\rho)}\big\|\boldsymbol{I}_n^\sharp - \boldsymbol{I}_{n-1}^\sharp\big\|_\infty,
\end{aligned}$$

where $S^+ = \max_{t\in[0,T]}\|S(t,\cdot)\|_{\mathbb{X}_T}$. After passing the iteration, we have

$$\big\|\boldsymbol{I}_{n+1}^\sharp - I_n^\sharp\big\|_\infty \leqslant M_\rho\big\|\boldsymbol{I}_n^\sharp - \boldsymbol{I}_{n-1}^\sharp\big\|_\infty \leqslant \cdots \leqslant M_\rho^n\big\|\boldsymbol{I}_1^\sharp - \boldsymbol{I}_0^\sharp\big\|_\infty,$$

where $M_\rho = (\beta S^+ \rho + kr^+)/\rho(\mu_I + k + \rho)$. It is easy to see that

$$\|I^\sharp_m - I^\sharp_n\|_\infty \leqslant \frac{M^n_\rho}{1 - M_\rho}\|I^\sharp_1 - I^\sharp_0\|_\infty, \quad m, n \in \mathbb{N}.$$

As a result, we can select a sufficiently small $\rho > 0$ such that $M_\rho < 1$ ensuring that $\|I^\sharp_m - I^\sharp_n\|_\infty \to 0$ as $n \to \infty$. This tells us that $\lim_{n\to\infty} \boldsymbol{I}_n(t, x) = I(t, x)$ on $(t, x) \in [0, T) \times \Omega$. The positivity of I directly follows from the positivity of $\boldsymbol{I}_n$. This proves Lemma 2. □

We are now in a position to confirm that the solution (S, I) of (4) exists globally. We shall confirm this by checking that the solution is bounded in $[0, T)$.

Lemma 3. *For each $\phi \in \mathbb{X}^+ \times \mathbb{X}^+$, the solution (S, I) of* (4) *is bounded in* $[0, T)$.

Proof. It is well known that

$$\begin{aligned}
&\frac{\partial w}{\partial t} = d_w \Delta w + \lambda - \mu_S w, \quad x \in \Omega,\ t > 0,\\
&\frac{\partial w}{\partial n} = 0, \quad x \in \partial\Omega,\ t > 0,
\end{aligned}$$

admits a unique positive steady state $w^* = \lambda/\mu_S$, which is globally attractive in $\mathbb{X}$. By the standard comparison principle, S is bounded above by λ/μ_S.

Assume for the contrary that I is unbounded, that is, there exist $t^* > 0$ and $x^* \in \Omega$ such that $\lim_{t\to t^*} I(t, x^*) = +\infty$. Then by S-equation of (4), we know that $\lim_{t\to t^*} \partial_t S(t, x^*) = -\infty$. Hence, $S(t, x^*)$ becomes negative near the t^*, a contradiction with the positivity of S. Hence, I is bounded in $[0, T)$. □

Based on the above lemmas, we now briefly prove Theorem 1.

Proof of Theorem 1. From Theorem 1 we establish the local existence and uniqueness of the solution (S, I) of (4). Lemma 2 confirms that the solution (S, I) of (4) is positive. By Lemma 3, the solution (S, I) of (4) is bounded in $[0, T)$. Hence, the first assertion in Theorem 1 holds directly. The second assertion is a direct consequence of applying the general results in [8, Thm. 2.4.6]. This proves Theorem 1. □

3 Basic reproduction number and equilibria

Obviously, system (4) admits a disease-free equilibrium $E_0 = (S^0, 0)$, where $S^0 = \lambda/\mu_S$. Linearizing (4) at E_0, we obtain

$$\begin{aligned}
&\frac{\partial S}{\partial t} = d_1 \Delta S + \lambda - \mu_S S - \beta S^0 I,\\
&\frac{\partial I}{\partial t} = d_2 \Delta I + \beta S^0 I - (\mu_I + k) I + \mathfrak{F}_1
\end{aligned}$$

for $t > 0$ and $x \in \Omega$, where $\mathfrak{F}_1$ is defined in (4).

Next, we consider only the infectious disease compartment and solve it directly, yielding that

$$
\begin{aligned}
I = F_I + \beta S^0 \int_0^t \mathrm{e}^{-(\mu_I+k)s} \int_\Omega \Gamma_2(s,x,y) I(t-s,\, y)\,\mathrm{d}y\,\mathrm{d}s \\
+ k \int_0^t \mathrm{e}^{-(\mu_I+k)s} \int_\Omega \Gamma_2(s,x,y) \int_0^{t-s} r(b)\Pi(b) \\
\times \int_\Omega \Gamma_3(b,y,z) I(t-s-b,\, z)\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}s.
\end{aligned} \tag{7}
$$

Note that (7) is a renewal equation. Performing Laplace transformation to (7) gives

$$
\begin{aligned}
\int_0^\infty \mathrm{e}^{-\omega t} I(t,x)\,\mathrm{d}t &= \beta S^0 \int_0^\infty \mathrm{e}^{-\omega t} \int_0^t \mathrm{e}^{-(\mu_I+k)s} \int_\Omega \Gamma_2(s,x,y) I(t-s,\, y)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}t \\
&\quad + \int_0^\infty \mathrm{e}^{-\omega t} \int_0^t \mathrm{e}^{-(\mu_I+k)s} \int_\Omega \Gamma_2(s,x,y) \int_0^{t-s} r(b)\Pi(b) \\
&\quad \times \int_\Omega \Gamma_3(b,y,z) I(t-s-b,\, z)\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}t \\
&:= \Psi_1(x) + \Psi_2(x).
\end{aligned}
$$

After passing multiple interchanging the order of integration, one will get

$$
\Psi_1(x) = \beta S^0 \int_0^\infty \mathrm{e}^{-(\mu_I+k)s} \mathrm{e}^{-\omega s} \int_\Omega \Gamma_2(s,x,y) \int_0^\infty \mathrm{e}^{-\omega t} I(t,y)\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s
$$

and

$$
\begin{aligned}
\Psi_2(x) = k \int_0^\infty \mathrm{e}^{-(\mu_I+k)s} \mathrm{e}^{-\omega s} \int_\Omega \Gamma_2(s,x,y) \int_0^\infty r(b)\Pi(b)\mathrm{e}^{-\omega b} \\
\times \int_\Omega \Gamma_3(b,y,z) \int_0^\infty \mathrm{e}^{-\omega t} I(t,z)\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}s.
\end{aligned}
$$

