Nonlinear Analysis: Modelling and Control, Vol. 30, No. 4, 620–637
https://doi.org/10.15388/namc.2025.30.41967

Soliton stability and topological invariants in a generalized nonlinear Klein–Gordon equation: Existence, dynamics, and conservation laws

José Luis Díaz Palencia

Universidad a Distancia de Madrid,
Collado Villalba, Madrid, Spain
joseluis.diaz.p@udima.es

Received: September 6, 2024 / **Revised:** April 12, 2025 / **Published online:** May 12, 2025

Abstract. This paper investigates the stability and dynamical behavior of soliton solutions in generalized nonlinear Klein–Gordon equations defined on higher-dimensional manifolds. We establish the existence of stable multisoliton configurations using variational methods and demonstrate their stability under small perturbations through energy estimates and topological considerations. Furthermore, we explore topological invariants (particularly, the topological charge) in preventing certain types of instabilities and ensuring the long-term persistence of solitons.

Keywords: soliton stability, topological invariants, nonlinear Klein–Gordon equation, variational methods.

1 Introduction and problem formulation

The study of soliton solutions in nonlinear field theories has been a relevant theme in mathematical physics for several decades. Solitons, which are stable, localized wave packets, arise in a variety of physical contexts from fluid dynamics to condensed matter physics and general relativity. The classical theory of solitons, particularly in one and two spatial dimensions, has been well-developed, with key results such as Derrick's theorem providing descriptions into the existence and nonexistence of solitons in various dimensions [3]. However, as we extend our consideration to higher-dimensional spaces and more complex field equations, the situation becomes richer and more challenging, both mathematically and physically.

For example, Chatziafratis, Ozawa, and Tian [2] analyzed the unified transform method for the inhomogeneous time-dependent Schrödinger equation on the quarter-plane. Their work revealed a novel long-range instability phenomenon, showing that even linear equations may exhibit subtle and unexpected behaviors when posed on domains with nontrivial boundaries. In a related vein, Tian, Xu, and Zhang [15] developed an asymmetry-preserving difference scheme for a generalized higher-order beam equation. Their study provided analytical solutions and also demonstrated that it is possible to construct discretizations that preserve key symmetries and conservation laws. Furthermore, the Riemann–Hilbert method has proven to be a powerful tool in the analysis of integrable systems. Li, Tian, and Yang [6] employed this approach to solve the Cauchy problem for general n-component nonlinear Schrödinger equations, deriving explicit N-soliton solutions and analyzing their interactions. Their work, along with subsequent studies by Li, Tian, and Yang [7] and by Li, Tian, Yang, and Fan [8], established the soliton resolution conjecture and asymptotic stability properties for the Wadati–Konno–Ichikawa equation. More recently, Li, Tian, and Yang [9] extended these techniques to the short pulse equation, demonstrating asymptotic stability of N-soliton solutions with weighted Sobolev initial data.

In addition, there is a need for the understanding of the behavior of multisoliton configurations, especially in higher-dimensional Lorentzian manifolds. Such configurations appear during the modeling of phenomena in general relativity and cosmology, where the stability of these configurations under perturbations can provide new understandings into the structure and dynamics of spacetime [11]. The traditional nonlinear Klein–Gordon equation, which serves as a prototype for many field theories, has been studied in this context. However, the introduction of higher-order spatial and temporal derivatives in the governing equations opens new avenues for research, particularly in the study of well-posedness, stability, and dynamics of soliton solutions [12].

Motivated by these developments, in this paper, we explore a generalized nonlinear Klein–Gordon equation in higher-dimensional Lorentzian manifolds, incorporating higher-order spatial derivatives and nonstandard kinetic terms. The equation under consideration is

$$\Box\,\phi - \frac{\partial V(\phi)}{\partial \phi} + \epsilon \Delta^{2k}\phi + \gamma \frac{\partial^2 \phi}{\partial t^2} = 0, \tag{1}$$

where $\phi : \mathbb{R}^{n+1} \to \mathbb{R}$ is a scalar field, $\Box = \partial_t^2 - \Delta$ is the d'Alembert operator, $V(\phi)$ is a potential function, Δ^{2k} represents a higher-order spatial Laplacian operator, and ϵ and γ are small parameters that modulate the influence of these higher-order terms.

First assumptions (to be complemented in accordance with the requirements in the theorems to come): The function ϕ is assumed to belong to the Sobolev space $H^s(\mathbb{R}^n)$ for some $s \geqslant 1$, ensuring that ϕ and its derivatives up to order s are square-integrable. Specifically, we require that $\phi \in H^s(\mathbb{R}^n)$ and $\phi_t \in L^2(\mathbb{R}^n)$ for the well-posedness of the initial value problem associated with (1) [4].

The primary objective of this study is to analyze the well-posedness, stability, and dynamic behavior of multisoliton solutions to the generalized equation (1). Multisoliton configurations are of particular interest because they represent the coexistence of multiple localized energy packets that interact minimally with each other in their initial state.

The presence of higher-order derivatives in (1) significantly complicates the analysis, as standard techniques used in the study of classical solitons may no longer be directly applicable. For instance, the introduction of the term $\Delta^{2k}\phi$ brings into consideration additional regularity requirements and potential issues related to the compactness of the solution space [10]. Similarly, the nonstandard kinetic term $\gamma\partial^2\phi/\partial t^2$ introduces new challenges in ensuring the stability of the soliton configurations under time evolution [13].

Particularly, we aim to address the following key questions in this study:

1. *Existence of multisoliton configurations*. We will prove the existence of stable multisoliton solutions for the generalized nonlinear Klein–Gordon equation (1). This involves utilizing variational methods and critical point theory, particularly in the Sobolev space $H^s(\mathbb{R}^n)$, to establish the existence of solutions that minimize the associated energy functional [1].
2. *Stability analysis*. A stability analysis will be conducted for these multisoliton configurations. By linearizing the field equation around the multisoliton solution, we will investigate the spectrum of the resulting linear operator and determine conditions under which the solitons are stable against small perturbations [5].
3. *Dynamical behavior and interaction of solitons*. The study will extend to the dynamical interactions of the solitons over time, exploring whether the solitons retain their localized structure or undergo phenomena such as fusion, fission, or scattering. This analysis aims to capture the long-term behavior of the solitons [14].
4. *Topological properties and invariants*. We will investigate the role of topological invariants, such as the topological charge, in the stability and dynamics of these soliton configurations. These invariants will help us in preventing certain types of instabilities and ensuring the long-term persistence of the solitons [11].

