Nonlinear Analysis: Modelling and Control, Vol. 30, No. 4, 771–791
https://doi.org/10.15388/namc.2025.30.42686

Numerical algorithms to solve one inverse problem for Navier–Stokes equations*

Raimondas Čiegis

Vilnius Gediminas Technical University
Saulėtekio av. 11, Vilnius, Lithuania
rc@vgtu.lt

Received: January 8, 2025 / **Revised:** June 2, 2025 / **Published online:** July 1, 2025

Abstract. This model describes the Poiseuille type solution in the nonstationary case of the Navier–Stokes problem. An equivalent form of PDE problem is defined as the first-kind Volterra integral equation. The main aim is to analyze a possible ill-posedness of the given problem. For some problems the first-kind Volterra integral equation can be modified to the integral equation of the second kind and the letter equation is well-posed. Different regularization techniques also can be used to control the influence of error pollution with not equal efficiency. Thus we made an extensive analysis and compared classical discretization schemes for PDE and integral Navier–Stokes models and regularization algorithms.

The regularization methods are applied to control the influence of the noise in data. The numerical experiment was aimed at obtaining new information about the stability of schemes for the inverse problems. Different approximations methods are used to solve PDE and integral versions of the equation. Results of computational experiments are presented, they confirm the theoretical error analysis and stability estimates.

Keywords: inverse problems, numerical approximation, Navier–Stokes problem, Volterra equation, regularization methods.

1 Introduction

The development of mathematical models based on the Navier–Stokes equations and simulation of various real world applications requires construction of efficient numerical solvers. It is a general trend in applied numerical analysis to consider specific cases of the classical Navier–Stokes equations and to propose accurate approximation methods targeted for such restricted subsets of problems. Here we can mention analysis of models with weak and very weak solutions [8, 11, 15], problems with a specific structure of solutions, including the Poiseuille type solutions. This class of problems is the main goal of our paper. An important approach to reduce the general Navier–Stokes problem

*This research was supported by a grant from the Research Council of Lithuania (LMTLT), agreement No. S-MIP-23-43.

to inverse parabolic problems was initiated in [14] and continued in [15]. The problem of finding the solution of inverse parabolic problems with unknown linear and non-linear source terms and nonstationary integral constraints were considered in many papers. A good review of numerical schemes is given in [4].

The main technique of papers mentioned above is based on a possibility to rewrite a PDE model to the first-kind integral Volterra equations.

It is quite well known, that the continuous problem can reformulated as Volterra equation with a weakly singular kernel. Space discretization regularize this kernel, making it smooth. Those two facts are very important in formulation and analysis of mathematical problems described in this article.

Our main aim is to compare the accuracy and solution costs of numerical schemes developed for the solution of parabolic PDEs and integral equations. In order to make this comparison honestly justified we restricted to the analysis of basic classical discretization algorithms fitted for both classes of problems.

The cited papers include problems defined in spaces of $n = 2, 3$ dimensions. We have restricted to Navier–Stokes problems defined in $n = 2$ dimensions, since the reduction of 3D problems to integral Volterra equations lead to kernels of the equations with very similar properties. The complexity of the discrete scheme implementation for the parabolic PDEs depend on the dimension of the space, but in the case of 3D problems we can use efficient ADI schemes.

It is well known that inverse problems, including the first-kind integral Volterra equations, can be ill-posed [9, 15].

In definitions of these Navier–Stokes problems some nonclassical additional local and nonlocal conditions, boundary conditions can be formulated. The existence and uniqueness of the solutions, stability of discrete approximation methods are very sensitive to influence of such conditions. Often they require development and analysis of special discrete approximation techniques. For a case of smooth data the first-kind integral Volterra equations can be reduced to the second-kind integral equations which are well-posed, but in the case of non-smooth coefficients such a transformation can't be done. In real world applications the coefficients are perturbed by measured noise pollution. Thus our second goal was to compare the efficiency of various regularization techniques, including specific methods for the integral equations and general variational Tikhonov type methods [9].

A very interesting problem is investigated in [1, 2], where numerical solution of the viscous flows in a network of thin tubes are considered. These equations are defined on the graph and its one-dimensional approximation is proposed. We note, that the aim of this analysis was to develop efficient algorithms to discretize the obtained weakly singular kernels and to compare the accuracy of the new scheme to the direct numerical solution of the full 3D Navier–Stokes mathematical model. The influence of noise perturbations was not considered in these cited papers.

We note that in general numerical modeling of thin tubes has a long history It is sufficient to mention applications/modelling of blood flow in arteries. A deep and extensive review of these results and different numerical techniques is given e.g. in [12, 16–18].

These problems are not considered in detail in our paper since we have restricted our goal to a more narrow topic of comparing the discrete schemes for two classes of ill-posed problems, i.e. the inverse PDEs and integral first-kind Volterra equations. Still the indicated applications and validation of mathematical models for the given medical applications will be considered in the second part of this project.

We also note that the implicit Euler scheme and a midpoint scheme are used in our paper as typical standard methods for the approximation of inverse parabolic PDEs and integral Volterra equations. Both these schemes (the implicit Euler and a Galerkin type scheme) are also used in [1] to simulate the viscous flows in a network of thin tubes, when equations are specified on the graph. The main goal of [1,2] is to propose accurate approximations of weakly singular kernels in the obtained set of integral equations on the graph and to give some convergence bounds. Our aim is to compare different approximation schemes with respect to errors introduced by noise pollution of data. We agree that more accurate approximations for kernels of Volterra ill-posed equations can be useful in order to regularize the stability of constructed discrete schemes. We plan to test this approach in future papers also put a note on this question in the revised version of the recent paper.