By letting $\omega = 0$, we have

$$
\int_0^\infty I(t,x)\,\mathrm{d}t = \beta S^0 \int_0^\infty \mathrm{e}^{-(\mu_I+k)s} \int_\Omega \Gamma_2(s,x,y) \int_0^\infty I(t,y)\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s
$$

$$+ k \int_0^\infty \mathrm{e}^{-(\mu_I+k)s} \int_\Omega \Gamma_2(s,x,y) \int_0^\infty r(b)\Pi(b) \times \int_\Omega \Gamma_3(b,y,z) \int_0^\infty I(t,z)\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}s.$$

This allows us to define

$$\mathcal{L}[\varphi](x) := \beta S^0 \int_0^\infty \mathrm{e}^{-(\mu_I+k)s} \int_\Omega \Gamma_2(s,x,y)\varphi(y)\,\mathrm{d}y\,\mathrm{d}s + k \int_0^\infty \mathrm{e}^{-(\mu_I+k)s} \int_\Omega \Gamma_2(s,x,y) \int_0^\infty r(b)\Pi(b) \times \int_\Omega \Gamma_3(b,y,z)\varphi(z)\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}s \tag{8}$$

for $\varphi \in \mathbb{X}$. By [27], $\mathcal{L}$ is called the next-generation operator, and its spectral radius is referred as the BRN of (4) denoted by $\mathfrak{R}_0 := r(\mathcal{L})$.

In order to get the explicit expression of $\mathfrak{R}_0$, we need prove the following result.

Lemma 4. *$\mathcal{L}$ is strictly positive and compact.*

Proof. Clearly, $\mathcal{L}$ is positive. Let $\psi_n = \mathcal{L}[\varphi_n]$, where $\{\varphi_n\}_{n\in\mathbb{N}}$, $n \in \mathbb{N}$, is a bounded sequence in $\mathbb{X}$ in the sense that for some $B > 0$, $|\varphi_n|_{\mathbb{X}} \leqslant B$. For all $x \in \Omega$,

$$\psi_n(x) \leqslant \beta S^0 \int_0^\infty \mathrm{e}^{-(\mu_I+k)s} \int_\Omega \Gamma_2(s,x,y)\,\mathrm{d}y\,\mathrm{d}sB + k \int_0^\infty \mathrm{e}^{-(\mu_I+k)s} \int_\Omega \Gamma_2(s,x,y) \int_0^\infty r(b)\Pi(b) \int_\Omega \Gamma_3(b,y,z)\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}sB.$$

This proves the uniform boundedness of $\{\psi_n\}_{n\in\mathbb{N}}$. Further, for $x, \tilde{x} \in \Omega$ with $|x - \tilde{x}| < \delta$, we have

$$\bigl|\psi_n(x) - \psi_n(\tilde{x})\bigr| \leqslant \beta S^0 B \int_0^\infty \mathrm{e}^{-(\mu_I+k)s} \int_\Omega \bigl|\Gamma_2(s,x,y) - \Gamma_2(s,\widetilde{x},y)\bigr|\,\mathrm{d}y\,\mathrm{d}a + kB \int_0^\infty \mathrm{e}^{-(\mu_I+k)s} \int_\Omega \bigl|\Gamma_2(s,x,y) - \Gamma_2(s,\widetilde{x},y)\bigr| \int_0^\infty r(b)\Pi(b) \times \int_\Omega \Gamma_3(b,y,z)\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}s.$$

Due to the uniform continuity of Γ_2, we can choose $\varepsilon > 0$ such that

$$\left|\Gamma_2(s,x,y) - \Gamma_2(s,\widetilde{x},y)\right| \leqslant \frac{\mu_S \varepsilon(\mu_I + k)}{(\beta\lambda + k\mu_S Q)B|\Omega|},$$

where $Q = \int_0^\infty r(b)\Pi(b)\,\mathrm{d}b$. This leads to $|\psi_n(x) - \psi_n(\tilde{x})| < \varepsilon$, that is, $\psi_n(x)_{n\in N}$ is equicontinuous. Hence, $\mathcal{L}$ is compact. This proves Lemma 4. □

Further, from Krein–Rutman theorem [2, Thm. 3.2] we substitute $\varphi(x) \equiv [\hbar] > 0$ into (8) and use $\int_\Omega \Gamma_i(\cdot,x,y) = 1$ ($i = 2,3$) obtaining that

$$\begin{aligned}
\mathcal{L}[\hbar] &= \beta S^0 \int_0^\infty \mathrm{e}^{-(\mu_I+k)s} \int_\Omega \Gamma_2(s,x,y)\,\mathrm{d}y\,\mathrm{d}s[\hbar] \\
&\quad + k\int_0^\infty \mathrm{e}^{-(\mu_I+k)s} \int_\Omega \Gamma_2(s,x,y) \int_0^\infty r(b)\Pi(b) \int_\Omega \Gamma_3(b,y,z)\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}s[\hbar] \\
&= \beta S^0 \int_0^\infty \mathrm{e}^{-(\mu_I+k)s}\,\mathrm{d}s[\hbar] + k\int_0^\infty \mathrm{e}^{-(\mu_I+k)s} \int_0^\infty r(b)\Pi(b)\,\mathrm{d}b\,\mathrm{d}s[\hbar].
\end{aligned}$$

Hence, $\mathfrak{R}_0$ can be obtained by

$$\mathfrak{R}_0 = \beta S^0 \int_0^\infty \mathrm{e}^{-(\mu_I+k)s}\,\mathrm{d}s + k\int_0^\infty \mathrm{e}^{-(\mu_I+k)s} \int_0^\infty r(b)\Pi(b)\,\mathrm{d}b\,\mathrm{d}s = \frac{\beta S^0 + kQ}{\mu_I + k}. \tag{9}$$

Note that this $\mathfrak{R}_0$ is similar to the one obtained in [6, 12]. We next claim that if $\mathfrak{R}_0 > 1$, then there exists a positive space-independent equilibrium for (2), denoted by $E^* = (S^*, I^*, R^*(a))$, which fulfills

$$\begin{aligned}
&0 = \lambda - \mu_S S^* - \beta S^* I^*, \qquad 0 = \beta S^* I^* - (\mu_I + k)I^* + \int_0^{+\infty} r(a)R^*(a)\,\mathrm{d}a, \\
&\frac{\mathrm{d}R^*(a)}{\mathrm{d}a} = -\big(\mu_R + r(a)\big)R^*(a), \qquad R^*(0) = kI^*.
\end{aligned} \tag{10}$$

In fact, from the last two equations of (10) we have

$$R^*(a) = kI^*\Pi(a).$$

This, combined with the second equation of (10), gives that

$$S^* = \frac{\lambda}{\mu_S + \beta I^*} = \frac{\mu_I + k - kQ}{\beta}. \tag{11}$$

Consequently, by the first equation of (10) and the expression of $\mathfrak{R}_0$, we have

$$I^* = \frac{\mu_S(\mu_I + k)(\mathfrak{R}_0 - 1)}{\beta(\mu_I + k - kQ)}.$$

4 Local stability of equilibria

Clearly, system (2) admits a disease-free equilibrium $E_0 = (S^0, 0, 0)$. We shall prove that the equilibria of system (2) are locally asymptotically stable (LAS) in terms of the sign of $\mathfrak{R}_0 - 1$.