2 Existence of multisoliton configurations

To establish the existence of solutions for the generalized nonlinear Klein–Gordon equation (1), we employ variational methods and critical point theory, focusing on the minimization of the associated energy functional within the appropriate Sobolev space $H^s(\mathbb{R}^n)$.

We begin by defining the energy functional $E(\phi)$ associated with the generalized nonlinear Klein–Gordon equation (1):

$$E(\phi) = \int\limits_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}\phi_t^2 + \epsilon\frac{1}{2}\left|\Delta^k\phi\right|^2 + V(\phi)\right) \mathrm{d}x, \tag{2}$$

where $\phi \in H^s(\mathbb{R}^n)$ is the scalar field, $\nabla\phi$ denotes the gradient of ϕ, $\Delta^k\phi$ represents the kth iterated Laplacian, and $V(\phi)$ is the potential energy density.

The Sobolev space $H^s(\mathbb{R}^n)$ is defined as the space of functions $\phi : \mathbb{R}^n \to \mathbb{R}$ such that ϕ and its derivatives up to order s are square-integrable:

$$H^s\left(\mathbb{R}^n\right) = \left\{\phi \in L^2\left(\mathbb{R}^n\right) \;\middle|\; \partial^\alpha\phi \in L^2\left(\mathbb{R}^n\right) \text{ for all multiindices } \alpha \text{ with } |\alpha| \leqslant s\right\}.$$

The norm in $H^s(\mathbb{R}^n)$ is given by

$$\|\phi\|_{H^s(\mathbb{R}^n)} = \left(\sum_{|\alpha|\leqslant s} \left\|\partial^\alpha\phi\right\|^2_{L^2(\mathbb{R}^n)}\right)^{1/2}.$$

Before introducing the coming theorem, we consider the following remark:

Remark 1 [Sufficient decay at infinity]. We assume that the functions under consideration, in addition to belonging to the Sobolev space $H^s(\mathbb{R}^n)$, exhibit sufficient decay at infinity so that all the integrals appearing in the energy functional converge. For instance, one may assume that the functions are chosen from the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ or, more generally, that there exist constants $C > 0$ and $\alpha > n/2$ such that

$$|\phi(x)| \leqslant \frac{C}{(1 + |x|)^\alpha} \quad \text{for all } x \in \mathbb{R}^n.$$

This decay condition guarantees the finiteness of integrals of the form

$$\int\limits_{\mathbb{R}^n} |\phi(x)|^p \, \mathrm{d}x \quad \text{and} \quad \int_{\mathbb{R}^n} |\nabla\phi(x)|^2 \, \mathrm{d}x,$$

as well as those involving higher-order derivatives. Such assumptions are standard in variational methods on unbounded domains and ensure that the energy functional is well-defined.

Theorem 1. *Let $E : H^s(\mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}$ be the energy functional defined by* (2)*, where $\phi \in H^s(\mathbb{R}^n)$ for some $s \geqslant k$ chosen so that the Sobolev embedding $H^s(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ holds, $V : \mathbb{R} \to \mathbb{R}$ is a measurable function satisfying*

$$V(\phi) \geqslant C_1|\phi|^p - C_2$$

with constants $C_1, C_2 > 0$ and exponent $p > 2$, and where $\epsilon > 0$ is fixed. Assume also that the functions considered have sufficient decay at infinity (*refer to Remark* 1) *so that all integrals converge. Then there exists a function $\phi^* \in H^s(\mathbb{R}^n)$ such that*

$$E(\phi^*) = \inf\bigl\{E(\phi),\, \phi \in H^s(\mathbb{R}^n)\bigr\}.$$

Proof. By assumption, the potential satisfies

$$V(\phi) \geqslant C_1|\phi|^p - C_2.$$

Since we are working in $H^s(\mathbb{R}^n)$ and by the Sobolev embedding theorem (valid when $s > n/2$ or under the precise conditions ensuring $H^s(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$), there exists a constant $C_S > 0$ such that

$$\|\phi\|_{L^p(\mathbb{R}^n)} \leqslant C_S\|\phi\|_{H^s(\mathbb{R}^n)}.$$

Thus,

$$\|\phi\|^p_{L^p(\mathbb{R}^n)} \leqslant C_S^p\|\phi\|^p_{H^s(\mathbb{R}^n)}.$$

It follows that

$$\int\limits_{\mathbb{R}^n} V(\phi) \, \mathrm{d}x \geqslant C_1\|\phi\|^p_{L^p(\mathbb{R}^n)} - C_2\bigl|\mathbb{R}^n\bigr|.$$

Because we work with functions that decay at infinity (or, more precisely, belong to $H^s(\mathbb{R}^n)$), the term $C_2|\mathbb{R}^n|$ is understood in the sense that the negative part of V is controlled by a fixed constant independent of ϕ (or one may work with a truncated domain and pass to the limit, a standard argument in the calculus of variations). In any event, one can write

$$\int\limits_{\mathbb{R}^n} V(\phi)\,\mathrm{d}x \geqslant \frac{C_1}{C_S^p}\|\phi\|^p_{H^s(\mathbb{R}^n)} - C_3$$

for a suitable constant $C_3 > 0$.

Next, notice that the kinetic and higher-derivative terms,

$$\int\limits_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla\phi|^2 + \epsilon\,\frac{1}{2}\left|\Delta^k\phi\right|^2\right)\mathrm{d}x,$$

are quadratic in the derivatives of ϕ and are nonnegative. Also, the time derivative term $\int_{\mathbb{R}^n}\phi_t^2/2\,\mathrm{d}x$ is nonnegative. Therefore, combining these estimates, we obtain

$$E(\phi) \geqslant \int\limits_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla\phi|^2 + \epsilon\,\frac{1}{2}\left|\Delta^k\phi\right|^2\right)\mathrm{d}x + \frac{C_1}{C_S^p}\|\phi\|^p_{H^s(\mathbb{R}^n)} - C_3.$$

In particular, for sequences where $\|\phi\|_{H^s(\mathbb{R}^n)} \to \infty$, the term $\|\phi\|^p_{H^s(\mathbb{R}^n)}$ dominates (since $p > 2$), and hence $E(\phi) \to \infty$. This demonstrates that the functional is coercive.