In this paper we have developed and analyzed finite-volume discrete schemes to solve the inverse parabolic problems with unknown source function which defines a pressure of Poiseuille problem solution and an additional linear integral flux condition to specify this function. The numerical experiment was aimed at obtaining new information about the stability of discrete schemes for these inverse problems. Second, we have developed and solved numerically the first-kind integral Volterra equation which defines an equivalent mathematical model for the Poiseuille problem. The stability, efficiency (CPU time) and accuracy of the proposed discrete solvers is analyzed and compared in the cases when test solutions are smooth and non-smooth functions.

Next, the ill-posedness of the Poiseuille problem is investigated by adding the noise perturbations for specified flux functions. The regularization of discrete schemes is mainly done for the Volterra integral equation. Results of numerical experiments by using simple transformations to the second-kind Volterra problems [9, 15] and the regularization by using the classical variational Tikhonov method are presented and analyzed.

The definition of variational Tikhonov method algorithm is also presented for the inverse parabolic problem.

The rest of the paper is organized in the following way. In Section 2 the problem is formulated. For a general Navier–Stokes problem we are interested to define the non-stationary Poiseuille type solution. Two different approaches are used. First, the inverse parabolic PDE problem is formulated and in addition the flow rate (flux) is defined. The discretization of space operators is done. Second, this problem is reduced to the integral first-kind Volterra equation.

In Section 3 the fully discrete schemes are constructed for the inverse PDE model. The approximation accuracy and stability of these schemes for solution of the direct parabolic problems are investigated. A connection of these discrete algorithms with specific discrete schemes for the Volterra integral equation is also analyzed. The stability of the inverse parabolic problem is studied only by making the computational experiments.

In Section 4 the popular discrete schemes are used to approximate the given first kind Volterra equation. The approximation accuracy and stability of these algorithms are investigated.

Results of computational experiments are presented and compared in Section 5. In the first test problem the smooth flux function is used and a non-smooth flux function is defined in the second test problem. For all algorithms experiments are done for data without noise and for two different noise levels. The noise is generated according a uniform random number distribution. For noisy data application of some popular efficient regularization methods is considered. Some final conclusions are done in Section 6.

2 Problem formulation

Let us consider an initial-boundary value problem for the Navier–Stokes equations

$$\begin{aligned}
&\frac{\partial \mathbf{u}}{\partial t} - \nu \sum_{j=1}^{2} \frac{\partial^2 \mathbf{u}}{\partial x_j^2} + \big(\mathbf{u}(x,t)\cdot\nabla\big)\mathbf{u}(x,t) + \nabla p(x,t) = 0, \qquad (1)\\
&\nabla\cdot\mathbf{u}(x,t) = 0,\\
&\mathbf{u}(x,t)|_{\Pi} = 0, \qquad \mathbf{u}(x,0) = \mathbf{u_0}(x).
\end{aligned}$$

In (1), $\mathbf{u}$ is the fluid velocity, p is the pressure function and $\nu > 0$ is the constant kinematic viscosity of the fluid. It describes the flow of an incompressible fluid in an infinite straight pipe

$$\Pi = \big\{x = (x_1, x_2) \in \mathbb{R}^2\colon\ 0 \leqslant x_1 \leqslant 1,\ -\infty < x_2 < \infty\big\}.$$

We are interested to define the Poiseuille type solution in the nonstationary case. Assume that the initial data $\mathbf{u_0}(x)$ has only the one component:

$$\mathbf{u_0}(x) = \big(0, u_{02}(x_1)\big).$$

Then we look for the solution of problem (1) in the form [8]

$$\mathbf{u}(x,t) = \big(0, U_2(x_1,t)\big), \qquad p(x,t) = -q(t)x_2 + p_0(t), \qquad (2)$$

where $p_0(t)$ is an arbitrary function. Next, we substitute (2) into (1) and get for functions $U(x_1,t) := U_2(x_1,t)$ and $q(t)$ the following initial-boundary value problem

$$\begin{aligned}
&\frac{\partial U}{\partial t} - \nu\frac{\partial^2 U}{\partial x_1^2} = -q(t), \qquad (3)\\
&U(0,t) = 0, \qquad U(1,t) = 0,\\
&U(x_1,0) = u_{02}(x_1), \quad 0 \leqslant x_1 \leqslant 1.
\end{aligned}$$

In addition the flow rate (flux) is defined

$$\int_0^1 U(x_1,t)\,\mathrm{d}x_1 = F(t). \qquad (4)$$

Thus we solve an inverse problem: for given initial condition u_{02} and flow rate $F(t)$ we must find a pair of functions $(U(x_1,t), q(t))$ solving the parabolic problem (3) and satisfying the flux condition (4).

Now we restrict to the analysis of discrete in space operators. Thus a uniform discrete grid in space is defined

$$\begin{aligned}
&\omega_h = \{x_{1j}\colon x_{1j} = jh,\ j = 1,\dots,J-1\}, \quad x_{1J} = 1,\\
&\bar{\omega}_h = \omega_h \cup \{0,1\}.
\end{aligned}$$

Semi-discrete functions $V(x_1,t) = (v_0, v_1, \dots, v_J)$ are approximations of $U(x_1,t)$, where $v_j = V(x_{1j},t)$, $j = 0,\dots,J$.