Theorem 2. *Let* $\mathfrak{R}_0$ *be defined by* (9). *Then we have*:

(i) *If* $\mathfrak{R}_0 < 1$, *then* E_0 *is LAS*;
(ii) *If* $\mathfrak{R}_0 > 1$, *then* E^* *is LAS.*

Proof. We begin with the proof of (i). It is crucial to determine the characteristic equation of E_0. To this end, we linearize (2) around $E_0 = (S^0, 0, 0)$ obtaining that

$$\begin{aligned}
&\frac{\partial S}{\partial t} = d_1 \Delta S - \mu_S S - \beta S^0 I,\\
&\frac{\partial I}{\partial t} = d_2 \Delta I + \beta S^0 I - (\mu_I + k) I + \int_0^\infty r(a) R \,\mathrm{d}a,\\
&\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) R = d_3 \Delta R - \big(\mu_R + r(a)\big) R,\\
&R(t, 0, x) = k I(t, x).
\end{aligned} \tag{12}$$

By [3], we let ζ_j ($j = 1, 2, \dots$) be the eigenvalues of $-\Delta$ with (1), i.e., $\Delta z(x) = -\zeta_i z(x)$. Plugging $(\mathrm{e}^{\eta t}(\phi(x), \psi(x), \xi(a, x))$ into (12) gives

$$\begin{aligned}
&\eta \phi(x) = -d_1 \zeta_i \phi(x) - \mu_S \phi(x) - \beta S^0 \psi(x),\\
&\eta \psi(x) = -d_2 \zeta_i \psi(x) + \beta S^0 \psi(x) - (\mu_I + k) \psi(x) + \int_0^\infty r(a) \xi(a, x) \,\mathrm{d}a,\\
&\eta \xi(a, x) + \frac{\partial \xi(a, x)}{\partial a} = -d_3 \zeta_i \xi(a, x) - \big(\mu_R + r(a)\big) \xi(a, x),\\
&\xi(0, x) = k \psi(x).
\end{aligned} \tag{13}$$

Directly solving $\xi(a, x)$ of (13) and substituting it into the second equation of (13) allow us to rewrite (13) in terms of $(\phi(x), \psi(x))$. Then we have

$$\begin{vmatrix} \eta + d_1 \zeta_i + \mu_S & \beta S^0 \\ 0 & \mathcal{C}(\eta, \zeta_i) \end{vmatrix} = 0,$$

$$\mathcal{C}(\eta, \zeta_i) := \eta + d_2 \zeta_i - \beta S^0 + \mu_I + k - k \int_0^\infty r(a) \mathrm{e}^{-\eta a} \tilde{\Pi}(a) \,\mathrm{d}a$$

and $\tilde{\Pi}(a) = \mathrm{e}^{-d_3 \zeta_i a} \Pi(a)$. Then we only pay attention to the roots of $\mathcal{C}(\eta, \zeta_i) = 0$, that is,

$$1 = \frac{\beta S^0 + k \int_0^\infty r(a) \mathrm{e}^{-\eta a} \mathrm{e}^{-d_3 \zeta_i a} \Pi(a) \,\mathrm{d}a}{\eta + d_2 \zeta_i + \mu_I + k}.$$

If we assume that $\mathcal{C}(\eta,\zeta_i)=0$ admits an eigenvalue η with $\mathrm{Re}(\eta)\geqslant 0$, we then have

$$\begin{aligned}1&=\left|\frac{\beta S^0+k\int_0^\infty r(a)\mathrm{e}^{-\eta a}\mathrm{e}^{-d_3\zeta_i a}\varPi(a)\,\mathrm{d}a}{\eta+d_2\zeta_i+\mu_I+k}\right|\\&\leqslant\frac{\beta S^0+k\int_0^\infty r(a)\varPi(a)\,\mathrm{d}a}{\mu_I+k}=\frac{\beta S^0+kQ}{\mu_I+k}=\mathfrak{R}_0,\end{aligned}$$

a contradiction with $\mathfrak{R}_0<1$. It then follows that all the real eigenvalues of $\mathcal{C}(\eta,\zeta_i)=0$ are negative. On the other hand, if we let $\eta=m\pm ni$ with $m\geqslant 0$ and $n>0$ be a pair of complex roots of $\mathcal{C}(\eta,\zeta_i)=0$, it follows that

$$\begin{aligned}1&=\left|\frac{\beta S^0(m+d_2\zeta_i+\mu_I+k)+k(m+d_2\zeta_i+\mu_I+k)H_1-knH_2}{(m+d_2\zeta_i+\mu_I+k)^2+n^2}\right|\\&\leqslant\frac{\beta S^0(m+d_2\zeta_i+\mu_I+k)+k(m+d_2\zeta_i+\mu_I+k)H_1}{(m+d_2\zeta_i+\mu_I+k)^2}\\&\leqslant\frac{\beta S^0+k\int_0^\infty r(a)\varPi(a)\,\mathrm{d}a}{\mu_I+k}=\mathfrak{R}_0,\end{aligned}$$

where $H_1=\int_0^\infty r(a)\mathrm{e}^{-ma}\cos(na)\tilde{\varPi}(a)\,\mathrm{d}a$ and $H_2=\int_0^\infty r(a)\mathrm{e}^{-ma}\sin(na)\tilde{\varPi}(a)\,\mathrm{d}a$. This again results in a contradiction with $\mathfrak{R}_0<1$. This proves (i).