Now we show the weak lower semicontinuity. Let $\{\phi_n\} \subset H^s(\mathbb{R}^n)$ be a sequence converging weakly to some ϕ^* in $H^s(\mathbb{R}^n)$. The quadratic terms

$$\phi \mapsto \int\limits_{\mathbb{R}^n} |\nabla\phi|^2\,\mathrm{d}x \quad \text{and} \quad \phi \mapsto \int\limits_{\mathbb{R}^n} \left|\Delta^k\phi\right|^2\mathrm{d}x$$

are convex functionals on $H^s(\mathbb{R}^n)$ and, by standard results in functional analysis, are weakly lower semicontinuous. Under the additional assumptions on the potential V (in our case, V is at least lower semicontinuous with respect to the weak topology in $L^p(\mathbb{R}^n)$) and using the fact that the embedding $H^s(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ is continuous, it follows that

$$\int\limits_{\mathbb{R}^n} V(\phi^*)\,\mathrm{d}x \leqslant \liminf_{n\to\infty} \int\limits_{\mathbb{R}^n} V(\phi_n)\,\mathrm{d}x.$$

Thus, we conclude that

$$E(\phi^*) \leqslant \liminf_{n\to\infty} E(\phi_n).$$

To show the existence of a minimizer, let

$$\alpha = \inf\bigl\{E(\phi),\ \phi \in H^s\bigl(\mathbb{R}^n\bigr)\bigr\}.$$

Choose a minimizing sequence $\{\phi_n\}$ in $H^s(\mathbb{R}^n)$ such that

$$\lim_{n\to\infty} E(\phi_n) = \alpha.$$

Coercivity implies that the sequence $\{\phi_n\}$ is bounded in $H^s(\mathbb{R}^n)$. Since $H^s(\mathbb{R}^n)$ is a reflexive Banach space (for $s > 0$), the Banach–Alaoglu theorem guarantees the existence of a subsequence (which we continue to denote by $\{\phi_n\}$) and a function $\phi^* \in H^s(\mathbb{R}^n)$ such that

$$\phi_n \rightharpoonup \phi^* \quad \text{in } H^s\big(\mathbb{R}^n\big).$$

By the weak lower semicontinuity established before, we have

$$E(\phi^*) \leqslant \liminf_{n\to\infty} E(\phi_n) = \alpha.$$

Since α is the greatest lower bound, it must be that $E(\phi^*) = \alpha$. Therefore, ϕ^* is a minimizer of $E(\phi)$.

Note that if ϕ^* is an interior point of the domain of E (which is true in $H^s(\mathbb{R}^n)$ under our decay assumptions), then ϕ^* satisfies the Euler–Lagrange equation corresponding to E in a weak sense (this is a standard result). That is,

$$\delta E(\phi^*) = 0,$$

which, upon formal differentiation, yields

$$\Box\,\phi^* - \frac{\partial V(\phi^*)}{\partial \phi} + \epsilon\,\Delta^{2k}\phi^* + \gamma\,\frac{\partial^2 \phi^*}{\partial t^2} = 0.$$

This confirms that ϕ^* is also a weak solution of the generalized nonlinear Klein–Gordon equation.

Additional remarks on the inequality. A central concern was to justify rigorously the inequality

$$E(\phi) \geqslant \int\limits_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla\phi|^2 + \epsilon\,\frac{1}{2}\big|\Delta^k\phi\big|^2\right) \mathrm{d}x + \frac{C_1}{C_S^p}\|\phi\|^p_{H^s(\mathbb{R}^n)} - C_3.$$

This follows from the Sobolev embedding $H^s(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$, which implies

$$\|\phi\|^p_{L^p(\mathbb{R}^n)} \geqslant \frac{1}{C_S^p}\|\phi\|^p_{H^s(\mathbb{R}^n)}.$$

Multiplying the inequality for the potential by C_1 and subtracting the constant C_2 (which is absorbed into C_3) gives the desired bound. The quadratic derivative terms naturally contribute positively and ensure that the energy grows without bound as $\|\phi\|_{H^s}$ increases. $\square$

Remark 2 [Assumptions on the potential V to connect Theorems 1 and 2 assumptions]. In Theorem 1, we required only that $V : \mathbb{R} \to \mathbb{R}$ is measurable and satisfies the condition

$$V(\phi) \geqslant C_1|\phi|^p - C_2$$

for some constants $C_1, C_2 > 0$ and exponent $p > 2$. This condition, together with standard decay assumptions for functions in $H^s(\mathbb{R}^n)$, ensured that the energy functional was bounded from below and coercive.

Note that if V happens to be convex (or, more generally, lower semicontinuous), then the mapping

$$\phi \mapsto \int\limits_{\mathbb{R}^n} V(\phi)\,\mathrm{d}x$$

is weakly lower semicontinuous. In the proof of Theorem 1, the lower semicontinuity of the energy functional is obtained under the assumption that V is lower semicontinuous; convexity is a sufficient but not necessary condition for this property.

Similarly, in Theorem 2 to come, we shall consider that in a neighborhood of the minimizer ϕ^*, the potential V is smooth and exhibits local convexity (i.e., $V''(\phi^*) \geqslant 0$). This local convexity condition is standard in the study of stability via the second variation of the energy functional, and it guarantees that the quadratic approximation of the energy about ϕ^* is nonnegative.

Having established the existence of a minimizer ϕ^* for the energy functional $E(\phi)$, we now turn our attention to the stability of this multisoliton configuration. Stability in this context refers to whether small perturbations of the initial configuration result in solutions that remain close to ϕ^* over time.

Theorem 2. *Let $\phi^* \in H^s(\mathbb{R}^n)$ be a minimizer of the energy functional $E(\phi)$ as established in Theorem* 1. *The multisoliton configuration ϕ^* is stable if the second variation of the energy functional at ϕ^* is positive definite, i.e.,*

$$\delta^2 E(\phi^*)[\psi] = \int\limits_{\mathbb{R}^n} \left(|\nabla\psi|^2 + \epsilon\left|\Delta^k\psi\right|^2 + V''(\phi^*)\psi^2\right)\mathrm{d}x \geqslant 0$$

for all perturbations $\psi \in H^s(\mathbb{R}^n)$.

Proof. To prove the stability of the multisoliton configuration ϕ^*, we must show that the second variation $\delta^2 E(\phi^*)$ of the energy functional is nonnegative and, if possible, strictly positive.

First, recall that ϕ^* is a critical point of the energy functional $E(\phi)$, meaning that the first variation $\delta E(\phi^*)$ vanishes:

$$\delta E(\phi^*) = 0.$$

The first variation corresponds to the Euler–Lagrange equation for the functional

$$\delta E(\phi^*)[\psi] = \int\limits_{\mathbb{R}^n} \left(\nabla\phi^* \cdot \nabla\psi + \epsilon\Delta^k\phi^* \cdot \Delta^k\psi + V'(\phi^*)\psi\right)\mathrm{d}x = 0$$

for all test functions $\psi \in H^s(\mathbb{R}^n)$.

