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On stability and convergence of difference schemes for one class of parabolic equations with nonlocal condition

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Abstract. In this paper, we construct and analyze the finite-difference method for a two-dimensional nonlinear parabolic equation with nonlocal boundary condition. The main objective of this paper is to investigate the stability and convergence of the difference scheme in the maximum norm. We provide some approaches for estimating the error of the solution. In our approach, the assumption of the validity of the maximum principle is not required. The assumption is changed to a weaker one: the difference problem's matrix is the M-matrix. We present numerical experiments to illustrate and supplement theoretical results.

Keywords: nonlocal boundary conditions, finite-difference method, stability and convergence, majorant, M-matrices.

1 Introduction and problem statement

We consider the finite-difference method for the following initial boundary value problem for the nonlinear two-dimensional parabolic equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - f(u) + p(x,y,t), \quad (x,y) \in \Omega,\ t \in (0,T], \tag{1}$$

where $\Omega = (0,1) \times (0,1)$, with boundary and initial conditions

$$u(0,y,t) = 0, \tag{2}$$

$$\gamma \frac{\partial u(0,y,t)}{\partial x} = \frac{\partial u(1,y,t)}{\partial x}, \tag{3}$$

$$u(x,0,t) = \mu_1(x,t), \qquad u(x,1,t) = \mu_2(x,t), \tag{4}$$

$$u(x,y,0) = \varphi(x,y), \tag{5}$$

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where $(x,y) \in \overline{\Omega} := [0,1] \times [0,1]$, $t \in [0,T]$. Functions $f(u)$, $p(x,y,t)$, $\varphi(x,y)$, $\mu_1(x,t)$, and $\mu_2(x,t)$ are given. Functions $\varphi(x,y)$, $\mu_1(x,t)$, and $\mu_2(x,t)$ must satisfy the compatibility conditions on the Ω boundaries

$$\mu_1(x,0) = \varphi(x,0), \qquad \mu_2(x,0) = \varphi(x,1),$$

and, also, function $\varphi(x,y)$ must satisfy boundary conditions (2), (3)

$$\varphi(0,y) = 0, \qquad \gamma \frac{\partial \varphi(0,y)}{\partial x} = \frac{\partial \varphi(1,y)}{\partial x}.$$

We assume that $\partial f/\partial u \geqslant 0$ and there exists a unique sufficiently smooth solution of problem (1)–(5).

The thorough study of numerical methods for one-dimensional parabolic problems with the nonlocal condition of type (3) started in 1970's [16, 17] when the new numerical models for nonlocal problems of various types had been massively created. Initial boundary value problems for parabolic equations with nonlocal conditions of type (3) arise, for example, in exploring diffusion of particles in turbulent plasma, as well as in the investigation of heat conduction in a thin heated rod when the flow change law is specified at the ends of the rod [16]. The primary purpose of solving such models using the finite-difference method was the stability of finite schemes in the special energy norms [14].

Creating new mathematical models for scientific and technical problems stimulates the further development of numerical methods for differential equations with nonlocal conditions. In recent years, increased attention has been paid to new mathematical models, including multidimensional and time–space fractional equations and inverse problems with nonlocal conditions [1, 7, 13, 18]. This growing interest motivated us to investigate the two-dimensional parabolic problem with nonlocal condition of type (3).

The intensive application and theoretical analysis of these numerical methods for the new mathematical models with nonlocal boundary conditions started at the end of the last century. One of the most used numerical techniques is the finite-difference method.

The stability of finite-difference scheme for the one-dimensional parabolic equation with integral conditions

$$u(0,t) = \int_0^1 \alpha(x) u(x,t)\,\mathrm{d}x + g_0(t), \qquad u(1,t) = \int_0^1 \beta(x) u(x,t)\,\mathrm{d}x + g_1(t)$$

is investigated in articles [3, 9]. The sufficient stability conditions considered in these papers afterward were improved, analyzed, compared, and commented in [20, 27, 30] and references therein.

Two-dimensional parabolic equations with various nonlocal boundary conditions were also investigated in many papers [6, 19, 22]. Several directions of numerical methods for solving differential equations with nonlocal conditions have been formed. One of the directions is investigating the difference operator's spectrum structure and applications for the difference schemes' analysis [4, 9, 14, 21, 23, 27, 33, 34]. The other direction is implicit alternating direction methods (ADI) for solving the systems of difference equations with nonlocal conditions [11, 28, 31, 35, 37]. Some applications and comments are presented in [5, 25].

A few years ago, it was noticed that the modified approach of constructing a majorant function could be effectively applied to some new problems with nonlocal boundary conditions. The reasoning behind the modification is that the M-matrices theory is the base instead of the maximum principle. This fact has prevented one from requiring the difference equations system's matrix to be diagonally dominant.

The main objective of this article is to investigate the stability and convergence of the difference scheme. For this purpose, we use the spectrum analysis of the difference problem and the M-matrices theory. This approach is used for the theoretical investigation of various type difference problems for elliptic and parabolic equations with nonlocal conditions [10, 29, 32, 36].

2 Difference problem and approximation error

We solve the differential problem (1)–(5) using finite-difference method. We assume that the solution $u(x,t)$ of (1)–(5) exists, is unique, and the derivatives $\partial^k u/\partial x^k$, $k=\overline{1,4}$, and $\partial^l u/\partial t^l$, $l=1,2$, are continuous and bounded. We define

$$\max\left|\frac{\partial^k u}{\partial x^k}\right| \leqslant M_k, \quad k=\overline{1,4}; \qquad \max\left|\frac{\partial^l u}{\partial t^l}\right| \leqslant C_l, \quad l=1,2.$$

Let U_{ij}^n be the finite-difference approximation of $u(x,y,t)$. We denote

$$U_{ij}^n = U\bigl(x_i, y_j, t^n\bigr),$$

where $x_i = ih$, $y_j = jh$, $i,j=\overline{0,N}$, $h=1/N$; $t^n = n\tau$, $n=\overline{0,M}$, $\tau = T/M$; $N,M\in\mathbb{N}$.

We denote

$$\begin{aligned}
\partial_x U_{ij}^n &= \frac{U_{i+1,j}^n - U_{ij}^n}{h}, \qquad \partial_{\bar{x}} U_{ij}^n = \frac{U_{ij}^n - U_{i-1,j}^n}{h},\\
\partial_{\bar{t}} U_{ij}^n &= \frac{U_{ij}^n - U_{ij}^{n-1}}{\tau}, \qquad \partial_t U_{ij}^n = \frac{U_{ij}^{n+1} - U_{ij}^n}{\tau},\\
\partial_x^2 U_{ij}^n &= \frac{U_{i-1,j}^n - 2U_{ij}^n + U_{i+1,j}^n}{h^2}, \qquad \partial_y^2 U_{ij}^n = \frac{U_{i,j-1}^n - 2U_{ij}^n + U_{i,j+1}^n}{h^2}.
\end{aligned}$$