For any V, such that $v_0 = 0$, $v_J = 0$ we define the discrete diffusion operator

$$A_h V = -\frac{v_{j+1} - 2v_j + v_{j-1}}{h^2}, \quad j = 1,\dots,J-1.$$

The eigenvectors $V_k = (v_{k1},\dots,v_{k,J-1})$ and eigenvalues λ_k of A_h are well known [7]:

$$\begin{aligned}
&A_h V_k = \lambda_k V_k, \quad k = 1,\dots,J-1,\\
&v_{kj} = \sqrt{2}\sin(\pi k x_j), \qquad \lambda_k = \frac{4}{h^2}\sin\left(\frac{h}{2}\pi k\right).
\end{aligned}$$

Let us define a scalar product and the L_2 norm for the discrete functions

$$(V,W) = \sum_{j=1}^{J-1} v_j w_j h, \qquad \|V\| = (V,V)^{1/2}.$$

The set of eigenvectors $\{V_k\}$, $k = 1,\dots,J-1$, make an orthonormal and complete basis.

The semi-discrete approximation of problem (3)-(4) is given by

$$\begin{aligned}
&\frac{\partial V}{\partial t} + \nu A_h V = -q(t), \qquad (5)\\
&v_0(t) = 0, \qquad v_J(t) = 0,\\
&v(x_j,0) = u_{02}(x_j), \ \ j = 0,\dots,J,
\end{aligned}$$

and the discrete flux condition is defined as

$$\big(V(t),1\big) := \sum_{j=1}^{J-1} v_j h = F(t). \qquad (6)$$

We got the one dimensional semi-discrete nonstationary parabolic PDE problem (5). Since it is an inverse problem, we are interested to analyze the conditioning of this formulation and it is well known, that inverse problems can be ill-posed. In the case of ill-posed problems the main aim is to apply the regularization techniques in order to construct robust discrete schemes.

Next, we define the second approach, when the given problem is reformulated as an integral first-kind Volterra equation. In fact, our main aim is to compare the accuracy and efficiency of numerical schemes constructed following both approaches: when the inverse PDE and the integral Volterra equation are solved. The efficiency of algorithms was mainly estimated by comparing CPU times required to solve the given problems.

Let us consider the spectral representation of the solution

$$v_j = \sum_{k=1}^{J-1} w_k(t) v_{kj}, \quad j = 1, \dots, J-1. \tag{7}$$

Next, we derive a spectral representation of the identity function I

$$I(x_j) = \sum_{k=1}^{J-1} \beta_k v_{kj}, \quad \beta_k = \sum_{j=1}^{J-1} v_{kj} h,$$

where

$$I(x_j) := \begin{cases} 1, & 1 \leqslant j \leqslant J-1, \\ 0, & j = \{0, J\}. \end{cases}$$

It is sufficient to consider the case of homogeneous initial condition $u_{02} = 0$. Substituting solution (7) into equation (5) we get linear ODEs for coefficients w_k, $k = 1, \dots, J-1$:

$$\frac{\mathrm{d}w_k(t)}{\mathrm{d}t} + \nu\lambda_k w_k(t) = \beta_k q(t), \qquad w_k(0) = 0. \tag{8}$$

The solution of this equation can be written in an explicit form

$$w_k(t) = \beta_k \int_0^t \exp\bigl(-\nu\lambda_k(t-s)\bigr) q(s)\, \mathrm{d}s.$$

Substituting it into the spectral representation of the discrete flow condition (6), which is written as

$$\sum_{k=1}^{J-1} w_k(t) \left(\sum_{j=1}^{J-1} v_{kj} h \right) = F(t),$$

we get the integral first-kind Volterra equation for unknown function $q(t)$:

$$\int_0^t \left(\sum_{k=1}^{J-1} \beta_k^2 \exp\bigl(-\nu\lambda_k(t-s)\bigr) \right) q(s)\, \mathrm{d}s = F(t). \tag{9}$$

It can be written in a compact form as

$$\int_0^t K(t,s) q(s)\, \mathrm{d}s = F(t),$$

where the function $K(t,s)$ in the integral is called the kernel.

It is well known that in many cases the first-kind Volterra equations define the ill-posed mathematical problems [9]. Application of regularization methods is required in order to solve such problems efficiently and to achieve a maximum possible accuracy for a noisy data.

The main aim of this paper is to compare the accuracy of numerical schemes for solution of the inverse parabolic problem (5) and the integral first-kind Volterra equation (9). The comparison is done for similar sizes of discrete problems and different algorithms.

3 Discretization in time of problem (5)

In this section we construct fully discrete approximations of the given semi-discrete parabolic problem (5). Again we restrict to the uniform discrete grid in time

$$\omega_\tau = \{t^n\colon\ t^n = n\tau,\ n = 0, \dots, N\}, \quad t^N = T.$$

Vectors $V^n = (v_1^n, v_1^n, \dots, v_{J-1}^n)$ and $\bar{V}^n = (v_0^n, v_1^n, \dots, v_J^n)$ are approximations of the semi-discrete solution $V(t^n)$ at $t = t^n$.

Let us consider a family of discrete schemes

$$\frac{V^{n+1} - V^n}{\tau} + A_h V^{n+\sigma} = q^{n+\sigma}, \tag{10}$$

$$v_0^n = 0, \qquad v_J^n = 0, \tag{11}$$

where $0 \leqslant \sigma \leqslant 1$ is the weight parameter and

$$V^{n+\sigma} = \sigma V^{n+1} + (1 - \sigma) V^n.$$

We are interested in two schemes: the Backward Euler (BE) scheme for $\sigma = 1$ and Crank–Nicolson (CN) method for $\sigma = 0.5$.