We next prove (ii). Linearizing (2) around $E^*=(S^*,I^*,R^*(a))$, we obtain

$$\begin{aligned}&\frac{\partial S}{\partial t}=d_1\Delta S-\mu_S S-\beta S^*I-\beta SI^*,\\&\frac{\partial I}{\partial t}=d_2\Delta I+\beta S^*I+\beta SI^*-(\mu_I+k)I+\int_0^\infty r(a)R\,\mathrm{d}a,\\&\left(\frac{\partial R}{\partial t}+\frac{\partial}{\partial a}\right)R=d_3\Delta R-\big(\mu_R+r(a)\big)R,\\&R(t,0,x)=kI.\end{aligned}\tag{14}$$

Substituting $\mathrm{e}^{\eta t}(\phi_1(x),\psi_1(x),\xi_1(a,x))$ into (14), we obtain

$$\begin{aligned}&\eta\phi_1(x)=-d_1\zeta_i\phi_1(x)-\mu_S\phi_1(x)-\beta S^*\psi_1(x)-\beta I^*\phi_1(x),\\&\eta\psi_1(x)=-d_2\zeta_i\psi_1(x)+\beta S^*\psi_1(x)+\beta I^*\phi_1(x)-(\mu_I+k)\psi_1(x)\\&\qquad\qquad+\int_0^\infty r(a)\xi_1(a,x)\,\mathrm{d}a,\\&\eta\xi_1(a,x)+\frac{\partial\xi_1(a,x)}{\partial a}=-d_3\zeta_i\xi_1(a,x)-\big(\mu_R+r(a)\big)\xi_1(a,x),\\&\xi_1(0,x)=k\psi_1(x).\end{aligned}\tag{15}$$

Directly solving $\xi_1(a,x)$ of (15) and substituting it into the second equation of (15) allow us to rewrite (15) in terms of $(\phi_1(x),\psi_1(x))$. Hence, it is sufficient to consider the following characteristic equation:

$$\begin{aligned}&(\eta + d_1\zeta_i + \mu_S)\beta S^*\\&\quad + (\eta + d_1\zeta_i + \mu_S + \beta I^*)\big[kH_3(\eta) - d_2\zeta_i - (\eta + \mu_I + k)\big] = 0,\end{aligned} \tag{16}$$

where $H_3(\eta) = \int_0^\infty r(a)\mathrm{e}^{-\eta a}\tilde{\Pi}(a)\,\mathrm{d}a$. Furthermore, (16) can also be rewritten as

$$\frac{\frac{\eta + d_1\zeta_i + \mu_S}{\eta + d_1\zeta_i + \mu_S + \beta I^*} \cdot \beta S^* + kH_3(\eta) - d_2\zeta_i}{\eta + \mu_I + k} = 1. \tag{17}$$

Assume that $\mathrm{Re}(\eta) \geqslant 0$, it then follows that $|H_3(\eta)| \leqslant Q$. This, together with $|(\eta + d_1\zeta_i + \mu_S)/(\eta + d_1\zeta_i + \mu_S + \beta I^*)| < 1$ and (11), implies that

$$\left|\frac{\frac{\eta + d_1\zeta_i + \mu_S}{\eta + d_1\zeta_i + \mu_S + \beta I^*} \cdot \beta S^* + kH_3(\eta) - d_2\zeta_i}{\eta + \mu_I + k}\right| < \frac{\beta S^* + kQ}{\mu_I + k} = 1,$$

which leads to a contradiction with (17). That is to say, all the real eigenvalues of (17) are negative. If we let $\eta = m \pm n\mathrm{i}$ with $m \geqslant 0$ and $n > 0$ be a pair of complex roots (17), it follows that

$$\begin{aligned}&\left|\frac{\frac{\eta + d_1\zeta_i + \mu_S}{\eta + d_1\zeta_i + \mu_S + \beta I^*} \cdot \beta S^* + kH_3(\eta) - d_2\zeta_i}{\eta + \mu_I + k}\right|\\&\quad < \frac{\beta S^* + k\int_0^\infty r(a)\mathrm{e}^{-\eta a}\tilde{\Pi}(a)\,\mathrm{d}a}{\eta + \mu_I + k}\\&\quad = \frac{\beta S^*(m + \mu_I + k) + k(m + \mu_I + k)H_1 - knH_2}{(m + \mu_I + k)^2 + n^2}\\&\quad < \frac{\beta S^* + k\int_0^\infty r(a)\Pi(a)\,\mathrm{d}a}{\mu_I + k} = 1,\end{aligned}$$

a contradiction with (17). This proves (ii). □

5 Disease persistence

By [18, Sect. 9.4], we will establish the dynamics of the solution of system (2) when $\mathfrak{R}_0 > 1$. Let us first introduce the following conclusion about the solution of system (2), whose proof method is similar to [4, Lemma 5.1], which will not be repeated here.

Lemma 5. *Let $\phi = (\phi_1, \phi_2, \phi_3) \in \mathbb{X}^+ \times \mathbb{X}^+ \times \mathbb{Y}^+$. System* (2) *admits a continuous semiflow, which is written by $\Theta(t, \phi_1, \phi_2, \phi_3) := (S(t,\cdot), I(t,\cdot), R(t,\cdot,\cdot)) \in \mathbb{X}^+ \times \mathbb{X}^+ \times \mathbb{Y}^+$ for all $t \geqslant 0$.*

Let $D := \{(\phi_1, \phi_2, \phi_3) \in \mathbb{X}^+ \times \mathbb{X}^+ \times \mathbb{Y}^+\colon\ \phi_2 > 0$ for some $x \in \Omega\}$. Following the idea of [1, Lemma 6.1], we obtain the following conclusion.

Lemma 6. *Assume that* $\mathfrak{R}_0 > 1$. *Then there exists a* $\varepsilon_1 > 0$ *such that* $I(t,\cdot)$ *with* $\phi_2 \in D$ *satisfies* $\limsup_{t\to\infty} |I(t,\cdot)|_{\mathbb{X}} > \varepsilon_1$.