We approximate Eq. (1) and conditions (2), (4), and (5) by difference one using standard method:

$$\partial_{\bar{t}} U_{ij}^n = \bigl(\partial_x^2 + \partial_y^2\bigr) U_{ij}^n - f\bigl(U_{ij}^n\bigr) + p_{ij}^n, \quad i,j=\overline{1,N-1}, \tag{6}$$

$$U_{0j}^n = 0, \quad j=\overline{0,N}, \tag{7}$$

$$U_{i0}^n = (\mu_1)_i^n, \quad U_{iN}^n = (\mu_2)_i^n, \quad i=\overline{0,N}, \tag{8}$$

$$U_{ij}^0 = \varphi_{ij}, \quad i,j=\overline{0,N}. \tag{9}$$

The approximation error of (6) is

$$r(h,\tau) = R_{1,ij}(h) + R_{2,ij}(\tau), \quad i,j = \overline{1,N-1}, \tag{10}$$

where

$$\left|R_{1,ij}(h)\right| \leqslant \frac{h^2 M_4}{6}, \qquad \left|R_{2,ij}(\tau)\right| \leqslant \frac{\tau C_2}{2}. \tag{11}$$

To approximate nonlocal condition (3) with accuracy $O(h^2+\tau)$, we rewrite it in the following form:

$$\gamma\left(\partial_x u_{0j}^n - \frac{h}{2}\frac{\partial^2 u_{0j}^n}{\partial x^2} - \frac{h^2}{6}\frac{\partial^3 \widetilde{u}_{0j}^n}{\partial x^3}\right) = \partial_{\bar{x}} U_{Nj}^n + \frac{h}{2}\frac{\partial^2 u_{Nj}^n}{\partial x^2} - \frac{h^2}{6}\frac{\partial^3 \widetilde{u}_{Nj}^n}{\partial x^3}.$$

Assume that differential equation (1) is defined in the interval $x \in (0,1)$ and on the boundaries $x=0$ and $x=1$. Now, substitute into latter equality expressions of $\partial^2 u_{0j}^n/\partial x^2$ and $\partial^2 u_{Nj}^n/\partial x^2$ from (1). Using $\partial U_{0j}^n/\partial t$ and $\partial U_{Nj}^n/\partial t$ approximations of order $O(\tau)$, after elementary rearrangements, we have

$$\begin{aligned}&\gamma\left(\partial_x u_{0j}^n + \frac{h}{2}\left(p_{0j}^n - f\left(u_{0j}^n\right)\right)\right)\\ &\quad = \partial_{\bar{x}} u_{Nj}^n + \frac{h}{2}\left(\partial_{\bar{t}} u_{Nj}^n - \partial_y^2 u_{Nj}^n + f\left(u_{Nj}^n\right) - p_{Nj}^n + r_{Nj}(h,\tau)\right).\end{aligned}$$

Using the previous expression, we have the following form of nonlocal condition (3):

$$\begin{aligned}\partial_{\bar{t}} U_{Nj}^n = {}& \frac{2}{h}\left(\gamma \partial_x U_{0j}^n - \partial_{\bar{x}} U_{Nj}^n\right) + \partial_y^2 U_{Nj}^n - f\left(U_{Nj}^n\right)\\ &+ \widetilde{p}_{Nj}^n + r_{Nj}^n, \quad j = \overline{1,N-1},\end{aligned} \tag{12}$$

where $\widetilde{p}_{Nj}^n = -\gamma f(U_{0j}^n) + \gamma p_{0j}^n + p_{Nj}^n$. The approximation error is

$$r_{Nj} = R_{1,Nj}(h) + R_{2,Nj}(\tau), \tag{13}$$

$$\left|R_{1,Nj}(h)\right| \leqslant \frac{h^2 M_4}{6} + \frac{h(\gamma+1)M_3}{3}, \qquad \left|R_{2,Nj}(\tau)\right| \leqslant \frac{\tau C_2}{2}. \tag{14}$$

Therefore, the differential problem (1)–(5) is approximated by the difference problem (6)–(9), (12) with approximation order defined by formulas (10) and (11).

As a result, we get the system of difference equations approximating problem (1)–(5) with accuracy $O(h^2+\tau)$

$$\begin{aligned}\partial_{\bar{t}} U_{ij}^n &= \left(\partial_x^2 + \partial_y^2\right) U_{ij}^n - f\left(U_{ij}^n\right) + p_{ij}^n, \quad i,j = \overline{1,N-1},\\ \partial_{\bar{t}} U_{Nj}^n &= \frac{2}{h}\left(\gamma \partial_x U_{0j}^n - \partial_{\bar{x}} U_{Nj}^n\right) - f\left(U_{Nj}^n\right) + \widetilde{p}_{Nj}^n, \quad j = \overline{1,N-1},\\ U_{0j}^n &= 0, \quad U_{i0}^n = (\mu_1)_i^n, \quad U_{iN}^n = (\mu_2)_i^n, \quad U_{ij}^n = \varphi_{ij} \quad i,j = \overline{0,N}.\end{aligned} \tag{15}$$

Remark 1. In the difference expression (6), approximating differential equation (1) as well as in expression (12), approximating nonlocal condition (3), one can take terms $f(U_{ij}^{n-1})$ and $f(U_{Nj}^{n-1})$ instead of terms $f(U_{ij}^{n})$ and $f(U_{Nj}^{n})$. The approximation error is the same $O(h^2+\tau)$. In this way, instead of solving a nonlinear system of equations, we get a linear one

$$\begin{aligned}
\partial_{\bar t} U_{ij}^n &= \left(\partial_x^2+\partial_y^2\right)U_{ij}^n - f\left(U_{ij}^{n-1}\right) + p_{ij}^n, \quad i,j=\overline{1,N-1},\\
\partial_{\bar t} U_{Nj}^n &= \frac{2}{h}\left(\gamma\partial_x U_{0j}^n - \partial_{\bar x} U_{Nj}^n\right) - f\left(U_{Nj}^{n-1}\right) + \widetilde{p}_{Nj}^n, \quad j=\overline{1,N-1},\\
U_{0j}^n &= 0, \quad U_{i0}^n = (\mu_1)_i^n, \quad U_{iN}^n = (\mu_2)_i^n, \quad U_{ij}^n = \varphi_{ij} \quad i,j=\overline{0,N}.
\end{aligned}$$

We denote

$$z_{ij}^n = u_{ij}^n - U_{ij}^n,$$

where U_{ij}^n is the solution of difference problem (15), u_{ij}^n is the solution of differential problem at the point (x_i,y_j) at time $t=t_n$.

Using system (15) and formulas (10) and (13), we get that z_{ij}^n is the solution of the following system:

$$\begin{aligned}
\partial_{\bar t} z_{ij}^n &= \partial_x^2 z_{ij}^n + \partial_y^2 z_{ij}^n - d_{ij}^n z_{ij}^n + r_{ij}^n, \quad i,j=\overline{1,N-1},\\
\partial_{\bar t} z_{Nj}^n &= \frac{2}{h}\left(\gamma\partial_x z_{0j}^n - \partial_{\bar x} z_{Nj}^n\right) + \partial_y^2 z_{Nj}^n - d_{Nj} z_{Nj}^n + r_{Nj}, \quad j=\overline{1,N-1},\\
z_{0j}^n &= 0, \quad z_{i0}^n = 0, \quad z_{iN}^n = 0, \quad i,j=\overline{0,N},
\end{aligned} \tag{16}$$

where $d_{Nj} = f'(\widetilde{U}_{Nj})$.

To investigate the difference systems (15) and (16), we rewrite them on the nth layer in the matrix form regarding to unknowns U_{ij}^n, $i=\overline{0,N}$, $j=\overline{1,N}$.