The additional flux condition is defined at t^{n+1} as

$$\sum_{j=1}^{J-1} V_j^{n+1} h = F^{n+1}. \tag{12}$$

The system of equations (10)–(12) can be solved very efficiently. For simplicity, we restrict the description of the numerical algorithm only to the CN scheme. This scheme can be written as

$$\frac{V^{n+1/2} - V^n}{0.5\tau} + A_h V^{n+1/2} = q^{n+1/2}.$$

Next, we compute an auxiliary function W which solves the following discrete problem:

$$\frac{W}{0.5\tau} + A_h W = 1, \quad 1 \leqslant j < J,$$

$$w_0 = 0, \qquad w_J = 0.$$

Then the solution $V^{n+1/2}$ is computed as

$$V^{n+1/2} = W^{n+1/2} + q^{n+1/2} W,$$

where the nonstationary discrete function $W^{n+1/2}$ takes into account the dynamics of the solution from the previous time level

$$\begin{aligned} &\frac{W^{n+1/2} - V^n}{0.5\tau} + A_h W^{n+1/2} = 0, \quad 1 \leqslant j < J, \\ &w_0^{n+1/2} = 0, \qquad w_J^{n+1/2} = 0. \end{aligned}$$

The value of the source function $q^{n+1/2}$ is computed from the flow condition

$$q^{n+1/2} = \frac{\frac{1}{2}(F^{n+1} + F^n) - \sum_{j=1}^{J-1} W_j^{n+1/2} h}{\sum_{j=1}^{J-1} W_j h}.$$

The existence and stability of the solution of the semi-discrete problem (5) is done by the authors of referenced papers. Thus for the goals of this work it is sufficient to use numerical schemes which approximate the non-homogeneous parabolic problem by preserving the stability of the solution. For the discrete integral Volterra equations the stability of the discrete solutions follow from the general theory.

It is well known that the constructed family of schemes define unconditionally stable methods if [7]

$$\sigma \geqslant \frac{1}{2} - \frac{h^2}{4\tau}. \tag{13}$$

It follows from (13) that the Explicit Euler (EE) scheme for $\sigma = 0$ is stable if the condition

$$\tau \leqslant \frac{h^2}{2}$$

is valid and BE and CN schemes are unconditionally stable.

The discrete scheme (10) also can be interpreted as a specific discretization of the Volterra equation (9). Let us consider the spectral representation of the discrete solution

$$v_j^n = \sum_{k=1}^{J-1} w_k^n v_k(x_j).$$

Then we get from (10) the discrete approximation of equation (8)

$$\frac{w_k^{n+1} - w_k^n}{\tau} + \nu\lambda_k w_k^{n+\sigma} = \beta_k q^{n+\sigma}, \quad k = 1, \dots, J-1.$$

We restrict to the BE scheme (10), when $\sigma = 1$. Substituting the function

$$w_k^{n+1} = \frac{1}{1 + \tau\nu\lambda_k}\left(1 + \tau\beta_k q^{n+1}\right)$$

into the conjugation equation (12) gives

$$\sum_{k=1}^{J-1} \frac{\beta_k}{1+\tau\nu\lambda_k}\bigl(1+\tau\beta_k q^{n+1}\bigr) = F^{n+1}.$$

Then we compute the solution q^{n+1} of the inverse PDE problem

$$\tau q^{n+1} \sum_{k=1}^{J-1} \frac{\beta_k^2}{1+\tau\nu\lambda_k} = F^{n+1} - \sum_{k=1}^{J-1} \beta_k \frac{1-\tau\nu\lambda_k}{1+\tau\nu\lambda_k}\, w_k^n.$$

Next, we can write the obtained solution in the form which mimics the solution of the integral Volterra equation. The algorithm is presented in an iterative way. The initial condition is given as

$$w_k^0 = 0, \quad k = 1, \dots, J-1.$$

The first iteration defines a solution at $t = \tau$

$$w_k^1 = w_k^0 + \beta_k \frac{\tau}{1+\tau\nu\lambda_k}\, q^1 = \beta_k \frac{\tau}{1+\tau\nu\lambda_k}\, q^1, \quad k = 1, \dots, J-1.$$

After n iterations we get the following equality:

$$w_k^n = \beta_k \sum_{l=1}^{n} \frac{\tau}{(1+\tau\nu\lambda_k)^{n+1-l}}\, q^l, \quad k = 1, \dots, J-1.$$

Substituting these equations into (9) we get a nonstandard discrete approximation of the given first-kind Volterra equation

$$\sum_{l=1}^{n} \sum_{k=1}^{J-1} \beta_k^2 \frac{\tau}{(1+\tau\nu\lambda_k)^{n+1-l}}\, q^l = F^n.$$

4 Direct discretization of Volterra equation (9)

In this section we present classical approximations of the first-kind Volterra equations. Our aim is to compare these numerical methods with special approximations obtained using discrete approximations of parabolic problem.

We present the numerical approximations for integral equations written in a compact form

$$\int_0^t K(t,s)\, q(s)\, \mathrm{d}s = F(t), \tag{14}$$

where the kernel K is defined as

$$K(t,s) = \sum_{k=1}^{J-1} \beta_k^2 \exp\bigl(-\nu\lambda_k(t-s)\bigr).$$

In fact this linear Volterra integral equation is a convolution equation but we consider numerical algorithms targeted for general kernels.