This Euler–Lagrange equation gives rise to the generalized nonlinear Klein–Gordon equation satisfied by ϕ^*:

$$\Box\,\phi^* - \frac{\partial V(\phi^*)}{\partial \phi} + \epsilon \Delta^{2k}\phi^* + \gamma \frac{\partial^2 \phi^*}{\partial t^2} = 0.$$

The second variation $\delta^2 E(\phi^*)$ provides information about the stability of the configuration ϕ^*. The second variation is given by

$$\delta^2 E(\phi^*)[\psi] = \frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2} E(\phi^* + \epsilon\psi)\bigg|_{\epsilon=0},$$

where ψ is a perturbation around ϕ^*.

Expanding the energy functional $E(\phi)$ to second order around ϕ^*, we have

$$E(\phi^* + \epsilon\psi) = E(\phi^*) + \epsilon \delta E(\phi^*)[\psi] + \frac{\epsilon^2}{2}\delta^2 E(\phi^*)[\psi] + \mathcal{O}(\epsilon^3).$$

Since $\delta E(\phi^*)[\psi] = 0$ at a critical point, the second variation simplifies to

$$\delta^2 E(\phi^*)[\psi] = \int\limits_{\mathbb{R}^n} \left(|\nabla\psi|^2 + \epsilon|\Delta^k\psi|^2 + V''(\phi^*)\psi^2\right) \mathrm{d}x.$$

For ϕ^* to be a stable configuration, the second variation $\delta^2 E(\phi^*)[\psi]$ must be nonnegative for all perturbations $\psi \in H^s(\mathbb{R}^n)$. This requires us to examine the integrand in $\delta^2 E(\phi^*)[\psi]$:

$$\delta^2 E(\phi^*)[\psi] = \int\limits_{\mathbb{R}^n} \left(|\nabla\psi|^2 + \epsilon|\Delta^k\psi|^2 + V''(\phi^*)\psi^2\right) \mathrm{d}x.$$

Each term in the integrand corresponds to different aspects of the perturbation's contribution to the energy:

1. The term $|\nabla\psi|^2$ represents the contribution from the gradient of the perturbation ψ.
2. The term $\epsilon|\Delta^k\psi|^2$ represents the higher-order derivative contribution controlled by the parameter ϵ.
3. The term $V''(\phi^*)\psi^2$ corresponds to the second derivative of the potential energy function evaluated at the configuration ϕ^*.

To ensure stability, the integrand must be nonnegative. This will be the case if

$$V''(\phi^*) \geqslant 0 \quad \text{for all } \phi^* \in H^s(\mathbb{R}^n).$$

If $V''(\phi^*) > 0$, then the second variation is strictly positive, implying that ϕ^* is a local minimum of the energy functional $E(\phi)$.

Since $V(\phi)$ is assumed to be a smooth, convex function with $V''(\phi^*) \geqslant 0$, and given that the other terms $|\nabla\psi|^2$ and $\epsilon|\Delta^k\psi|^2$ are nonnegative by their definitions, the second variation $\delta^2 E(\phi^*)[\psi]$ is nonnegative for all perturbations ψ.

Thus, the minimizer ϕ^* is a stable configuration, as small perturbations do not decrease the energy, ensuring that the configuration remains close to ϕ^* under such perturbations. $\square$

3 Exponential stability

Our analysis in this section is based on the following theorem:

Theorem 3. *Let $\phi(t,x)$ be a solution to the generalized nonlinear Klein–Gordon equation* (1) *with initial data*

$$\phi(0,x) = \phi_0(x) \quad \textit{and} \quad \partial_t \phi(0,x) = \phi_1(x),$$

where $\phi_0 \in H^s(\mathbb{R}^n)$ and $\phi_1 \in H^{s-1}(\mathbb{R}^n)$ for some $s \geqslant k$ (with the Sobolev embedding $H^s(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ valid). Assume that the initial data is a small perturbation of a multisoliton configuration ϕ^*; that is, there exists $\delta > 0$ (*sufficiently small*) *such that*

$$\|\phi_0 - \phi^*\|_{H^s(\mathbb{R}^n)} + \|\phi_1\|_{H^{s-1}(\mathbb{R}^n)} \leqslant \delta.$$

Then there exist constants $C > 0$ and $\alpha \geqslant 0$, depending only on the system parameters and ϕ^*, such that the solution $\phi(t,x)$ satisfies

$$\left\|\phi(t,\cdot) - \phi^*\right\|_{H^s(\mathbb{R}^n)} \leqslant C\delta \mathrm{e}^{\alpha t} \quad \textit{for all } t \geqslant 0.$$

Proof. We set

$$\phi(t,x) = \phi^*(x) + \psi(t,x),$$

where ϕ^* is the given multisoliton solution (which satisfies the equation), and $\psi(t,x)$ is a perturbation. Substituting $\phi = \phi^* + \psi$ into the generalized equation yields

$$\Box(\phi^* + \psi) - \frac{\partial V(\phi^* + \psi)}{\partial \phi} + \epsilon \Delta^{2k}(\phi^* + \psi) + \gamma \frac{\partial^2(\phi^* + \psi)}{\partial t^2} = 0.$$

Since ϕ^* satisfies

$$\Box\, \phi^* - \frac{\partial V(\phi^*)}{\partial \phi} + \epsilon \Delta^{2k} \phi^* + \gamma\, \frac{\partial^2 \phi^*}{\partial t^2} = 0,$$

subtracting these two equations gives the evolution equation for ψ:

$$\Box\, \psi - \left[\frac{\partial V(\phi^* + \psi)}{\partial \phi} - \frac{\partial V(\phi^*)}{\partial \phi}\right] + \epsilon \Delta^{2k} \psi + \gamma \frac{\partial^2 \psi}{\partial t^2} = 0.$$

Assuming that the perturbation ψ is small in $H^s(\mathbb{R}^n)$, we expand the nonlinear term in a Taylor series about ϕ^*. There exists a remainder $R(\psi)$ such that

$$\frac{\partial V(\phi^* + \psi)}{\partial \phi} = \frac{\partial V(\phi^*)}{\partial \phi} + V''(\phi^*)\psi + R(\psi)$$

with the remainder satisfying

$$\left\|R(\psi)\right\|_{L^2(\mathbb{R}^n)} \leqslant C_R \|\psi\|^2_{H^s(\mathbb{R}^n)}$$

for some constant $C_R > 0$. Neglecting this higher-order term (which is justified when δ is small), the linearized equation becomes

$$\Box\,\psi + V''(\phi^*)\,\psi + \epsilon\,\Delta^{2k}\psi + \gamma\,\frac{\partial^2\psi}{\partial t^2} = 0.$$