3 Systems of difference equations in matrix form

We rewrite system (16) in the matrix form. We define $N\times N$ matrix Λ_x and $(N-1)\times(N-1)$ matrix Λ_y as follows:

$$\Lambda_x = \frac{1}{h^2}\begin{pmatrix}
2 & -1 & 0 & \cdots & 0 & 0\\
-1 & 2 & -1 & \ddots & 0 & 0\\
0 & -1 & 2 & \ddots & 0 & 0\\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\
0 & 0 & 0 & -1 & 2 & -1\\
-2\gamma & 0 & 0 & \cdots & -2 & 2
\end{pmatrix},$$

$$\Lambda_y = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \ddots & 0 & 0 \\ 0 & -1 & 2 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

We denote by $\mathbf{I}_x$ the $N \times N$ identity matrix and by $\mathbf{I}_y$ the $(N-1)\times(N-1)$ identity matrix. Now, we construct two $N(N-1)\times N(N-1)$ matrices

$$\mathbf{A}_1 = \begin{pmatrix} \Lambda_x & & & & \\ & \Lambda_x & & & \\ & & \ddots & & \\ & & & \Lambda_x & \\ & & & & \Lambda_x \end{pmatrix}, \tag{17}$$

$$\mathbf{A}_2 = h^{-2}\begin{pmatrix} 2\mathbf{I}_x & -\mathbf{I}_x & & \cdots & & \\ -\mathbf{I}_x & 2\mathbf{I}_x & -\mathbf{I}_x & & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\mathbf{I}_x & 2\mathbf{I}_x & -\mathbf{I}_x \\ & & & \cdots & -\mathbf{I}_x & 2\mathbf{I}_x \end{pmatrix}. \tag{18}$$

The number of diagonal blocks of matrices (17) and (18) is $N-1$.

Now, we rewrite system (16) in the matrix form (on the nth time layer) as

$$\frac{z^n - z^{n-1}}{\tau} = -\mathbf{A}z^n - \mathbf{D}_n z^n + r_n, \tag{19}$$

where z^n, z^{n-1}, and r_n are the vectors of order $N(N-1)$, $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2$, $\mathbf{D}_n$ is the $N(N-1)\times N(N-1)$ diagonal matrix with elements

$$d_{ij}^n = \frac{\partial f(\widetilde{U}_{ij}^n)}{\partial U} \geqslant 0.$$

Using the same method, by rewriting system (15), we get

$$\frac{U^n - U^{n-1}}{\tau} = -\mathbf{A}U^n - f\bigl(U^n\bigr) + p. \tag{20}$$

Now we find the eigenvalues of matrix $\mathbf{A}$.

Lemma 1. *If $\gamma \in (0,1)$, then all eigenvalues of matrix* $\mathbf{A}$ *are positive.*

Proof. We formulate two eigenvalue problems for matrices Λ_x and Λ_y

$$\Lambda_x V_k = \mu_k V_k, \quad k = \overline{1,N}$$
$$\Lambda_y W_l = \eta_l W_l, \quad l = \overline{1,N-1}.$$

If $\gamma \in (0,1)$, then all eigenvalues of matrix Λ_x are positive [17, 29]. The eigenvalues of matrix Λ_y are of the form

$$\eta_l = \frac{4}{h^2}\sin^2\frac{\pi l h}{2}, \quad l = \overline{1, N-1}.$$

We rewrite matrix $\mathbf{A}$ in another form using Kronecker (tensor) product:

$$\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2 = \mathbf{I}_y \otimes \Lambda_x + \Lambda_y \otimes \mathbf{I}_x.$$

Using properties of tensor product (see [38]), we have

$$(\mathbf{I}_y \otimes \Lambda_x + \Lambda_y \otimes \mathbf{I}_x)(\mathbf{W} \otimes \mathbf{V}) = (\mu + \eta)(\mathbf{W} \otimes \mathbf{V})$$

or

$$\mathbf{A}(\mathbf{W} \otimes \mathbf{V}) = \lambda(\mathbf{W} \otimes \mathbf{V}),$$

where $\lambda := \lambda_{kl} = \mu_k + \eta_l$, $k = \overline{1, N}$, $l = \overline{1, N-1}$, are the eigenvalues of matrix $\mathbf{A}$. So, $\lambda_{kl} > 0$. □

4 M-matrices and systems of difference equations

We investigate the system of difference equations (19) using the theory of M-matrices. For this purpose, we provide the definition and some properties of M-matrices [2, 38].

Definition 1. A square matrix with real elements $\mathbf{A} = \{a_{kl}\}$, $k, l = \overline{1, m}$, is called an M-matrix if $a_{kl} \leqslant 0$ when $k \neq l$ and the inverse $\mathbf{A}^{-1}$, whose all elements are nonnegative ($\mathbf{A}^{-1} \geqslant 0$), exists.

It follows from the definition that $a_{kk} > 0$. We also use the notation $\mathbf{A} > 0$ ($\mathbf{A} \geqslant 0$) if $a_{kl} > 0$ ($a_{kl} \geqslant 0$) for all k, l and $\mathbf{A} < \mathbf{B}$ ($\mathbf{A} \leqslant \mathbf{B}$) if $a_{kl} < b_{kl}$ ($a_{kl} \leqslant b_{kl}$).

We formulate some properties of M-matrices that will be used later.

Property 1. *If $a_{kl} \leqslant 0$ ($k \neq l$), then two next statements are equivalent*:

(i) *The matrix $\mathbf{A}^{-1}$ exists, and $\mathbf{A}^{-1} \geqslant 0$,*

(ii) *The real parts of each eigenvalue of $\mathbf{A}$ are positive:* $\operatorname{Re}\lambda(\mathbf{A}) > 0$.

Property 2. *If $\mathbf{A}_1$ is an M-matrix, $\mathbf{A}_2 \geqslant \mathbf{A}_1$, and all nondiagonal elements of the matrix $\mathbf{A}_2$ are nonpositive, then $\mathbf{A}_2$ is also an M-matrix, and*

$$\mathbf{A}_2^{-1} \leqslant \mathbf{A}_1^{-1}.$$

Property 3. *If an M-matrix $\mathbf{A}$ has a regular splitting $\mathbf{A} = \mathbf{B} - \mathbf{C}$, where $\mathbf{A}^{-1} \geqslant 0$, $\mathbf{C} \geqslant 0$, then*

$$\rho(\mathbf{B}^{-1}\mathbf{C}) = \max_{1 \leqslant k \leqslant m} \left|\lambda_k(\mathbf{B}^{-1}\mathbf{C})\right| < 1.$$

The next corollary is valid according to all three properties.

Corollary 1. *If* $a_{kk} > 0$, $a_{kl} \leqslant 0$, $k \neq l$, *and* $\operatorname{Re}\lambda(\mathbf{A}) > 0$, *then* $\mathbf{A}^{-1} \geqslant 0$.

Lemma 2. *If* $\gamma \in (0,1)$, *then the matrix* $\mathbf{A}$ *of system* (19) *is an M-matrix.*

Proof. It follows from the definition of the matrices Λ_x and Λ_y that all diagonal elements are positive and nondiagonal are nonnegative. So, the same property is valid for matrix $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2$. Since all eigenvalues of the matrix $\mathbf{A}$ are positive, then all listed properties are sufficient for the matrix $\mathbf{A}$ to be an M-matrix [38]. □

Remark 2. If $\mathbf{A}$ is an M-matrix, then $\mathbf{A} + \mathbf{D}$ (if $\partial f/\partial U \geqslant 0$) and $\mathbf{I} + \tau\mathbf{A}$ (if $\tau > 0$) are also M-matrices.