Let us assume that some standard requirements are satisfied for coefficients of (14):

- kernel $K(t,t)$ is nonsingular, i.e. $K(t,t) > 0$,
- functions $K(t,s)$ and $F(t)$ are sufficiently smooth and bounded, i.e. all derivatives of them, which are used in the theoretical analysis exist.

Then it is well known that the first-kind integral Volterra equation (14) has a unique and continuous solution.

Next, we define three popular approximation methods used in applications [3, 9].

The rectangular method. The discrete approximation of the integral Volterra equation (14) is defined as

$$\tau \sum_{l=0}^{n-1} K\big(t^n, t^l\big) q^l = F(t^n), \quad n = 1, \dots, N, \tag{15}$$

where the discrete solution q^n is defined on the discrete time mesh ω_τ.

Let us assume that for sufficiently small time steps τ the estimate $|K(t^n, t^{n-1})| > 0$ is valid, then the discrete solution is computed as

$$\tau K\big(t^n, t^{n-1}\big) q^{n-1} = F\big(t^n\big) - \tau \sum_{l=0}^{n-2} K\big(t^n, t^l\big) q^l, \quad n = 1, \dots, N.$$

It follows that the initial value q^0 of the discrete solution is also computed by using the basic rectangular method algorithm

$$\tau K\big(t^1, t^0\big) q^0 = F\big(t^1\big).$$

A discrete method is said to be of order p if there exists a finite constant C such that

$$\max_{0 \leqslant n \leqslant N} \big|q^n - q\big(t^n\big)\big| \leqslant C\tau^p.$$

A standard convergence analysis show that the accuracy of approximation (15) is of order $p = 1$.

The classical stability analysis takes into account the stability with respect to approximation errors when all data is smooth. We are interested to investigate the possible ill-posedness of the given problem (14) with respect to noise perturbations of function F.

Let us assume that the source term is defined as

$$\widetilde{F}(t) = F(t) + \varepsilon(t),$$

where $\varepsilon(t^n)$ values are generated according a uniform random numbers distribution.

Let us denote the global error $e^n = q(t^n) - q^n$. The applied stability analysis is based on three general techniques.

First, the discrete problem for the global error e^n is derived. It depends on the classical truncation error.

Second, the obtained equation for e^n is transformed to the discrete second-kind Volterra equation. This step mimics the well-known technique used for continuous Volterra equations (14). Assuming that the kernel $K(t,s)$ and source function $F(t)$ are smooth and $K(t,t) \geqslant \nu > 0$ we may differentiate this integral equation with respect to t to obtain [9]

$$K(t,t)q(t) + \int_0^t \frac{\partial}{\partial t} K(t,s)q(s)\,\mathrm{d}s = F'(t).$$

It is known that a Volterra equation of the second kind is a well-posed problem.

Third, using the discrete Gronwall's inequality [3] the global error estimate in a uniform norm is proved.

We will apply this analysis for the rectangular method. First, by substituting $q^j = q(t^j) - e^n$ into the discrete equation (15) we get the problem

$$\tau \sum_{j=0}^{n-1} K\big(t^n, t^j\big)e^j = \psi^n - \varepsilon_n, \quad n = 1,\dots,N, \tag{16}$$

where ψ^j defines the truncation error

$$\psi^n = \tau \sum_{j=0}^{n-1} K\big(t^n, t^j\big)q^j - F(t^n).$$

Next, we subtract equation (16) from a similar equation for t^{n+1} and divide by τ, we get

$$K\big(t^{n+1}, t^n\big)e^n + \tau \sum_{j=0}^{n-1} \left(\frac{K(t^{n+1}, t^j) - K(t^n, t^j)}{\tau}\right) e^j = \frac{\psi^{n+1} - \psi^n}{\tau} - \frac{\varepsilon^{n+1} - \varepsilon^n}{\tau}.$$

In order to keep the main idea of the stability proof for a general cases, we also assume that the following estimate

$$\left|\frac{K(t^{n+1}, t^j) - K(t^n, t^j)}{\tau}\right| \leqslant C_1, \quad j = 0,\dots,n-1,$$

is valid. This assumption must tested for any specific applied problem and given kernels of discrete operators.

Then, assuming that $K^{-1}(t^{n+1}, t^n) < C$, all data is smooth and the truncation error is of order $O(\tau)$ the following stability estimate is obtained

$$\big|e^n\big| \leqslant \tau C_1 \sum_{j=0}^{n-1} \big|e^j\big| + C_2\tau + \frac{1}{2}C_3 \max_{1\leqslant j\leqslant n} \frac{|\varepsilon(t^j) - \varepsilon(t^{j-1})|}{\tau}.$$

Since $|\varepsilon(t^j)| \leqslant 1$, then the estimate

$$\left|e^n\right| \leqslant \tau C_1 \sum_{j=0}^{n-1} \left|e^j\right| + C_2\tau + C_3\frac{1}{\tau}$$

is valid, where C_1, C_2 and C_3 are bounded constants. Assuming that $e^0 = C_4\tau$ and using the discrete Gronwall's inequality we prove the following error estimate:

$$\left|e^n\right| \leqslant \exp\bigl(C_1 t^n\bigr)\left(C_2\tau + C_4\tau^2 + \frac{C_3}{\tau}\right).$$

Thus, for the case of no noise pollution $C_3 = 0$ in data the accuracy of approximation is of order $p = 1$, but if the noise error is included into the formulation of the integral equation, then the ill-posedness of the discrete rectangular method should be expected.