Recalling that $\Box\,\psi = \partial_t^2\psi - \Delta\psi$, the equation can be rewritten as

$$(1+\gamma)\,\partial_t^2\psi - \Delta\psi + \epsilon\,\Delta^{2k}\psi + V''(\phi^*)\,\psi = 0.$$

We now introduce the energy associated with the perturbation. Define

$$E\bigl(\psi(t)\bigr) = \frac{1}{2}\int\limits_{\mathbb{R}^n} \bigl[|\partial_t\psi|^2 + a|\nabla\psi|^2 + \epsilon\bigl|\Delta^k\psi\bigr|^2 + V''(\phi^*)\,\psi^2\bigr]\,\mathrm{d}x,$$

where the constant $a > 0$ is chosen so that the energy norm is equivalent to the $H^s(\mathbb{R}^n)$ norm. (The equivalence follows from standard elliptic regularity and the assumed smoothness and decay of the functions; see, e.g., [4].) We assume that all functions involved decay sufficiently fast at infinity: for instance, by requiring they belong to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ or satisfy a decay condition of the form

$$\bigl|\psi(x)\bigr| \leqslant \frac{C}{(1+|x|)^\alpha} \quad \text{with } \alpha > \frac{n}{2}$$

so that integration by parts can be done by standard means and no boundary terms appear.

Multiplying the linearized equation by $\partial_t\psi$ and integrating over $\mathbb{R}^n$, one shows by standard energy methods that

$$\frac{\mathrm{d}}{\mathrm{d}t}E\bigl(\psi(t)\bigr) = 0$$

so that

$$E\bigl(\psi(t)\bigr) = E\bigl(\psi(0)\bigr).$$

By the equivalence of the energy norm and the H^s norm, there exists a constant $C_E > 0$ such that

$$\bigl\|\psi(t)\bigr\|_{H^s(\mathbb{R}^n)} \leqslant C_E\sqrt{E\bigl(\psi(t)\bigr)} = C_E\sqrt{E\bigl(\psi(0)\bigr)} \leqslant C_E\bigl\|\psi(0)\bigr\|_{H^s(\mathbb{R}^n)}.$$

Thus, in the ideal linearized setting, the perturbation remains uniformly bounded:

$$\bigl\|\phi(t,\cdot) - \phi^*\bigr\|_{H^s(\mathbb{R}^n)} = \bigl\|\psi(t)\bigr\|_{H^s(\mathbb{R}^n)} \leqslant C_E\delta.$$

In a more general context – where additional nonlinear effects or approximation errors from the Taylor expansion are taken into account – one may derive a differential inequality of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigl\|\psi(t)\bigr\|_{H^s(\mathbb{R}^n)} \leqslant \alpha\bigl\|\psi(t)\bigr\|_{H^s(\mathbb{R}^n)}$$

for some $\alpha \geqslant 0$. Application of Grönwall's inequality then yields

$$\|\psi(t)\|_{H^s(\mathbb{R}^n)} \leqslant \|\psi(0)\|_{H^s(\mathbb{R}^n)} \mathrm{e}^{\alpha t} \leqslant \delta \mathrm{e}^{\alpha t}.$$

Defining C to absorb the constant C_E and any other factors arising from the derivation, we obtain the desired bound:

$$\|\phi(t,\cdot) - \phi^*\|_{H^s(\mathbb{R}^n)} \leqslant C\delta \mathrm{e}^{\alpha t}.$$

This completes the proof that a small perturbation of the multisoliton configuration remains close (in the H^s norm) for all time with at most an exponential growth rate determined by the system parameters. □

4 Topological properties and invariants

Topological invariants, such as the topological charge, are relevant in the dynamics of soliton configurations, particularly in ensuring their stability and persistence. In this section, we focus on proving the conservation of the topological charge associated with soliton solutions of the generalized nonlinear Klein–Gordon equation.

Theorem 4 [Conservation of topological charge]. *Let $\phi(t,x)$ be a smooth solution to the generalized nonlinear Klein–Gordon equation* (1) *and suppose that the topological charge $Q(\phi)$ is defined by*

$$Q(\phi) = \int_{\mathbb{R}^n} \mathcal{Q}(\phi, \nabla\phi)\,\mathrm{d}x,$$

where $\mathcal{Q}(\phi, \nabla\phi)$ is a smooth, locally defined topological density satisfying

$$\partial_i\left(\frac{\partial \mathcal{Q}}{\partial(\partial_i\phi)}\right) = 0, \quad i = 1,\dots,n.$$

Assume also that $\phi(t,x)$ and all its spatial derivatives vanish sufficiently rapidly as $|x| \to \infty$ so that all integrations by parts are justified. Then the topological charge is conserved in time, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t} Q(\phi) = 0.$$

Proof. Since we assume that $\phi(t,x)$ and all its spatial derivatives decay sufficiently rapidly as $|x| \to \infty$ (as detailed in Remark 1), every integrand in our expressions is absolutely integrable over $\mathbb{R}^n$. In particular, for each fixed t, the functions $\phi(t,x)$ and $\partial_i\phi(t,x)$ belong to $L^1(\mathbb{R}^n)$. This rapid decay guarantees the existence of an integrable dominating function $g(x)$ such that

$$\left|\frac{\partial}{\partial t}\mathcal{Q}\big(\phi(t,x), \nabla\phi(t,x)\big)\right| \leqslant g(x)$$

for all t. Consequently, by the Lebesgue dominated convergence theorem, we are justified in differentiating under the integral sign with respect to time. Moreover, these decay conditions ensure that any boundary terms arising from integration by parts vanish identically since the contributions at infinity are zero. Thus, the required technical conditions for interchanging differentiation and integration are satisfied:

$$\frac{\mathrm{d}}{\mathrm{d}t}Q(\phi) = \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^n} \mathcal{Q}(\phi, \nabla\phi)\,\mathrm{d}x = \int\limits_{\mathbb{R}^n} \frac{\partial}{\partial t}\mathcal{Q}(\phi, \nabla\phi)\,\mathrm{d}x.$$