Remark 3. For the matrix $\mathbf{A} + \mathbf{D}$ to be an M-matrix, the condition $\partial f/\partial u \geqslant 0$ can be improved. Indeed, consider the new condition

$$\frac{\partial f}{\partial u} \geqslant -\eta_1,$$

where $\eta_1 = (4/h^2)\sin^2(\pi h/2)$ is the smallest eigenvalue of the matrix Λ_y. We construct the matrix $\mathbf{A} - \eta_1\mathbf{I}$. This matrix's diagonal elements are positive, nondiagonal are nonnegative, and all the eigenvalues are positive. According to Definition 1 and Property 1, the matrix $\mathbf{A} - \eta_1\mathbf{I}$ is an M-matrix. Now we take two matrices $\mathbf{A} - \eta_1\mathbf{I}$ and $\mathbf{A} + \mathbf{D}$ with the condition $\partial f/\partial u \geqslant -\eta_1$. Since $\mathbf{A} + \mathbf{D} \geqslant \mathbf{A} - \eta_1\mathbf{I}$, then, according to Property 2, the matrix $\mathbf{A} + \mathbf{D}$ is an M-matrix.

5 An existence and uniqueness of the difference solution

We investigate existence conditions and construct an iterative method for finding the unique solution of system (20). We do not write the index of the time layer t^n for simplicity. We rearrange system (20) in the form

$$\mathbf{A}_1 U = -f(U) + \widetilde{p},$$

where $\widetilde{p} = p + U^{n-1}/\tau$, $\mathbf{A}_1 = (1/\tau)\mathbf{I} + \mathbf{A}$ is an M-matrix. We define the inner product in the vector space $\mathbb{R}_{N(N-1)}$ as

$$(U,V) = h^2 \sum_{i=1}^{N}\sum_{j=1}^{N-1} U_{ij}V_{ij}$$

and corresponding norm as

$$\|U\| = (U,U)^{1/2}.$$

We also use the other compatible vector and matrix norms

$$\|U\|_* = \left(\left(\mathbf{M}\mathbf{M}^\top\right)^{-1}U, U\right), \qquad \|\mathbf{A}\|_* = \rho(\mathbf{A}),$$

where $\mathbf{M}$ is a matrix with rows constructed using the linearly independent eigenvectors of matrix $\mathbf{A}$. The purpose of norm $\|\mathbf{A}\|$ is such: if $\rho(\mathbf{A}) < 1$ and the eigenvectors of matrix $\mathbf{A}$ are linearly independent, then $\rho(\mathbf{A})$ is the norm whether or not matrix $\mathbf{A}$ is symmetric. The following theorem is proved in [26].

Theorem 1. *If matrix $\mathbf{A}_1$ in the system*

$$\mathbf{A}_1 U = -f(U) + p \tag{21}$$

is an M-matrix and $0 \leqslant \partial f/\partial U < \beta < \infty$, then there exists a unique solution of system (21)*. This system could always be rewritten in a form where the nonlinear operator is the contraction operator.*

The following is the iteration method for solving system (21):

$$(\mathbf{A}_1 + \beta\mathbf{I})U^{k+1} = \beta U^k - f\big(U^k\big) + p, \tag{22}$$

where β is a finite constant such that $0 \leqslant \partial f/\partial U < \beta < \infty$.

We denote by U^* the unique solution of system (21). It follows from (22)

$$\begin{aligned} \big\|U^* - U^{k+1}\big\| &= \big\|(\mathbf{A}_1 + \beta\mathbf{I})^{-1}(\beta\mathbf{I} - \widetilde{\mathbf{D}})\big(U^* - U^k\big)\big\| \\ &\leqslant \big\|(\mathbf{A}_1 + \beta\mathbf{I})^{-1}\beta\big\| \cdot \big\|U^* - U^k\big\|, \end{aligned}$$

where $\widetilde{\mathbf{D}} = \operatorname{diag}\{\partial\widetilde{f}/\partial U\}$. Thus, the iteration method (22) converges in $\|U\|_*$ if

$$\big\|(\mathbf{A}_1 + \beta\mathbf{I})^{-1}\beta\big\|_* = \rho\big((\mathbf{A}_1 + \beta\mathbf{I})^{-1}\beta\big) < 1.$$

So, we have

$$\begin{aligned} \rho\big((\mathbf{A}_1 + \beta\mathbf{I})^{-1}\beta\big) &= \rho\bigg(\bigg(\frac{1}{\tau}\mathbf{I} + \mathbf{A} + \beta\mathbf{I}\bigg)^{-1}\beta\bigg) \\ &= \frac{\beta}{\frac{1}{\tau} + \lambda(\mathbf{A}) + \beta} = \frac{\beta\tau}{\beta\tau + 1 + \lambda(\mathbf{A})\tau} < 1. \end{aligned}$$

The iteration method (22) converges if the following sufficient conditions are satisfied:

(i) $\mathbf{A}$ is a simple structure M-matrix,
(ii) $0 \leqslant \partial f/\partial U < \beta < \infty$.

Remark 4. If, instead of the iteration method (22), we take the simpler structure method

$$\mathbf{A}_1 U^{k+1} = -f\big(U^k\big),$$

then, applying the same technique, we get

$$\rho\big(\mathbf{A}_1^{-1}\beta\big) = \rho\bigg(\bigg(\frac{1}{\tau}\mathbf{I} + \mathbf{A}\bigg)^{-1}\beta\bigg) = \frac{\beta\tau}{1 + \tau\lambda(\mathbf{A})}.$$

So, the sufficient convergence condition is

$$\beta < \frac{1}{\tau}.$$

6 Comparison theorem

Now, we formulate and prove the comparison theorem for system (19) using the properties of M-matrices.

Let V^n be a vector with coordinates V_{ij}^n. We denote the vector with coordinates $|V_{ij}^n|$ by $|V^n|$, i.e.,

$$|V^n| = \{|V_{ij}^n|\}.$$

Theorem 2 [Comparison theorem]. *Let $z^n = \{z_{ij}^n\}$ and $w^n = \{w_{ij}^n\}$ be two different solutions of the difference system* (19)

$$\frac{z^n - z^{n-1}}{\tau} = -\mathbf{A}z^n - \mathbf{D}_n z^n + f_n, \tag{23}$$

$$\frac{w^n - w^{n-1}}{\tau} = -\mathbf{A}w^n + g_n, \tag{24}$$

where $\mathbf{D}_n$ is the diagonal matrix with elements $d_{ij}^n \geqslant 0$, $w^0 \geqslant 0$, $g^n \geqslant 0$ $(n \geqslant 1)$. If $|z^0| \leqslant w^0$, $|f^n| \leqslant g^n$, then

$$|z^n| \leqslant w^n, \quad n \geqslant 1.$$

Proof. First, notice that according to the theorem's formulation, the following inequality is valid:

$$w^n \geqslant 0.$$

Indeed, using system (24), we have

$$(\mathbf{I} + \tau\mathbf{A})w^n = w^{n-1} + \tau g^n.$$

It follows

$$w^n = (\mathbf{I} + \tau\mathbf{A})^{-1}w^{n-1} + (\mathbf{I} + \tau\mathbf{A})^{-1}\tau g^n,$$

where $(\mathbf{I} + \tau\mathbf{A})^{-1} \geqslant 0$, $g^n \geqslant 0$. So, if $w^{n-1} \geqslant 0$, then $w^n \geqslant 0$ for all $n = 1, 2, \dots$.