Proof. Since $\mathfrak{R}_0 > 1$, we can choose $\varepsilon_1 > 0$ such that

$$\frac{\beta\lambda}{\beta\epsilon_1 + \mu_S}\bigl(1 - \mathrm{e}^{-(\mu_S+\beta\varepsilon_1)t}\bigr)\int_0^\infty \mathrm{e}^{-(\mu_I+k)s}\,\mathrm{d}s + k\int_0^\infty \mathrm{e}^{-(\mu_I+k)s}\int_0^\infty r(b)\varPi(b)\,\mathrm{d}b\,\mathrm{d}s > 1. \tag{18}$$

We proceed the assertion indirectly and assume that there is $\hat{t} > 0$ such that $I \leqslant \varepsilon_1$ for $t > \hat{t}$, $x \in \varOmega$. By (18), there is sufficiently large $\hat{t}_1 > \hat{t}$ and small $\theta > 0$ such that

$$\widetilde{\mathfrak{R}} := \frac{\beta\lambda}{\beta\epsilon_1 + \mu_S}\bigl(1 - \mathrm{e}^{-(\mu_S+\beta\varepsilon_1)h}\bigr)\int_0^\infty \mathrm{e}^{-(\mu_I+k)s}\mathrm{e}^{-\theta s}\,\mathrm{d}s + k\int_0^\infty \mathrm{e}^{-(\mu_I+k)s}\mathrm{e}^{-\theta s}\int_0^\infty r(b)\varPi(b)\mathrm{e}^{-\theta b}\,\mathrm{d}b\,\mathrm{d}s > 1, \tag{19}$$

where $h = \hat{t}_1 - \hat{t}$. On the other hand, for all $t > \hat{t}_1$, $x \in \varOmega$, $\partial S/\partial t \geqslant d_1\Delta S + \lambda - \beta\varepsilon_1 S - \mu_S S$. From the standard comparison principle we then have

$$\begin{aligned} S &\geqslant \mathrm{e}^{-(\mu_S+\beta\varepsilon_1)(t-\hat{t})}\int_\varOmega \varGamma_1(t-\hat{t}, x, y)S(\hat{t}, y)\,\mathrm{d}y + \frac{\lambda}{\beta\epsilon_1 + \mu_S}\bigl(1 - \mathrm{e}^{-(\mu_S+\beta\varepsilon_1)(t-\hat{t})}\bigr) \\ &\geqslant \frac{\lambda}{\beta\epsilon_1 + \mu_S}\bigl(1 - \mathrm{e}^{-(\mu_S+\beta\varepsilon_1)h}\bigr) \end{aligned}$$

for all $t > \hat{t}_1$, $x \in \varOmega$. With the help of Lemma 5, we let $\hat{t}_1$ be the initial time. Hence,

$$\begin{aligned} I \geqslant{} & \frac{\beta\lambda}{\beta\epsilon_1 + \mu_S}\bigl(1 - \mathrm{e}^{-(\mu_S+\beta\varepsilon_1)h}\bigr)\int_0^t \mathrm{e}^{-(\mu_I+k)s}\int_\varOmega \varGamma_2(s,x,y)I(t-s, y)\,\mathrm{d}y\,\mathrm{d}s \\ & + k\int_0^t \mathrm{e}^{-(\mu_I+k)s}\int_\varOmega \varGamma_2(s,x,y)\int_0^{t-s} r(b)\varPi(b) \\ & \times \int_\varOmega \varGamma_3(b,y,z)I(t-s-b, z)\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}s. \end{aligned} \tag{20}$$

Due to the fact that $\int_0^\infty \mathrm{e}^{-\theta t}I(t,x)\,\mathrm{d}t < +\infty$ for any $x \in \varOmega$, we can find a $\tilde{x} \in \varOmega$ ensuring that

$$\int_0^\infty \mathrm{e}^{-\theta t}I(t,\tilde{x})\,\mathrm{d}t = \min_{x\in\varOmega}\int_0^\infty \mathrm{e}^{-\theta t}I(t,x)\,\mathrm{d}t.$$

This, together with (20), tells us that

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{-\theta t} I(t,\tilde{x})\,\mathrm{d}t \geqslant{}& \frac{\beta\lambda}{\beta\epsilon_1+\mu_S}\bigl(1-\mathrm{e}^{-(\mu_S+\beta\varepsilon_1)h}\bigr) \\
&\times \int_0^{+\infty} \mathrm{e}^{-\theta t} \int_0^t \mathrm{e}^{-(\mu_I+k)s} \int_\Omega \Gamma_2(s,\tilde{x},y) I(t-s,\,y)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}t \\
&+ k \int_0^{+\infty} \mathrm{e}^{-\theta t} \int_0^t \mathrm{e}^{-(\mu_I+k)s} \int_\Omega \Gamma_2(s,\tilde{x},y) \int_0^{t-s} r(b)\Pi(b) \\
&\quad\times \int_\Omega \Gamma_3(b,y,z) I(t-s-b,\,z)\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}t, \\
:={}& \Psi_3+\Psi_4.
\end{aligned}$$

After passing multiple interchanging the order of integration, one will get

$$\begin{aligned}
\Psi_3 ={}& \frac{\beta\lambda}{\beta\epsilon_1+\mu_S}\bigl(1-\mathrm{e}^{-(\mu_S+\beta\varepsilon_1)h}\bigr) \\
&\times \int_0^\infty \mathrm{e}^{-(\mu_I+k)s}\mathrm{e}^{-\theta s} \int_\Omega \Gamma_2(s,\tilde{x},y) \int_0^\infty \mathrm{e}^{-\theta t} I(s,y)\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}
\end{aligned}$$

and

$$\begin{aligned}
\Psi_4 ={}& k \int_0^\infty \mathrm{e}^{-(\mu_I+k)s}\mathrm{e}^{-\theta s} \int_\Omega \Gamma_2(a,\tilde{x},y) \int_0^\infty r(b)\Pi(b)\mathrm{e}^{-\theta b} \int_\Omega \Gamma_3(b,y,z) \\
&\times \int_0^\infty \mathrm{e}^{-\theta t} I(t,z)\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}s.
\end{aligned}$$

Consequently,

$$\int_0^\infty \mathrm{e}^{-\theta t} I(t,\tilde{x})\,\mathrm{d}t \geqslant \widetilde{\mathfrak{R}} \int_0^\infty \mathrm{e}^{-\theta t} I(t,\tilde{x})\,\mathrm{d}t. \tag{21}$$

In virtue of (19), inequality (21) leads to a contradiction. This proves Lemma 6. $\square$

Following the idea of [7, Thm. 1] and Lemma 6, then we have the strong $|\cdot|_{\mathbb{X}}$-persistence result about the solution of system (4), whose proof method is similar to [4, Prop. 5.3] and [24, Lemma 4.5], which will not be repeated here.