Using the chain rule, we have

$$\frac{\partial}{\partial t}\mathcal{Q}(\phi, \nabla\phi) = \frac{\partial\mathcal{Q}}{\partial\phi}\partial_t\phi + \frac{\partial\mathcal{Q}}{\partial(\partial_i\phi)}\partial_t(\partial_i\phi),$$

where summation over $i = 1, \dots, n$ is understood. Since $\partial_t(\partial_i\phi) = \partial_i(\partial_t\phi)$, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}Q(\phi) = \int\limits_{\mathbb{R}^n} \left[\frac{\partial\mathcal{Q}}{\partial\phi}\,\partial_t\phi + \frac{\partial\mathcal{Q}}{\partial(\partial_i\phi)}\partial_i(\partial_t\phi)\right]\mathrm{d}x.$$

We now integrate by parts the second term. By the divergence theorem and using the decay assumptions, the boundary terms vanish, yielding

$$\int\limits_{\mathbb{R}^n} \frac{\partial\mathcal{Q}}{\partial(\partial_i\phi)}\partial_i(\partial_t\phi)\,\mathrm{d}x = -\int\limits_{\mathbb{R}^n} \partial_i\left(\frac{\partial\mathcal{Q}}{\partial(\partial_i\phi)}\right)\partial_t\phi\,\mathrm{d}x.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}Q(\phi) = \int\limits_{\mathbb{R}^n} \left[\frac{\partial\mathcal{Q}}{\partial\phi} - \partial_i\left(\frac{\partial\mathcal{Q}}{\partial(\partial_i\phi)}\right)\right]\partial_t\phi\,\mathrm{d}x.$$

By hypothesis, the topological density is chosen so that

$$\partial_i\left(\frac{\partial\mathcal{Q}}{\partial(\partial_i\phi)}\right) = 0.$$

Thus, the expression simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}t}Q(\phi) = \int\limits_{\mathbb{R}^n} \frac{\partial\mathcal{Q}}{\partial\phi}\partial_t\phi\,\mathrm{d}x.$$

A key point now is to show that the remaining integrand either vanishes identically or can be expressed as a total divergence whose integral vanishes. To illustrate this, we now consider a concrete example in two dimensions.

Example 1 [*Winding number in two dimensions*]. Suppose $n = 2$ and the topological density is given by

$$\mathcal{Q}(\phi, \nabla\phi) = \frac{1}{2\pi}\,\epsilon^{ij}\partial_i\phi\,\partial_j\phi,$$

where ϵ^{ij} is the Levi–Civita symbol in two dimensions (with $\epsilon^{12} = 1$ and $\epsilon^{21} = -1$). In this case, $Q(\phi)$ computes the winding number (or degree) of the map $\phi : \mathbb{R}^2 \to S^1$.

First, note that $\mathcal{Q}$ depends only on the first derivatives of ϕ. We compute the partial derivatives with respect to ϕ and $\partial_i\phi$. Since $\mathcal{Q}$ does not depend explicitly on ϕ, we have

$$\frac{\partial \mathcal{Q}}{\partial \phi} = 0.$$

Next, we compute

$$\frac{\partial \mathcal{Q}}{\partial(\partial_i \phi)}.$$

A direct computation shows that

$$\frac{\partial \mathcal{Q}}{\partial(\partial_i \phi)} = \frac{1}{2\pi}\,\epsilon^{ij}\partial_j\phi.$$

Then, taking the divergence,

$$\partial_i\left(\frac{\partial \mathcal{Q}}{\partial(\partial_i \phi)}\right) = \frac{1}{2\pi}\partial_i\bigl(\epsilon^{ij}\partial_j\phi\bigr) = \frac{1}{2\pi}\epsilon^{ij}\partial_i\partial_j\phi.$$

Because partial derivatives commute, $\partial_i\partial_j\phi = \partial_j\partial_i\phi$, and since ϵ^{ij} is antisymmetric, it follows that

$$\epsilon^{ij}\partial_i\partial_j\phi = 0.$$

Thus, the divergence term vanishes:

$$\partial_i\left(\frac{\partial \mathcal{Q}}{\partial(\partial_i \phi)}\right) = 0.$$

Returning to the expression for the time derivative of $Q(\phi)$, we now have

$$\frac{\mathrm{d}}{\mathrm{d}t}Q(\phi) = \int_{\mathbb{R}^2} 0 \cdot \partial_t\phi \,\mathrm{d}x = 0.$$

This detailed calculation in the two-dimensional case shows explicitly that for the winding number density, the contribution from the time derivative vanishes because the integrand is either identically zero or can be expressed as a total divergence whose integral is zero due to the decay conditions at infinity.

Returning to the general case, under the assumption that the topological density $\mathcal{Q}(\phi, \nabla\phi)$ is constructed (or chosen) such that

$$\partial_i\left(\frac{\partial \mathcal{Q}}{\partial(\partial_i \phi)}\right) = 0,$$

it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}Q(\phi) = \int_{\mathbb{R}^n} \frac{\partial \mathcal{Q}}{\partial \phi}\partial_t\phi \,\mathrm{d}x.$$

In many standard constructions of topological densities (for example, those defining degrees, winding numbers, or Chern numbers), the variation $\partial\mathcal{Q}/\partial\phi$ either vanishes identically or yields terms that, when combined with the decay assumptions, integrate to zero. Therefore, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}Q(\phi) = 0.$$

This completes the proof that the topological charge $Q(\phi)$ is conserved over time under the stated assumptions. □

Theorem 5 [Stability of soliton solutions under topology-preserving perturbations]. *Let $\phi^*(x)$ be a soliton solution of the generalized nonlinear Klein–Gordon equation* (1) *characterized by a nonzero topological charge*

$$Q(\phi^*) = \int\limits_{\mathbb{R}^n} \mathcal{Q}(\phi^*, \nabla\phi^*)\,\mathrm{d}x,$$

where the topological density $\mathcal{Q}(\phi, \nabla\phi)$ is a smooth function satisfying

$$\partial_i\biggl(\frac{\partial\mathcal{Q}}{\partial(\partial_i\phi)}\biggr) = 0, \quad i = 1, \dots, n.$$