Further, using (23), we have

$$z^n = (\mathbf{I} + \tau\mathbf{A} + \tau\mathbf{D}_n)^{-1}z^{n-1} + \tau(\mathbf{I} + \tau\mathbf{A} + \tau\mathbf{D}_n)^{-1}f^n.$$

Using Property 2,

$$\begin{aligned} |z^n| &\leqslant (\mathbf{I} + \tau\mathbf{A})^{-1}|z^{n-1}| + \tau(\mathbf{I} + \tau\mathbf{A})^{-1}|f^n| \\ &\leqslant (\mathbf{I} + \tau\mathbf{A})^{-1}w^{n-1} + \tau(\mathbf{I} + \tau\mathbf{A})^{-1}g^n = w^n. \end{aligned}$$ □

Remark 5. Theorem 2 can be proven using the monotone matrix property: if $\mathbf{A}$ is a monotone matrix, then from the inequality $\mathbf{A}v \leqslant \mathbf{A}w$ it follows $v \leqslant w$.

We note that the comparison theorem for the finite-difference method is usually formulated as a corollary of the maximum principle (see e.g. [8, 12, 24]). It means that the system of difference equations is diagonally dominant. In the M-matrices theory, this property is optional.

Function $w(x, y, t)$ satisfying Theorem 2 conditions is usually called the majorant of the difference problem (23) solution. The majorant is typically not unique for the considered problem regardless of the method it obtained (whether as the conclusion of the maximum principle or as the property of M-matrices). Several forms of majorants could be constructed.

In the article [10], the comparison theorem is formulated and proved for the difference system of the following form (unlike the (23) form):

$$\mathbf{A}z^n = \mathbf{B}z^{n-1} + f^n,$$

where $\mathbf{A}$ is the M-matrix, and $\mathbf{B} \geqslant 0$. System (23) could also be rewritten in the same form. However, as it was noticed in [29], the (23) majorant form is preferable for certain cases.

7 Stability of difference scheme

First, we define the stability notion used in this article for the difference scheme analysis. We use the general stability concept for a nonlinear problem. We consider two problems. One is system (6)–(9), (12) with the data $p(x, y, t)$ and $\varphi(x, y)$. We denote the solution of this problem by U_{ij}^n. The other problem is the same system (6)–(9), (12) with perturbed functions $\widetilde{p}(x, y, t)$ and $\widetilde{\varphi}(x, y)$. We denote the solution of the perturbed problem by $\widetilde{U}_{ij}$.

Definition 2. Difference scheme (6)–(9), (12) is stable if for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ not dependent on h and τ such that

$$\left|U_{ij}^n - \widetilde{U}_{ij}^n\right| \leqslant \varepsilon$$

for $n = 1, 2, \dots$ if

$$\left|p_{ij}^n - \widetilde{p}_{ij}^n\right| \leqslant \delta, \quad |\varphi_{ij} - \widetilde{\varphi}_{ij}| \leqslant \delta.$$

The stability defined by this method is usually called the stability concerning the initial data and the right-hand side stability [24, pp. 403, 411].

We denote

$$\widetilde{z}_{ij}^n = U_{ij}^n - \widetilde{U}_{ij}^n, \qquad \widetilde{f}_{ij}^0 = \varphi_{ij} - \widetilde{\varphi}_{ij}, \qquad \widetilde{f}_{ij}^n = p_{ij}^n - \widetilde{p}_{ij}^n.$$

Vector $\widetilde{z}_{ij}^n$ is the solution of the following system (equivalent to system (16)):

$$\partial_{\bar{t}}\widetilde{z}_{ij}^n = \partial_x^2\widetilde{z}_{ij}^n + \partial_y^2\widetilde{z}_{ij}^n - d_{ij}^n\widetilde{z}_{ij}^n + \widetilde{f}_{ij}^n, \quad i, j = \overline{1, N-1}, \tag{25}$$

$$\partial_{\bar{t}}\widetilde{z}_{Nj}^n = \frac{2}{h}\left(\gamma\partial_x\widetilde{z}_{0j}^n - \partial_{\bar{x}}\widetilde{z}_{Nj}^n\right) + \partial_y^2\widetilde{z}_{Nj}^n - d_{Nj}\widetilde{z}_{Nj}^n + \widetilde{f}_{Nj}, \quad j = \overline{1, N-1}, \tag{26}$$

$$\widetilde{z}_{0j}^n = 0, \quad \widetilde{z}_{i0}^n = 0, \quad \widetilde{z}_{iN}^n = 0, \quad \widetilde{z}_{ij}^0 = \widetilde{f}_{ij}^0, \quad i, j = \overline{0, N}. \tag{27}$$

The matrix form of system (25)–(27) is equivalent to (23)

$$\frac{\widetilde{z}^n - \widetilde{z}^{n-1}}{\tau} = -\mathbf{A}\widetilde{z}^n - \mathbf{D}_n\widetilde{z}^n + \widetilde{f}_n. \tag{28}$$

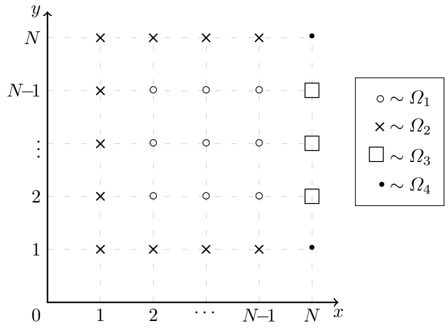

Figure 1. Split of grid points into groups.

Now, in order to investigate the stability of problem (6)–(9), (12), we find the majorant of problem's (28) solution in the following form:

$$w(x,y,t) = \frac{\delta}{5}\left(2 - x^2 - y^2 + \frac{2x}{1-\gamma}\right) + \frac{\delta t}{5}, \tag{29}$$

where $\delta > 0$ is a fixed, still undefined, number.

According to Theorem 2, the majorant (29) is the solution of system (24). We rewrite this system in the coordinates form

$$\frac{w_{ij}^n - w_{ij}^{n-1}}{\tau} = \partial_x^2 w_{ij}^n + \partial_y^2 w_{ij}^n + g_{ij}^n, \quad n = \overline{1, N-1}, \tag{30}$$

$$\frac{w_{Nj}^n - w_{Nj}^{n-1}}{\tau} = \frac{2}{h}\left(\gamma \partial_x w_{0j}^n - \partial_{\bar{x}} w_{Nj}^n\right) + \partial_y^2 w_{Nj}^n + g_{Nj}, \quad n = \overline{1, N-1}. \tag{31}$$

The g_{ij}^n values still need to be found in this system. The values must be such that the function $w(x,y,t)$, defined by formula (29), is the solution of system (30), (31).