Lemma 7. *If* $\mathfrak{R}_0 > 1$, *then* $\liminf_{t\to\infty} |I(t,\cdot)|_{\mathbb{X}} > \varepsilon_2$ *for any* $(\phi_1,\phi_2) \in D$.

6 Global dynamics

This section is spent on designing the Lyapunov functionals to solve the global attractivity of equilibria of system (2). We further can conclude that the equilibria of system (2) are globally asymptotically stable (GAS) in terms of the sign of $\Re_0 - 1$. For ease of notations, we always use

$$G(u,v) = u - v - v\ln\frac{u}{v} \quad \text{for } u, v \in \mathbb{R}_+.$$

Clearly, $G(u,v) = 0$ if and only if $u = v$.

Theorem 3. *For any $(\phi_1, \phi_2, \phi_3) \in D$. The following statements hold true*:

(i) *If $\Re_0 < 1$, then E_0 is GAS*;

(ii) *If $\Re_0 > 1$, then E^* is GAS.*

Proof. We first prove (i). Let us set $L^1_{E_0} = G(S, S^0)$. Then differentiating L_1 along the solution of (2) yields

$$\frac{\partial L^1_{E_0}}{\partial t} = d_1\left(1 - \frac{S^0}{S}\right)\Delta S - \mu_S\frac{(S-S^0)^2}{S} + \beta S^0 I - \beta SI. \tag{22}$$

Let $L^2_{E_0} = I$. Then the derivative of $L^2_{E_0}$ along the solution of (2) is just the I equation of (2), namely,

$$\frac{\partial L^2_{E_0}}{\partial t} = d_2\Delta I + \beta SI - (\mu_I + k)I + \int_0^\infty r(a)R(t,a,x)\,\mathrm{d}a. \tag{23}$$

We further set $L^3_{E_0} = \int_0^\infty \Psi(a)R\,\mathrm{d}a$, where $\Psi(a)$ will be determined later. With the help of (3), we rewrite $L^3_{E_0}$ as

$$\begin{aligned}
L^3_{E_0} &= \int_0^t \Psi(r)\Pi(r)\int_\Omega \Gamma_3(r,x,y)kI(t-r,\,y)\,\mathrm{d}y\,\mathrm{d}r \\
&\quad + \int_t^\infty \Psi(r)\frac{\Pi(r)}{\Pi(r-t)}\int_\Omega \Gamma_3(t,x,y)\phi_3(r-t,\,y)\,\mathrm{d}y\,\mathrm{d}r \\
&= \int_0^t \Psi(t-a)\Pi(t-a)\int_\Omega \Gamma_3(t-a,\,x,y)kI(a,y)\,\mathrm{d}y\,\mathrm{d}a \\
&\quad + \int_0^\infty \Psi(a+t)\frac{\Pi(a+t)}{\Pi(a)}\int_\Omega \Gamma_3(t,x,y)\phi_3(a,y)\,\mathrm{d}y\,\mathrm{d}a.
\end{aligned}$$

Then we have

$$\frac{\partial L^3_{E_0}}{\partial t} = \Psi(0)kI + \int_0^\infty \big[\Psi'(a) - \big(\mu_R + r(a) - d_3\Delta\big)\Psi(a)\big]R\,\mathrm{d}a. \tag{24}$$

Let us define a Lyapunov functional for E_0 as

$$L_{E_0}(t) = \int_\Omega \sum_{i=1}^{3} L_{E_0}^i \,\mathrm{d}x.$$

This, combined with (22), (23) and (24), gives

$$\begin{aligned}
\frac{\mathrm{d}L_{E_0}(t)}{\mathrm{d}t} &= \int_\Omega \left\{ d_1\left(1 - \frac{S^0}{S}\right)\Delta S - \mu_S \frac{(S-S^0)^2}{S} + \beta S^0 I - \beta S I \right\} \mathrm{d}x \\
&\quad + \int_\Omega \left\{ d_2 \Delta I + \beta S I - (\mu_I + k) I + \int_0^\infty r(a) R(t,a,x)\,\mathrm{d}a \right\} \mathrm{d}x \\
&\quad + \int_\Omega \left\{ \Psi(0) k I + \int_0^\infty \bigl[\Psi'(a) - \bigl(\mu_R + r(a) - d_3\Delta\bigr)\Psi(a)\bigr] R\,\mathrm{d}a \right\} \mathrm{d}x \\
&= \int_\Omega d_1 \frac{(S-S^0)\Delta S}{S}\,\mathrm{d}x - \int_\Omega \mu_S \frac{(S-S^0)^2}{S}\,\mathrm{d}x \\
&\quad + \int_\Omega d_2 \Delta I\,\mathrm{d}x + \int_\Omega \bigl[\beta S^0 I + \Psi(0) k I(t,x) - (\mu_I + k) I\bigr]\,\mathrm{d}x \\
&\quad + \int_\Omega \int_0^\infty \bigl[r(a) + \Psi'(a) - \bigl(\mu_R + r(a) - d_3\Delta\bigr)\Psi(a)\bigr] R\,\mathrm{d}a\,\mathrm{d}x.
\end{aligned} \tag{25}$$

We are now in a position to define

$$\Psi(a) = \int_a^\infty r(\theta) \frac{\Pi(\theta)}{\Pi(a)}\,\mathrm{d}\theta.$$

Obviously, $\Psi'(a) = -r(a) + (\mu_R + r(a))\Psi(a)$ and $\Psi(0) = \int_0^\infty r(a)\Pi(a)\,\mathrm{d}a = Q$. Hence, (25) becomes

$$\frac{\mathrm{d}L_{E_0}(t)}{\mathrm{d}t} = -d_1 \int_\Omega \frac{|\nabla S|^2}{S^2}\,\mathrm{d}x - \mu_S \int_\Omega \frac{(S-S^0)^2}{S}\,\mathrm{d}x + \int_\Omega (\mu_I + k)(\mathfrak{R}_0 - 1) I\,\mathrm{d}x.$$

If $\mathfrak{R}_0 < 1$, then by the invariance principle [23, Thm. 4.2] and Theorem 2, E_0 is GAS in D. This proves (i).