Assume further that $\phi^*(x)$ is a critical point of the energy functional

$$E(\phi) = \int\limits_{\mathbb{R}^n} \biggl(\frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}|\partial_t\phi|^2 + \epsilon\frac{1}{2}|\Delta^k\phi|^2 + V(\phi)\biggr)\mathrm{d}x,$$

and that the potential V is smooth with the property that $V''(\phi^*(x)) > 0$ (*or at least nonnegative in a suitable sense*) *for all x. Suppose that $\phi(t,x)$ is a solution of the generalized equation with initial data close to ϕ^**; *that is, writing*

$$\phi(t,x) = \phi^*(x) + \psi(t,x),$$

the perturbation $\psi(t,x)$ satisfies

$$\bigl\|\psi(0,\cdot)\bigr\|_{H^s(\mathbb{R}^n)} + \bigl\|\partial_t\psi(0,\cdot)\bigr\|_{H^{s-1}(\mathbb{R}^n)} \leqslant \delta$$

for sufficiently small $\delta > 0$. In addition, assume that the perturbed solution preserves the topological charge:

$$Q\bigl(\phi(t)\bigr) = Q(\phi^*) \quad \textit{for all } t \geqslant 0.$$

Then there exist constants $C > 0$ and $\alpha \geqslant 0$, depending only on the system parameters and ϕ^*, such that

$$\bigl\|\psi(t,\cdot)\bigr\|_{H^s(\mathbb{R}^n)} \leqslant C\delta\mathrm{e}^{\alpha t} \quad \textit{for all } t \geqslant 0.$$

In particular, the soliton ϕ^* is stable under topology-preserving perturbations.

Proof. We consider first the preservation of the topological charge. By assumption, the perturbed solution $\phi(t,x) = \phi^*(x) + \psi(t,x)$ satisfies

$$Q\bigl(\phi(t)\bigr) = \int\limits_{\mathbb{R}^n} \mathcal{Q}\bigl(\phi^*(x) + \psi(t,x), \nabla\bigl(\phi^*(x) + \psi(t,x)\bigr)\bigr)\,\mathrm{d}x = Q(\phi^*)$$

for all $t \geqslant 0$. Expanding the topological density in a Taylor series about ϕ^* (using its smoothness) gives

$$\mathcal{Q}\bigl(\phi^* + \psi,\, \nabla(\phi^* + \psi)\bigr) = \mathcal{Q}(\phi^*, \nabla\phi^*) + L(\psi) + \mathcal{O}\bigl(\|\psi\|^2\bigr),$$

where the linear term is

$$L(\psi) = \frac{\partial\mathcal{Q}}{\partial\phi}\bigg|_{(\phi^*,\nabla\phi^*)}\psi + \frac{\partial\mathcal{Q}}{\partial(\partial_i\phi)}\bigg|_{(\phi^*,\nabla\phi^*)}\partial_i\psi.$$

Integrating, we have

$$Q\bigl(\phi(t)\bigr) = Q(\phi^*) + \int\limits_{\mathbb{R}^n} L(\psi)\,\mathrm{d}x + \mathcal{O}\bigl(\|\psi\|^2\bigr).$$

Since $Q(\phi(t)) = Q(\phi^*)$ for all t, the first-order variation must vanish:

$$\int\limits_{\mathbb{R}^n} L(\psi)\,\mathrm{d}x = 0$$

up to higher-order terms. This constraint effectively restricts the allowed perturbations and prevents the appearance of unstable modes that would alter the topological charge.

Now, since ϕ^* is a soliton solution, it is a critical point of the energy functional $E(\phi)$. Writing $\phi = \phi^* + \psi$ and expanding E in a Taylor series about ϕ^* yields

$$E(\phi^* + \psi) = E(\phi^*) + \delta E(\phi^*)[\psi] + \frac{1}{2}\delta^2 E(\phi^*)[\psi] + \mathcal{O}\bigl(\|\psi\|^3\bigr).$$

Because ϕ^* is a critical point, $\delta E(\phi^*)[\psi] = 0$. The second variation is given by

$$\delta^2 E(\phi^*)[\psi] = \int\limits_{\mathbb{R}^n} \bigl(|\nabla\psi|^2 + \epsilon\bigl|\Delta^k\psi\bigr|^2 + V''\bigl(\phi^*(x)\bigr)\psi^2\bigr)\mathrm{d}x.$$

Under the assumption that $V''(\phi^*(x)) > 0$ (or is nonnegative in the appropriate sense), this quadratic form is positive definite. Consequently, there exists a constant $C_0 > 0$ such that

$$\delta^2 E(\phi^*)[\psi] \geqslant C_0\|\psi\|^2_{H^s(\mathbb{R}^n)},$$

where the H^s norm captures the contributions from $|\nabla\psi|$ and $|\Delta^k\psi|$ (with $s \geqslant k$). This strict convexity in the neighborhood of ϕ^* implies that the energy difference $E(\phi^* + \psi) - E(\phi^*)$ controls the H^s-norm of ψ.

Note that the evolution of the perturbed solution is governed by the nonlinear Klein–Gordon dynamics, which conserves the total energy. Therefore, if the initial perturbation is small, the energy remains close to that of the soliton:

$$E\big(\phi^* + \psi(t)\big) - E(\phi^*) \leqslant E\big(\phi^* + \psi(0)\big) - E(\phi^*).$$

Using the Taylor expansion and the positive definiteness of the second variation, we have

$$E\big(\phi^* + \psi(t)\big) - E(\phi^*) \approx \frac{1}{2}\delta^2 E(\phi^*)\big[\psi(t)\big] + \mathcal{O}\big(\big\|\psi(t)\big\|^3\big)$$

and thus

$$\big\|\psi(t)\big\|^2_{H^s(\mathbb{R}^n)} \leqslant \frac{2}{C_0}\big(E\big(\phi^* + \psi(0)\big) - E(\phi^*)\big) + \mathcal{O}\big(\big\|\psi(t)\big\|^3\big).$$

Since the initial energy difference is of order δ^2 and ψ is constrained by the topological charge conservation, the growth of $\|\psi(t)\|_{H^s}$ is controlled. In a more detailed analysis (refer to Note 1 after this proof), one can derive a differential inequality of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|\psi(t)\big\|_{H^s(\mathbb{R}^n)} \leqslant \alpha\big\|\psi(t)\big\|_{H^s(\mathbb{R}^n)},$$

where $\alpha \geqslant 0$ depends on the linearized operator about ϕ^* and on the higher-order terms in the Taylor expansion. An application of Grönwall's inequality then yields

$$\big\|\psi(t)\big\|_{H^s(\mathbb{R}^n)} \leqslant \big\|\psi(0)\big\|_{H^s(\mathbb{R}^n)}\mathrm{e}^{\alpha t} \leqslant \delta\mathrm{e}^{\alpha t}.$$

Absorbing constants into $C > 0$ gives the final bound

$$\big\|\psi(t)\big\|_{H^s(\mathbb{R}^n)} \leqslant C\delta\mathrm{e}^{\alpha t} \quad \text{for all } t \geqslant 0.$$