First, we note that function $w(x,y,t)$, defined by formula (29), is positive for all $x \in [0,1]$, $y \in [0,1]$, $\gamma \in [0,1]$, $\delta > 0$, and $t \geqslant 0$. Next, the solution w_{ij}^n of system (30), (31), unlike the solution $\tilde{z}_{ij}^n$ of system (25)–(27), does not satisfy the homogenous boundary conditions. It affects the choice of g_{ij}^n values. We split all the grid points, where Eqs. (30), (31) are defined: (i,j), $i = \overline{1,N}$, $j = \overline{1, N-1}$, into four groups (see Fig. 1):

(i) $(i,j) \in \Omega_1$ if $i = \overline{2, N-1}$, $j = \overline{2, N-2}$;
(ii) $(i,j) \in \Omega_2$ if $\{i = 1,\ j = \overline{2, N-2}\}$ and $\{i = \overline{1, N-1},\ j = 1, N-1\}$;
(iii) $(i,j) \in \Omega_3$ if $i = N,\ j = \overline{2, N-2}$;
(iv) $(i,j) \in \Omega_4$ if $i = N,\ j = 1, N-1$.

From the $w(x,y,t)$ definition (29) it follows that

$$\partial_x^2 w_{ij}^n = \frac{\partial^2 w_{ij}^n}{\partial x^2} = -\frac{2\delta}{5}, \quad \partial_y^2 w_{ij}^n = \frac{\partial^2 w_{ij}^n}{\partial y^2} = -\frac{2\delta}{5}, \quad \partial_t w_{ij}^n = \frac{\partial w_{ij}^n}{\partial t} = \frac{\delta}{5}.$$

Using Eq. (30), we have

$$g_{ij}^n = \frac{\delta}{5} + \frac{4\delta}{5} = \delta, \quad (i,j) \in \Omega_1.$$

Furthermore,

$$g_{ij}^n = \frac{\delta}{5} + \frac{4\delta}{5} + \frac{\widetilde{w}_{ij}^n}{h^2} > \delta, \quad (i,j) \in \Omega_2.$$

Here $\widetilde{w}_{ij}^n$ is the value of function $w(x,y,t)$ at the point on the Ω boundary. For example, in the case $i = 1$, $j = \overline{2, N-2}$,

$$\widetilde{w}_{ij}^n = w_{0j}^n > 0.$$

Similarly, after elementary transformations, from Eqs. (31) we have

$$g_{Nj}^n = \frac{\delta}{5} + \frac{2(\gamma+1)\delta}{5} + \frac{2\delta}{5} = \delta + \frac{2\gamma\delta}{5} > \delta, \quad (i,j) \in \Omega_3,$$
$$g_{Nj}^n = \frac{\delta}{5} + \frac{2(\gamma+1)\delta}{5} + \frac{2\delta}{5} + \frac{\widetilde{w}_{Nj}^n}{h^2} > \delta, \quad (i,j) \in \Omega_4,$$

where $\widetilde{w}_{Nj}^n = w_{N0}^n > 0$ if $j = 1$, and $\widetilde{w}_{NN}^n = w_{NN}^n > 0$ if $j = N-1$.

Thus, in all cases (for all (i,j)), the following inequality is valid:

$$g_{ij}^n \geqslant \delta. \tag{32}$$

Generalizing this investigation, we have

Conclusion 1. If system (25)–(27) satisfies following conditions

$$\left|\widetilde{f}_{ij}^0\right| = |\varphi_{ij} - \widetilde{\varphi}_{ij}| < \delta, \qquad \left|\widetilde{f}_{ij}\right| = \left|p_{ij}^n - \widetilde{p}_{ij}^n\right| < \delta, \tag{33}$$

then the function $w(x,y,t)$, defined by formula (29), is the majorant of the solution $\widetilde{z}_{ij}^n$ of this system, that is,

$$\left|\widetilde{z}_{ij}^n\right| \leqslant w_{ij}^n, \quad i,j = \overline{0, N}. \tag{34}$$

Actually, all the assumptions of Theorem 2 are fullfilled, i.e., $d_{ij}^n \geqslant 0$, $w_{ij}^n > 0$ for all i, j, and n values; $\widetilde{z}_{ij}^0 = 0$, and $w_{ij}^0 > 0$, therefore $|\widetilde{z}_{ij}^0| \leqslant w_{ij}^0$. Matrix $\mathbf{A}$, according to Lemma 1, is M-matrix if $\gamma \in [0,1)$. Next, from Eqs. (32) and (33) it follows that $|\widetilde{f}_{ij}^n| \leqslant g_{ij}$. Then inequality (34) is valid.

Theorem 3. *The difference scheme* (6)–(9), (12) *is stable if* $\gamma \in [0,1)$.

Proof. Using function $w(x,y,t)$ definition (29), it follows that $\partial w/\partial x > 0$ if $x \in [0,1]$ and $\gamma \in [0,1)$. Hence, for every $\varepsilon > 0$ value, the inequality

$$w_{ij}^n < \varepsilon$$

is true if

$$\delta < \frac{5\varepsilon}{\frac{2}{1-\gamma} + T}.$$

In other words, for every $\varepsilon > 0$, there exists $\delta := \delta(\varepsilon) > 0$ such that

$$\left|\widetilde{z}_{ij}^n\right| \leqslant w_{ij}^n \leqslant \varepsilon$$

if

$$\delta \leqslant \delta_0 = \frac{5\varepsilon}{\frac{2}{1-\gamma} + T},$$

which is the definition of stability. □

8 Error estimation and convergence of difference scheme

Now, using the Comparison theorem 2, we evaluate the error of the difference scheme (6)–(9), (12)

$$z_{ij}^n = u_{ij} - U_{ij},$$

where u_{ij} is the solution of the differential problem (1)–(5), and U_{ij} is the solution of the difference problem (6)–(9), (12). For this purpose, we consider the finite-difference system (16) and its matrix form (19).

Using formulas (10), (11), (13), and (14), we have the following estimates for the vector $r = \{r_{ij}^n\}$ of system (16):

$$\begin{aligned} |r_{ij}^n| &\leqslant \frac{h^2 M_4}{6} + \frac{\tau C_2}{2}, \quad i,j = \overline{1, N-1}, \\ |r_{Nj}^n| &\leqslant \frac{h^2 M_4}{6} + \frac{h(\gamma+1)M_3}{3} + \frac{\tau C_2}{2}, \quad j = \overline{1, N-1}. \end{aligned} \tag{35}$$

We define the function $w(x,y,t)$ as follows:

$$w(x,y,t) = \frac{h^2 M(\gamma+1)}{6}\left(2 - \frac{x^2}{2} - \frac{y^2}{2} + \frac{2x}{1-\gamma}\right) + \frac{\tau C_2(\gamma+1)t}{2}, \tag{36}$$

where $M = \max(M_3, M_4)$. Now, we prove that $w(x,y,t)$ is the difference problem (19) solution's majorant. For this purpose, like in Section 7, we calculate the values g_{ij}^n in system (30), (31) as $w(x,y,t)$ is defined by formula (36). Using (30), (36), we have

$$\begin{aligned} g_{ij}^n &= \frac{\tau C_2(\gamma+1)}{2} + \frac{h^2 M(\gamma+1)}{3}, \quad (i,j) \in \Omega_1, \\ g_{ij}^n &= \frac{\tau C_2(\gamma+1)}{2} + \frac{h^2 M(\gamma+1)}{3} + \frac{\widetilde{w}_{ij}^n}{h^2} \\ &> \frac{\tau C_2(\gamma+1)}{2} + \frac{h^2 M(\gamma+1)}{3}, \quad (i,j) \in \Omega_2, \end{aligned}$$

where $\widetilde{w}_{ij}^n > 0$ is the value of function $w(x,y,t)$ at the corresponding point on the Ω boundary.