The midpoint method. The discrete approximation of the Volterra integral equation (14) is defined as

$$\tau \sum_{l=0}^{n-1} K\bigl(t^n, t^{l+1/2}\bigr) q^{l+1/2} = F\bigl(t^n\bigr), \quad n = 1, \dots, N. \tag{17}$$

We see again that there is no need to specify separately the initial condition.

The accuracy of this algorithm can be investigated similarly to the analysis of the rectangular method. The error e^n satisfies the stability estimate

$$\left|e^{n-1/2}\right| \leqslant \tau C_1 \sum_{j=0}^{n-2} \left|e^{j+1/2}\right| + C_2\tau^2 + \frac{1}{2}C_3 \max_{1\leqslant j\leqslant n} \frac{|\varepsilon(t^j) - \varepsilon(t^{j-1})|}{\tau},$$

where the second term $C_2\tau^2$ defines the truncation error and the third term includes the influence of noise. Thus the truncation error of the midpoint method (17) is of order $p = 2$. Taking into account the estimate of $e^{1/2} = C_4\tau^2$ and using the discrete Gronwall's inequality we prove the following error estimate

$$\left|e^{n-1/2}\right| \leqslant \exp\bigl(C_1 t^{n-1/2}\bigr)\left(C_2\tau^2 + C_4\tau^3 + \frac{C_3}{\tau}\right).$$

If noise pollution $C_3 = 0$ is not included into measurements of F then the accuracy of approximate solution is of order $p = 2$, but if the noise error is included into the formulation of the integral equation, then the ill-posedness of the discrete rectangular method is expected.

For discontinuous initial conditions the Crank–Nicolson time scheme can result in oscillations which are are only weakly damped by the CN scheme. In this case a few regularization steps are needed to reduce these oscillations quite fast in order to keep the second order of convergence of the symmetric scheme. Such techniques are described in detail in [10, 13].

Still in our computational experiments the jump discontinuities are avoided in the initial conditions and such an additional regularization was not needed. This theoretical and experimental analysis can make an interesting part of future simulations of more general initial and boundary conditions.

The trapezoidal method. The discrete approximation of the Volterra integral equation (14) is defined as

$$\frac{\tau}{2}K\bigl(t^n,t^0\bigr)q^0+\tau\sum_{l=1}^{n-1}K\bigl(t^n,t^l\bigr)q^l+\frac{\tau}{2}K\bigl(t^n,t^n\bigr)q^n=F\bigl(t^n\bigr),\quad n=1,\dots,N. \tag{18}$$

The initial condition q^0 is computed exactly by [9]

$$q^0=\frac{\frac{\mathrm{d}F}{\mathrm{d}t}(0)}{K(0,0)}.$$

We restrict to the evaluation of the truncation error. The analysis based on Taylor's series proves that this error is of order two. The remaining convergence conclusions are similar to the case of the midpoint method.

5 Computational experiments

Consider problem (3)–(4) in the domain $[0,1]$. First, we solve this inverse problem till $T=1$ with given zero initial, boundary conditions and smooth flux (flow rate) function

$$\nu=0.5,\qquad u_{02}(x_1)=0,\qquad F(t)=10\bigl(\mathrm{e}^t-1\bigr).$$

All numerical tests have been performed on the computer with Intel®Xeon® processor.

First, we want to compare the accuracy of discrete methods developed for the solution of inverse parabolic problem (5)–(6) with the accuracy of methods developed for the first-kind integral Volterra equation (14). Second, the influence of noise perturbations also is tested in these series of computations. No special regularization algorithms are applied in this part of computations.

5.1 The accuracy of time integration of BE and CN schemes

First, we have tested the time integration accuracy of the BE solver (10)–(12), $\sigma=1$ for the discrete inverse parabolic PDE model (5)–(6).

A uniform space grid ω_h with $J=40$ is used. Table 1 gives for a sequence of decreasing time step widths τ the errors $e(\tau)$ and the experimental convergence rates $\rho(\tau)$ of the discrete solution for BE finite volume scheme:

$$e(\tau)=\bigl|q^N-q(T)\bigr|,\qquad \rho(\tau)=\log_2\frac{e(2\tau)}{e(\tau)},$$

where the reference solution $V(T),q(T)$ is computed by using the very small time step $\tau=10^{-5}$ and the second-order accurate CN scheme. Then the approximation error $e(\tau)$ of the BE scheme can be measured accurately.

Table 1. Errors $e(\tau)$ and experimental convergence rates $\rho(\tau)$ for the discrete solution of BE scheme (10), $\sigma = 1$ for a sequence of time steps $\tau = T/N$, without noise and two different noise levels $\varepsilon(t)$ for flux $F(t)$.

N	$e_0(\tau)$ $\delta_0 = 0$	$\rho(\tau)$	$e_1(\tau)$ $\delta_1 = 0.001$	$e_2(\tau)$ $\delta_2 = 0.0001$
25	0.6416646	—	0.6835583	0.6458539
50	0.3229603	0.99046	0.4189708	0.3325613
100	0.1620159	0.99522	0.3067941	0.1764937
200	0.0811422	0.99761	0.4157049	0.1145985
400	0.0406045	0.99881	0.6959599	0.1061401
800	0.0203104	0.99942	1.3141067	0.1275119
1600	0.0101571	0.99974	3.0556232	0.2964209

Table 2. Errors $e(\tau)$ and experimental convergence rates $\rho(\tau)$ for the discrete solution of CN scheme (10) $\sigma = 0.5$ for a sequence of time steps $\tau = T/N$, without noise and two different noise levels $\varepsilon(t)$ for flux $F(t)$.