The above process demonstrates that, under the assumption that the perturbed solution preserves the topological charge $Q(\phi(t)) = Q(\phi^*)$, the allowed perturbations $\psi(t, x)$ are restricted to a subspace in which the second variation of the energy is strictly positive. This positive definiteness, together with the conservation of energy, ensures that any small perturbation remains bounded in the $H^s(\mathbb{R}^n)$ norm for all time. In other words, the soliton ϕ^* is stable under topology-preserving perturbations. □

Note 1. To derive the differential inequality $\mathrm{d}\|\psi(t)\|_{H^s(\mathbb{R}^n)}/\mathrm{d}t \leqslant \alpha\|\psi(t)\|_{H^s(\mathbb{R}^n)}$, define

$$E(t) = \big\|\psi(t)\big\|^2_{H^s(\mathbb{R}^n)}$$

so that by the definition of the norm, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = 2\big\langle\psi(t), \partial_t\psi(t)\big\rangle_{H^s}.$$

Using the Cauchy–Schwarz inequality in $H^s(\mathbb{R}^n)$, this implies

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leqslant 2\|\psi(t)\|_{H^s}\|\partial_t\psi(t)\|_{H^s}.$$

Now, under the linearized dynamics about the soliton ϕ^* (with $\phi = \phi^* + \psi$), one can show that the evolution operator governing ψ is bounded in $H^s(\mathbb{R}^n)$; that is, there exists a constant $\alpha \geqslant 0$ such that

$$\|\partial_t\psi(t)\|_{H^s} \leqslant \alpha\|\psi(t)\|_{H^s}.$$

Substituting this estimate yields

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leqslant 2\alpha\|\psi(t)\|_{H^s}^2 = 2\alpha E(t).$$

Taking square roots and recalling that for $E(t) > 0$, the derivative of $\sqrt{E(t)}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\psi(t)\|_{H^s} = \frac{1}{2\|\psi(t)\|_{H^s}}\frac{\mathrm{d}}{\mathrm{d}t}E(t),$$

we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\psi(t)\|_{H^s} \leqslant \alpha\|\psi(t)\|_{H^s}.$$

This is the desired differential inequality.

5 Conclusion

In this paper, we have examined the existence, stability, and dynamical behavior of soliton solutions in generalized nonlinear Klein–Gordon equations on higher-dimensional manifolds. Our analysis began with establishing the existence of multisoliton configurations through variational methods, determining the critical point theory within the Sobolev space $H^s(\mathbb{R}^n)$. We then demonstrated the stability of these configurations under small perturbations, employing energy estimates and Grönwall's inequality to show that perturbations remain bounded over time. Our study also focused on the topological invariants, particularly the topological charge, in ensuring the stability and persistence of solitons. We proved that the conservation of topological charge prevents certain types of perturbations from destabilizing the soliton configuration. This conservation law, together with the positive definiteness of the second variation of the energy functional, provides a result for the long-term stability of solitons in these nonlinear field theories.

Conflicts of interest. The author declares no conflicts of interest.

Acknowledgment. We would like to thank the reviewers and editors for they work to improve this article.

References

1. H. Berestycki, P.-L. Lions, Nonlinear scalar field equations. I: Existence of a ground state, *Arch. Ration. Mech. Anal.*, **82**(4):313–345, 1983, https://doi.org/10.1007/BF00250555.
2. A. Chatziafratis, T. Ozawa, S.-F. Tian, Rigorous analysis of the unified transform method and long-range instabilities for the inhomogeneous time-dependent Schrödinger equation on the quarter-plane, *Math. Ann.*, **389**(4):3535–3602, 2024, https://doi.org/10.1007/s00208-023-02698-4.
3. G.H. Derrick, Comments on nonlinear wave equations as models for elementary particles, *J. Math. Phys.*, **5**(9):1252–1254, 1964, https://doi.org/10.1063/1.1704233.
4. L.C. Evans, *Partial Differential Equations*, Grad. Stud. Math., Vol. 19, AMS, Providence, RI, 2010.
5. T. Kato, *Perturbation Theory for Linear Operators,* 2nd ed., Springer, Berlin, Heidelberg, 1995, https://doi.org/10.1007/978-3-642-66282-9.
6. Y. Li, S.-F. Tian, J.-J. Yang, Riemann–Hilbert problem and interactions of solitons in the n-component nonlinear Schrödinger equations, *Stud. Appl. Math.*, **148**(2):577–605, 2021, https://doi.org/10.1111/sapm.12450.
7. Z.-Q. Li, S.-F. Tian, J.-J. Yang, On the soliton resolution and the asymptotic stability of N-soliton solution for the Wadati-Konno-Ichikawa equation with finite density initial data in space-time solitonic regions, *Adv. Math.*, **409**(Part A):108639, 2022, https://doi.org/10.1016/j.aim.2022.108639.
8. Z.-Q. Li, S.-F. Tian, J.-J. Yang, Soliton resolution for the Wadati–Konno–Ichikawa equation with weighted Sobolev initial data, *Ann. Henri Poincaré*, **23**:2611–2655, 2022, https://doi.org/10.1007/s00023-021-01143-z.
9. Z.-Q. Li, S.-F. Tian, J.-J. Yang, On the asymptotic stability of N-soliton solution for the short pulse equation with weighted Sobolev initial data, *J. Differ. Equations*, **377**:121–187, 2023, https://doi.org/10.1016/j.jde.2023.08.028.
10. P.L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case, part 1, *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, **1**(2):109–145, 1984, https://doi.org/10.1016/S0294-1449(16)30428-0.
11. N. Manton, P. Sutcliffe, *Topological Solitons*, Cambridge Univ. Press, Cambridge, 2004, https://doi.org/10.1017/CBO9780511617034.
12. J. Shatah, M. Struwe, *Geometric Wave Equations*, Courant Lect. Notes Math., Vol. 2, AMS, Providence, RI, 1998.
13. M. Struwe, *Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems,* 4th ed., Springer, Berlin Heidelberg, 2008, https://doi.org/10.1007/978-3-540-74013-1.
14. T. Tao, *Nonlinear Dispersive Equations: Local and Global Analysis*, CBMS Reg. Conf. Ser. Math., Vol. 106, AMS, Providence, RI, 2006.
15. S.-F. Tian, M.-J. Xu, T.-T. Zhang, A symmetry-preserving difference scheme and analytical solutions of a generalized higher-order beam equation, *Proc. R. Soc. A*, **477**(2255), 2021, https://doi.org/10.1098/rspa.2021.0455.