Further, using (31) and (36), we have

$$g_{ij}^n = \frac{\tau C_2(\gamma+1)}{2} + \frac{h^2 M(\gamma+1)}{3} + \frac{hM(\gamma+1)}{3}, \quad (i,j) \in \Omega_3,$$
$$g_{ij}^n > \frac{\tau C_2(\gamma+1)}{2} + \frac{h^2 M(\gamma+1)}{3} + \frac{hM(\gamma+1)}{3}, \quad (i,j) \in \Omega_4.$$

So, for system (19) with estimates (35) and for system (24) with function $w(x,y,t)$, defined by formula (36), all assumptions of Theorem 2 are valid.

Theorem 4. *Let the following assumptions hold for the differential problem* (1)–(5):

(i) $\gamma \in [0,1)$,
(ii) $\partial f/\partial u \geqslant 0$,
(iii) *there exists a unique, sufficiently smooth solution such that estimates* (10), (11) *and* (13), (14) *are valid for the approximation error.*

Then, for the error of the solution

$$z_{ij}^n = u_{ij}^n - U_{ij}^n,$$

where u_{ij}^n is the solution of differential problem (1)–(5), *and U_{ij}^n is the solution of difference problem* (6)–(9), (12), *the following estimate is valid*:

$$\|z^n\|_C = \max_{i,j}\left|z_{ij}^n\right| \leqslant \overline{C}_1 h^2 + \overline{C}_2 \tau,$$

where $\overline{C}_1$ and $\overline{C}_2$ do not depend on h and τ.

Proof. Function $w(x,y,t)$, defined by formula (36), is the majorant of the solution z_{ij}^n of difference problem (23). According to Theorem 2, we have

$$\left|z_{ij}^n\right| \leqslant w_{ij}^n \leqslant \frac{h^2 M(\gamma+1)}{6} \cdot \frac{2}{1-\gamma} + \frac{\tau C_2(\gamma+1)T}{2}$$
$$= \overline{C}_1 h^2 + \overline{C}_2 \tau. \qquad \square$$

Conclusion 2. If the assumptions of Theorem 4 are valid, then the difference scheme (6)–(9), (12) converges in maximum norm.

9 Numerical experiment

This section provides the results of the numerical experiment to comment on and supplement the theoretical investigation. We made several different numerical experiments to show the stability and estimate the approximation error of the investigated difference method. Specifically, we considered the differential problem with various h and τ values, different solution exponential growth variants, and various time intervals $t \in [0,T]$.

Table 1. Absolute and relative errors of the numerical experiment.

(a) Absolute error $\varepsilon(h,\tau)$

		$T=1,\ \gamma=0.5$			
		$a=0.1$		$a=2$	
h	τ	$\varepsilon(h,\tau)$	$\varepsilon(h,\tau)/\varepsilon(2h,4\tau)$	$\varepsilon(h,\tau)$	$\varepsilon(h,\tau)/\varepsilon(2h,4\tau)$
$1/10$	$1/10$	$3.04\cdot10^{-4}$		$2.02\cdot10^{-2}$	
$1/20$	$1/40$	$7.61\cdot10^{-5}$	3.99	$5.50\cdot10^{-3}$	3.67
$1/40$	$1/160$	$1.91\cdot10^{-5}$	3.98	$1.40\cdot10^{-3}$	3.92
$1/80$	$1/640$	$4.77\cdot10^{-6}$	4.00	$3.60\cdot10^{-4}$	3.89

(b) Relative error $\Delta(h,\tau)$

		$T=1,\ \gamma=0.5$			
		$a=0.1$		$a=2$	
h	τ	$\Delta(h,\tau)$	$\Delta(h,\tau)/\Delta(2h,4\tau)$	$\Delta(h,\tau)$	$\Delta(h,\tau)/\Delta(2h,4\tau)$
$1/10$	$1/10$	$1.00\cdot10^{-3}$		$1.94\cdot10^{-2}$	
$1/20$	$1/40$	$2.58\cdot10^{-4}$	3.88	$5.30\cdot10^{-3}$	3.67
$1/40$	$1/160$	$6.49\cdot10^{-5}$	3.96	$1.40\cdot10^{-3}$	3.79
$1/80$	$1/640$	$1.63\cdot10^{-5}$	3.98	$3.42\cdot10^{-4}$	4.09

We have chosen the simple mathematical model (1)–(5) with known solution

$$u(x,y,t)=\frac{2}{\pi}\left(\sin\left(\frac{\pi x}{2}\right)+\gamma\frac{\pi x^2}{4}\right)\sin\left(\frac{\pi y}{2}\right)\mathrm{e}^{at}, \tag{37}$$

where $\gamma\in[0,1)$; $a>0$ is a parameter regulating solution's exponentional growth as $t\to\infty$. Next,

$$f(u)=Ku^3.$$

Boundary and nonlocal conditions (2) and (3) are fulfilled for all γ and a values. Functions p, μ_1, μ_2, and φ are chosen according to expression (37).

The main results of the numerical experiment are presented in Tables 1–3. Absolute and relative errors ($\varepsilon(h,\tau)$ and $\Delta(h,t)$, accordingly) were considered for each case.

$$\varepsilon(h,\tau)=\max_{i,j}|U_{ij}-u^*_{ij}|,\qquad \Delta(h,t)=\max_{i,j}\left|\frac{U_{ij}-u^*_{ij}}{U_{ij}}\right|,$$

where U_{ij} is the difference problem solution, and u^*_{ij} is the differential problem solution as $t=T$.

Table 1 provides the relative and absolute errors of the numerical experiment with fixed parameters $T=1$, $\gamma=0.5$, $a=0.1$, and $a=2$. Each row of Table 1(a) represents absolute error values, while the value of h is reduced in half, and the values of τ are reduced four times. Also, the rate of decrease of absolute error (while h and τ are decreasing) is presented in each row. According to the theoretical investigation about the error rate of $O(h^2+\tau)$, the decrease rate must be equal to four times for every row. Numerical experiment fully confirms theoretical results. Each row of Table 1(b) represents

Table 2. Relative error estimates for different values of T for $u(x,y,t)$ defined by formula (37).

		$\Delta(h,\tau),\ \gamma=0.5,\ a=0.5$					
h	τ	$T=1$	$T=4$	$T=6$	$T=8$	$T=10$	$T=12$
$1/10$	$1/10$	$2.40\cdot10^{-3}$	$2.40\cdot10^{-3}$	$2.40\cdot10^{-3}$	$6.25\cdot10^{-1}$	$6.25\cdot10^{-1}$	$6.25\cdot10^{-1}$
$1/20$	$1/10$	$6.15\cdot10^{-4}$	$6.15\cdot10^{-4}$	$6.15\cdot10^{-4}$	$5.57\cdot10^{-1}$	$1.37\cdot10^{0}$	$1.37\cdot10^{0}$
$1/40$	$1/10$	$1.55\cdot10^{-4}$	$1.55\cdot10^{-4}$	$1.55\cdot10^{-4}$	$1.55\cdot10^{-4}$	$1.42\cdot10^{0}$	$1.42\cdot10^{0}$
$1/80$	$1/10$	$3.90\cdot10^{-5}$	$3.90\cdot10^{-5}$	$3.90\cdot10^{-5}$	$3.90\cdot10^{-5}$	$3.90\cdot10^{-5}$	$1.59\cdot10^{0}$

Table 3. Absolute and relative error estimates for different values of T for $u(x,y,t)$ defined by formula (39).