We next prove (ii). Let $L_{E^*}^1 = G(S, S^*)$. With the aid of $\lambda = \mu_S S^* + \beta S^* I^*$, we can obtain the derivative of $L_{E^*}^1$ as follows:

$$\frac{\partial L_{E^*}^1}{\partial t} = d_1\left(1 - \frac{S^*}{S}\right)\Delta S - \frac{\mu_S}{S}(S - S^*)^2 + \beta S^* I^* \left(1 - \frac{SI}{S^* I^*} - \frac{S^*}{S} + \frac{I}{I^*}\right). \tag{26}$$

Let $L^2_{E^*} = G(I, I^*)$. This, together with $\beta S^* I^* + \int_0^\infty r(a)R^*(a)\,\mathrm{d}a = (\mu_I + k)I^*$, gives that the derivative of $L^2_{E^*}$ satisfies

$$\begin{aligned}\frac{\partial L^2_{E^*}}{\partial t} &= \left(1 - \frac{I^*}{I}\right)\left(d_2\Delta I + \beta SI - \beta S^* I - \frac{I}{I^*}\int_0^\infty r(a)R^*(a)\,\mathrm{d}a + \int_0^\infty r(a)R\,\mathrm{d}a\right) \\ &= d_2\left(1 - \frac{I^*}{I}\right)\Delta I + \beta S^* I^*\left(1 + \frac{SI}{S^* I^*} - \frac{S}{S^*} - \frac{I}{I^*}\right) \\ &\quad + \int_0^\infty r(a)R^*(a)\left(1 - \frac{I}{I^*} - \frac{I^* R}{IR^*(a)} + \frac{R}{R^*(a)}\right)\mathrm{d}a. \end{aligned} \tag{27}$$

We further set $L^3_{E^*} = \int_0^\infty \Theta(a)G(R, R^*(a))\,\mathrm{d}a$, where $\Theta(a)$ will be determined later. By direct calculation of the derivative of $L^3_{E^*}$, we obtain

$$\begin{aligned}\frac{\partial L^3_{E^*}}{\partial t} &= -\int_0^\infty \Theta(a)\left(1 - \frac{R^*(a)}{R}\right)\left(\frac{\partial R}{\partial a} - d_3\Delta R + \bigl(\mu_R + r(a)\bigr)R\right)\mathrm{d}a \\ &= d_3\int_0^\infty \Theta(a)\left(1 - \frac{R^*(a)}{R}\right)\Delta R\,\mathrm{d}a \\ &\quad - \int_0^\infty \Theta(a)\left(1 - \frac{R^*(a)}{R}\right)\frac{\partial R}{\partial a}\,\mathrm{d}a - \int_0^\infty \Theta(a)\left(1 - \frac{R^*(a)}{R}\right)\bigl[\mu_R + r(a)\bigr]R\,\mathrm{d}a. \end{aligned}$$

Recall that

$$R^*(a)\frac{\partial}{\partial a}\left(g\left(\frac{R}{R^*(a)}\right)\right) = \left(1 - \frac{R^*(a)}{R}\right)\frac{\partial R}{\partial a} + \left(1 - \frac{R^*(a)}{R}\right)\bigl[\mu_R + r(a)\bigr]R,$$

where $g(u) = u - 1 - \ln u$ for $u \in \mathbb{R}_+$. This, together with $\mathrm{d}R^*(a)/\mathrm{d}a = -[\mu_R + r(a)] \times R^*(a)$, directly leads to

$$\begin{aligned}\frac{\partial L^3_{E^*}}{\partial t} &= d_3\int_0^\infty \Theta(a)\left(1 - \frac{R^*(a)}{R}\right)\Delta R\,\mathrm{d}a - \int_0^\infty \Theta(a)R^*(a)\frac{\partial}{\partial a}\left(g\left(\frac{R}{R^*(a)}\right)\right)\mathrm{d}a \\ &= d_3\int_0^\infty \Theta(a)\left(1 - \frac{R^*(a)}{R}\right)\Delta R\,\mathrm{d}a - \Theta(a)R^*(a)g\left(\frac{R}{R^*(a)}\right)\Bigg|_{a=0}^{a=\infty} \\ &\quad + \int_0^\infty g\left(\frac{R}{R^*(a)}\right)\bigl(\Theta'(a) - \bigl(\mu_R + r(a)\bigr)\Theta(a)\bigr)R^*(a)\,\mathrm{d}a. \end{aligned} \tag{28}$$

Let

$$\Theta(a) = \int_a^\infty r(\theta)\frac{\Pi(\theta)}{\Pi(a)}\,\mathrm{d}\theta.$$

Then it satisfies

$$\frac{\mathrm{d}\Theta(a)}{\mathrm{d}a} = \big[\mu_R + r(a)\big]\Theta(a) - r(a), \qquad \Theta(0) = \int_0^\infty r(a)\Pi(a)\,\mathrm{d}a = Q.$$

Hence, (28) becomes

$$\begin{aligned}\frac{\partial L^3_{E^*}}{\partial t} &= d_3 \int_0^\infty \Theta(a)\left(1 - \frac{R^*(a)}{R}\right)\Delta R\,\mathrm{d}a + \Theta(0)R^*(0)g\left(\frac{R(t,0,x)}{R^*(0)}\right)\\ &\quad - \Theta(a)R^*(a)g\left(\frac{R}{R^*(a)}\right)\bigg|_{a=\infty} - \int_0^\infty g\left(\frac{R}{R^*(a)}\right)r(a)R^*(a)\,\mathrm{d}a.\end{aligned}$$

Note that $R^*(0) = kI^*$ and $R(t,0,x) = kI$, thus

$$\begin{aligned}\frac{\partial L^3_{E^*}}{\partial t} &= d_3 \int_0^\infty \Theta(a)\left(1 - \frac{R^*(a)}{R}\right)\Delta R\,\mathrm{d}a - \Theta(a)R^*(a)g\left(\frac{R}{R^*(a)}\right)\bigg|_{a=\infty}\\ &\quad + kQI^*g\left(\frac{I}{I^*}\right) + \int_0^\infty r(a)R^*(a)\left(1 - \frac{R}{R^*(a)} + \ln\frac{R}{R^*(a)}\right)\mathrm{d}a. \qquad (29)\end{aligned}$$