N	$e_0(\tau)$ $\delta_0 = 0$	$\rho(\tau)$	$e_1(\tau)$ $\delta_1 = 0.001$	$e_2(\tau)$ $\delta_2 = 0.0001$
10	0.4505788	—	0.5099376	0.4565147
20	0.1199393	1.90947	0.2618352	0.1341289
40	0.0313501	1.93576	0.2380837	0.0520234
80	0.0079756	1.97481	0.4737014	0.0401921
160	0.0020053	1.99192	0.8205092	0.0802461
320	0.0005024	1.99677	1.8073408	0.1811863

Table 1 shows errors of the discrete solution when the flux F is defined without noise $F(t)$ as well with a noise perturbations $F(t) + \varepsilon_j(t)$, for two different noise levels $\varepsilon_1(t^n) = \delta_1 R(t^n)$ and $\varepsilon_2(t^n) = \delta_2 R(t^n)$, where $\delta_1 = 0.001$ and $\delta_2 = 0.0001$. Here $R(t)$ generates random numbers in the interval $[0, 1]$ according a uniform random number distribution. For each case of noise level parameters twenty calculations are done with different random noises and the worst error value is presented.

It follows from the presented results, that the convergence rate of the discrete solution agree well with theoretical estimate of accuracy order $p = 1$.

The second important conclusion states that the given inverse problem is ill-posed and the errors increase when the discrete time steps τ are decreased, i.e. the numbers of time steps N are increased.

Next, we did similar computation experiments by using the discrete CN scheme (10). The results are presented in Table 2.

It follows from the presented results, that the experimental convergence rate is equal to the second order and this conclusion agrees well with the theoretical predictions.

The second conclusion is that the CN scheme defines an ill-posed problem, this property is similar to the BE scheme properties.

Next, we test the accuracy of discrete CN scheme (10) when the flux function F is not smooth (non-differentiable)

$$F(t) = \begin{cases} 10(\exp(t) - 1), & 0 \leqslant t \leqslant 0.5, \\ 10(\exp(2(t - 0.25)) + 2), & 0.5 < t \leqslant 1. \end{cases} \tag{19}$$

Table 3. Errors $e(\tau)$and experimental convergence rates $\rho(\tau)$ for the discrete solution of the CN method (10) $\sigma = 0.5$ for a sequence of time steps $\tau = T/N$, without noise and two different noise levels $\varepsilon(t)$. The non-smooth flux $F(t)$ is defined in (19).

N	$e_0(\tau)$ $\delta_0 = 0$	$\rho(\tau)$	$e_1(\tau)$ $\delta_1 = 0.001$	$e_2(\tau)$ $\delta_2 = 0.0001$
40	0.12150671	—	0.3377168	0.1431277
80	0.05369016	1.17830	0.4279869	0.0792428
160	0.01590153	1.75548	0.8066129	0.0877509
320	0.00399501	1.99289	1.8108334	0.1846789
640	0.00099849	2.00037	3.6123721	0.3621359

Table 3 shows the results when the flux is defined without noise $F(t)$ as well with noise perturbations $F(t) + \varepsilon_j(t)$, for two different noise levels $\varepsilon_1(t^n) = \delta_1 R(t^n)$ and $\varepsilon_2(t^n) = \delta_2 R(t^n)$, where $\delta_1 = 0.001$ and $\delta_2 = 0.0001$. For each parameter case twenty calculations are done with different sets of random noises and the worst error value is presented.

It follows from the presented results, that convergence rate is smaller than the second order for time meshes with $N \leqslant 150$. Such a drop of the accuracy is expected due to the non-smooth flux function $F(t)$ (19). The asymptotic second-order accuracy in time is achieved for finer discrete time meshes. This behavior of the discrete solution is expected from the presented theoretical results.

Our second conclusion states that the ill-posedness of the given differential problem is clearly seen for the CN scheme (10) in the case of non-smooth data functions.

5.2 Discrete approximations of the Volterra equation

First, we consider results of numerical experiments, when the rectangular method (15) is used to approximate the integral equation (9). The smooth flux function F is selected in this part of computations. The results of numerical experiments are presented in Table 4.

It follows that the convergence order is equal to one, but the error of the discrete solution is twenty times larger than the error of the BE scheme (10) with $\sigma = 1$, which also has the first order of accuracy. As expected for the first-kind integral Volterra equations

Table 4. Errors $e(\tau)$ and experimental convergence rates $\rho(\tau)$ for the discrete solution of the rectangular method (15) for a sequence of time steps $\tau = T/N$, without noise and two different noise levels $\varepsilon(t)$ for the smooth flux $F(t)$.

N	$e_0(\tau)$ $\delta_0 = 0$	$\rho(\tau)$	$e_1(\tau)$ $\delta_1 = 0.001$	$e_2(\tau)$ $\delta_2 = 0.0001$
25	20.358405	—	20.471404	20.369705
50	10.053094	1.01798	10.273131	10.075098
100	5.0283119	0.99949	5.4285812	5.0683389
200	2.5255935	0.99345	3.3774841	2.6107825
400	1.2689509	0.99298	3.1515991	1.4572158
800	0.6367833	0.99476	3.2247893	0.8840511
1600	0.3190841	0.99686	7.5110672	1.0382824

Table 5. Errors $e(\tau)$ and experimental convergence rates $\rho(\tau)$ for the discrete solution of the trapezoidal method (18) for a sequence of time steps $\tau = T/N$, without noise and two different noise levels $\varepsilon(t)$ for the smooth flux $F(t)$. The errors $\widetilde{e}(\tau)$ for the discrete solution of the midpoint method (17) are also given.