		$a=0.5,\ \gamma=0.5,\ C=2$					
		$T=1$		$T=4$		$T=8$	
h	τ	$\varepsilon(h,\tau)$	$\Delta(h,\tau)$	$\varepsilon(h,\tau)$	$\Delta(h,\tau)$	$\varepsilon(h,\tau)$	$\Delta(h,\tau)$
$1/10$	$1/10$	$4.99\cdot10^{-4}$	$1.72\cdot10^{-4}$	$7.84\cdot10^{-4}$	$1.98\cdot10^{-4}$	$2.33\cdot10^{1}$	$9.49\cdot10^{-1}$
$1/20$	$1/40$	$1.26\cdot10^{-4}$	$4.36\cdot10^{-5}$	$1.99\cdot10^{-4}$	$5.01\cdot10^{-4}$	$3.20\cdot10^{1}$	$6.70\cdot10^{-1}$
$1/40$	$1/160$	$3.16\cdot10^{-5}$	$1.09\cdot10^{-5}$	$5.00\cdot10^{-5}$	$1.26\cdot10^{-5}$	$5.00\cdot10^{-5}$	$1.26\cdot10^{-5}$
$1/80$	$1/640$	$7.91\cdot10^{-6}$	$2.74\cdot10^{-6}$	$1.25\cdot10^{-5}$	$3.14\cdot10^{-6}$	$1.25\cdot10^{-5}$	$3.14\cdot10^{-6}$

the relative error values. Using the same principle, we proved that the decreasing rate of relative error is also stable.

Table 2 presents the relative and absolute errors for different T values (up to $T=12$). For higher T values, the value of absolute error is growing. It is natural since both the solution and derivatives, on which depends the value of error, are proportional to the multiplier e^{at}, $t>0$.

We notice that with increasing t value, the relative error stays the same ($\Delta(h,\tau)$ rows of Table 2). Usually, for nonstationary problems, the absolute error and the relative error grow with increasing t value. We investigated this situation in detail and discovered that this is due to the structure of the selected solution. The exact solution (37) of differential problem is of the following form:

$$u(x,y,t)=u_1(x,y)\cdot u_2(t). \tag{38}$$

Whenever the solutions of differential and difference problems are proportional to the multiplier e^{at}, the relative error, according to (38), is independent of t. Therefore, by adding a constant term, let us slightly change solution (37) of the differential problem

$$u(x,y,t)=\frac{2}{\pi}\left(\sin\left(\frac{\pi x}{2}\right)+\gamma\frac{\pi x^2}{4}\right)\sin\left(\frac{\pi y}{2}\right)\mathrm{e}^{at}+C. \tag{39}$$

Table 3 shows the results of the numerical experiment. Neither of the solution derivatives, nonlocal condition, and the structure of the equation has changed. Only the Dirichlet boundary condition, initial condition, and function $f(u)$ have changed. However, the solution is not in the form of (38). According to the experiment data, absolute and relative errors are growing with increasing T.

Table 4. Absolute and relative error estimates in the case of big $\partial f/\partial u$ values.

		$T=1,\ \gamma=0.5,\ a=0.5$					
		$K=1$		$K=10$		$K=20$	
h	τ	$\varepsilon(h,\tau)$	$\Delta(h,\tau)$	$\varepsilon(h,\tau)$	$\Delta(h,\tau)$	$\varepsilon(h,\tau)$	$\Delta(h,\tau)$
$1/10$	$1/10$	$1.90\cdot10^{-3}$	$2.80\cdot10^{-3}$	$4.35\cdot10^{-2}$	$4.80\cdot10^{-2}$	$1.68\cdot10^{-2}$	$2.31\cdot10^{-2}$
$1/20$	$1/40$	$4.68\cdot10^{-4}$	$7.09\cdot10^{-4}$	$1.99\cdot10^{-1}$	$1.73\cdot10^{-1}$	$9.40\cdot10^{-3}$	$1.06\cdot10^{-1}$
$1/40$	$1/160$	$1.17\cdot10^{-4}$	$1.78\cdot10^{-4}$	$1.06\cdot10^{-1}$	$8.56\cdot10^{-2}$	$8.70\cdot10^{-1}$	$8.62\cdot10^{-1}$
$1/80$	$1/640$	$2.93\cdot10^{-5}$	$4.48\cdot10^{-5}$	$1.97\cdot10^{-3}$	$1.59\cdot10^{-2}$	$2.26\cdot10^{0}$	$2.11\cdot10^{0}$

While investigating the iterative method of solving difference problem, we encountered a situation when the convergence condition of the iterative method is valid only subject to the additional condition

$$\beta = \max\frac{\partial f}{\partial u} < \frac{1}{\tau}.$$

This fact motivated us to make an additional numerical experiment artificially increasing $\partial f/\partial u$ value ($\partial f/\partial u = 3Ku^2$); see Table 4.

The numerical experiment results show that the difference scheme is unstable if $\tau > 0$ value is not small enough ($\beta\tau < 1$). Also, at sufficiently large h values, the instability does not occur as the matrix has a low order ($h = 1/10,\ 1/20$). Finally, the numerical experiment confirms the authors' theoretical observation that relatively high $\partial f/\partial u$ values, at relative high τ ($\tau|\partial f/\partial u| > 1$) values, may lead to difference scheme's instability.

10 Conclusions and comments

In this article, we investigated the stability and convergence of the implicit difference scheme for the two-dimensional nonlinear parabolic equation with nonlocal condition (3). Theoretic results are obtained using the system of difference equations solution's majorant, constructed for the error of solution (Gerschgorin method). Such an approach was used for the elliptic problem with Dirichlet boundary conditions difference method's error of solution estimates decades ago [8, 12, 24].

The differential problem (1)–(5) investigated in this paper has been the object of numerical analysis research. The two-dimensional linear parabolic equation (1) with nonlocal boundary condition (3) in the case $f = 0$ is solved using Crank–Nicolson method in [15]. It is proved that the used difference scheme is stable in a specific (weak) energy norm

$$\|u\|_D = (\mathbf{D}u, u)^{1/2},$$

where $\mathbf{D} = (\mathbf{M}\mathbf{M}^*)^{-1}$ is the positive definite matrix, and $\mathbf{M}$ is the matrix formed by the eigenvectors (in some instances adjoint vectors) of difference problem. Problem (1)–(5) in the case $f = f(x, y, t)$ was solved by Peaceman–Rachford alternating direction method in article [29]. It was proved that the method is stable in norm $\|u\|_D$. However, the stability in the $\|u\|_D$ norm does not imply the stability in the maximum norm and convergence of the scheme.

In this article, the stability in maximum norm and the convergence of difference scheme for the differential problem (1)–(5) is proved using M-matrices theory (see also [10, 29]).

We also note that it is essential to investigate the spectrum structure of the difference operator's matrix using M-matrices theory for the analysis of the difference method's solutions for the problems with nonlocal boundary conditions.

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