Let us define a Lyapunov functional for E^* as

$$L_{E^*}(t) = \int_\Omega \sum_{i=1}^3 L^i_{E^*}\,\mathrm{d}x.$$

This, combined with (26), (27) and (29), gives

$$\begin{aligned}&\frac{\partial(\sum_{i=1}^3 L^i_{E^*})}{\partial t}\\ &\quad = \bigotimes + \beta S^*I^*\left(1 - \frac{SI}{S^*I^*} - \frac{S^*}{S} + \frac{I}{I^*}\right) + \beta S^*I^*\left(1 + \frac{SI}{S^*I^*} - \frac{S}{S^*} - \frac{I}{I^*}\right)\\ &\qquad + \int_0^\infty r(a)R^*(a)\left(1 - \frac{I}{I^*} - \frac{I^*R}{IR^*(a)} + \frac{R}{R^*(a)}\right)\mathrm{d}a + kQI^*g\left(\frac{I}{I^*}\right)\\ &\qquad + \int_0^\infty r(a)R^*(a)\left(1 - \frac{R}{R^*(a)} + \ln\frac{R}{R^*(a)}\right)\mathrm{d}a,\end{aligned}$$

where

$$\bigotimes = d_1\left(1-\frac{S^*}{S}\right)\Delta S - \frac{\mu_S}{S}(S-S^*)^2 + d_2\left(1-\frac{I^*}{I}\right)\Delta I + d_3\int_0^\infty \Theta(a)\left(1-\frac{R^*(a)}{R(t,a,x)}\right)\Delta R\,\mathrm{d}a - \Theta(a)R^*(a)g\left(\frac{R(t,a,x)}{R^*(a)}\right)\bigg|_{a=\infty}.$$

Note that

$$kQI^* = \int_0^\infty r(a)R^*(a)\,\mathrm{d}a.$$

It then follows that

$$\begin{aligned}\frac{\partial(\sum_{i=1}^3 L^i_{E^*})}{\partial t} &= \bigotimes + \beta S^* I^*\left(2-\frac{S^*}{S}-\frac{S}{S^*}\right) \\ &\quad + \int_0^\infty r(a)R^*(a)\left(1-\ln\frac{I}{I^*}-\frac{I^*R(t,a,x)}{IR^*(a)}+\ln\frac{R(t,a,x)}{R^*(a)}\right)\mathrm{d}a \\ &= \bigotimes - \frac{\beta I^*}{S}(S-S^*)^2 - \int_0^\infty r(a)R^*(a)g\left(\frac{I^*R}{IR^*(a)}\right)\mathrm{d}a.\end{aligned}$$

Consequently, we have

$$\begin{aligned}\frac{\mathrm{d}L_{E^*}(t)}{\mathrm{d}t} &= \int_\Omega d_1\left(1-\frac{S^*}{S}\right)\Delta S\,\mathrm{d}x + \int_\Omega d_2\left(1-\frac{I^*}{I}\right)\Delta I\,\mathrm{d}x - \int_\Omega \frac{\mu_S}{S}(S-S^*)^2\,\mathrm{d}x \\ &\quad + \int_\Omega d_3\int_0^\infty \Theta(a)\left(1-\frac{R^*(a)}{R}\right)\Delta R\,\mathrm{d}a\,\mathrm{d}x - \int_\Omega \Theta(a)R^*(a)g\left(\frac{R}{R^*(a)}\right)\bigg|_{a=\infty}\mathrm{d}x \\ &\quad - \int_\Omega \frac{\beta I^*}{S}(S-S^*)^2\,\mathrm{d}x - \int_\Omega\int_0^\infty r(a)R^*(a)g\left(\frac{I^*R}{IR^*(a)}\right)\mathrm{d}a\,\mathrm{d}x \\ &= -d_1S^*\int_\Omega \frac{|\nabla S|^2}{S^2}\,\mathrm{d}x - d_2I^*\int_\Omega\frac{|\nabla I|^2}{I^2}\,\mathrm{d}x - \int_\Omega\frac{\mu_S}{S}(S-S^*)^2\,\mathrm{d}x \\ &\quad - d_3\int_0^\infty \Theta(a)R^*(a)\int_\Omega\frac{|\nabla R|^2}{R^2}\,\mathrm{d}x\,\mathrm{d}a - \int_\Omega \Theta(a)R^*(a)g\left(\frac{R}{R^*(a)}\right)\Big|_{a=\infty}\mathrm{d}x \\ &\quad - \int_\Omega\frac{\beta I^*}{S}(S-S^*)^2\,\mathrm{d}x - \int_\Omega\int_0^\infty r(a)R^*(a)g\left(\frac{I^*R}{IR^*(a)}\right)\mathrm{d}a\,\mathrm{d}x \leqslant 0.\end{aligned}$$

Then by the invariance principle [23, Thm. 4.2] and Theorem 2, E^* is GAS in D. This proves Theorem 3. □

7 Conclusion and discussion

This paper concerns with the analysis of the threshold-type result of an age-space structured disease model with age-dependent relapse rate. In contrast to [10,25] where the rate of change of $R(t,a)$ is controlled by a first-order PDE, we allow the reaction–diffusion equation for $R(t,a)$ in a bounded domain $\Omega \subset \mathbb{R}$ with smooth boundary $\partial\Omega$.

Mathematically, we first consider the well-posedness of the model (4). In Theorem 1, the existence of a unique local solution (S,I) to (4) is proved by designing a fixed point problem that was defined in (6). Then the positivity of the local solution of (4) is verified (see Theorem 2), where the proof of the positivity of $I(t,x)$ is nontrivial and is achieved by using the theory of Picard sequences and iteration method. Further, we extend the solution existence interval from $[0,T) \times \overline{\Omega}$ to $[0,+\infty) \times \overline{\Omega}$ through proving that the solution is bounded in $[0,T)$. By linearizing the model at E_0 and performing a Laplace transform, we can obtain the next-generation operator $\mathcal{L}$ (see (8)). Clearly, $\mathcal{L}$ is strictly positive. Further, by the Arzelà–Ascoli theorem, we can show that $\mathcal{L}$ is uniformly bounded and equicontinuous, that is, $\mathcal{L}$ is compact (see Lemma 4). Meanwhile, by the Krein–Rutman theorem, we are able to determine the explicit expression of BRN $\mathfrak{R}_0$ when the positive eigenvector is a constant.

Through the threshold dynamics analysis, $\mathfrak{R}_0$ is indeed a threshold parameter for determining whether the disease die out or not, which is useful to guide the disease control strategies. Especially, Theorem 2 reveals that both E_0 and E^* are LAS in terms of the sign of $\mathfrak{R}_0 - 1$. We also confirm that (4) is uniformly persistent through extending the weak persistence to strong persistence (see Lemmas 6 and 7). Finally, by designing the Lyapunov functionals, it is readily seen that if $\mathfrak{R}_0 < 1$, then E_0 is GAS, and if $\mathfrak{R}_0 > 1$, then E^* is GAS (see Theorem 3).

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