N	$e_0(\tau)$ $\delta_0 = 0$	$\rho(\tau)$	$e_1(\tau)$ $\delta_1 = 0.001$	$e_2(\tau)$ $\delta_2 = 0.0001$	$\widetilde{e}_0(\tau)$
10	9.9763765	—	10.023967	9.9811356	3.0586515
20	3.4529521	1.530685	3.5598865	3.4636456	1.0807689
40	1.1367817	1.602875	1.3014422	1.1532478	0.3798273
80	0.3602494	1.657887	0.9237413	0.4165986	0.1311862
160	0.1092939	1.720783	1.7748138	0.2758459	0.0437980
320	0.0313633	1.801059	3.2864789	0.3004209	0.0137335
640	0.0084609	1.890194	8.2681789	0.8344327	0.0039249

Table 6. Errors $e(\tau)$ and experimental convergence rates $\rho(\tau)$ for the discrete solution of the midpoint method (17) for a sequence of time steps $\tau = T/N$, without noise and two different noise levels $\varepsilon(t)$ for the non-smooth flux $F(t)$.

N	$e_0(\tau)$ $\delta_0 = 0$	$\rho(\tau)$	$e_1(\tau)$ $\delta_1 = 0.001$	$e_2(\tau)$ $\delta_2 = 0.0001$
40	1.2551493	—	1.3655095	1.2661852
80	0.4469026	1.48983	0.6893151	0.4711438
160	0.1522972	1.55307	0.4059690	0.1776643
320	0.0483563	1.65511	0.9920110	0.1427218
640	0.0139065	1.79795	1.8561645	0.1981323

the error of discrete solutions is sensitive to the noise perturbations of data and again the error of the discrete solution is essentially larger than the error of discrete solutions computed by using the BE discrete scheme (10).

Next, we present results of numerical experiments when the midpoint method (17) and the trapezoidal method (18) are used to approximate the integral equation (9). The smooth flux function F is selected in this part of computations. The results of numerical experiments are presented in Table 5.

The results of these computational experiments confirm conclusions of the theoretical analysis that the accuracy order of both schemes increases from 1.52 till 1.9. Still it follows from the given results that the error of the midpoint scheme solution is three times smaller.

It interesting to note that global errors of CN scheme (10), $\sigma = 0.5$ for solution of the inverse parabolic problem are from ten till fifteen times smaller than errors of the midpoint method (17).

The dynamics of the global errors when data is perturbed with noise show that the given first-kind discrete Volterra equation is ill-posed and the influence of noise increases when the number of discrete points N is increased.

For high-order methods it is interesting to solve the discrete problem with non-smooth flux F function. In Table 6 the results of computational experiments obtained with the midpoint method(17) are presented.

5.3 Regularization of discrete schemes

First, we consider the most simple regularization technique, which is based on a singular perturbation approach [9]. In order to stabilize an ill-posed problem (14) a term of the form $\alpha q(t)$ is added to the integral operator. Thus we consider a perturbed equation

$$\alpha q(t) + \int_0^t K(t,s)q(s)\,\mathrm{d}s = F(t), \tag{20}$$

which is a well-posed second-kind Volterra equation. It has a unique solution $q(t)$ depending continuously on data F.

We restrict to the analysis of the discrete approximation of the regularized Volterra integral equation (20) by using the midpoint method. The discrete scheme is defined as

$$\alpha q^{n-1/2} + \tau \sum_{l=0}^{n-1} K\bigl(t^n, t^{l+1/2}\bigr) q^{l+1/2} = F^{\varepsilon}(t^n), \quad n = 1, \dots, N, \tag{21}$$

where the flux function F is perturbed by a noise ε:

$$F^{\varepsilon}(t) := F(t) + \varepsilon(t).$$

Errors $e(\tau)$ for the discrete solution of the regularized midpoint method (21) for a sequence of time steps $\tau = T/N$, different regularization parameters $\alpha = \delta$ and noise levels $\varepsilon(t)$ for the smooth flux function F are presented in Table 7. In the last row of the table errors of the discrete solution are presented for the regularized problem with different values of regularization parameter α but without noise perturbations $\varepsilon = 0$.

It is recommended to select $\alpha = \alpha(\delta)$ satisfying $\delta/\alpha(\delta) \to 0$ in order to guarantee the convergence of the discrete solution, but our numerical experiments show that the selection $\alpha = \delta$ give even a better accuracy for this test problem.

The presented results confirm that this simple regularization method leads to robust and efficient computational solvers. Since the formulated discrete first-kind integral problem is ill-posed, some regularization technique is necessary in order to control the influence of noise perturbations in data.

Table 7. Errors $e(\tau)$ for the discrete solution of the regularized midpoint method (21) for a sequence of time steps $\tau = T/N$, different regularization parameters $\alpha = \delta$ and noise levels $\varepsilon(t)$ for the smooth flux function F.

N	$e_0(\tau)$ $\delta_0 = 0.01$	$e_1(\tau)$ $\delta_1 = 0.0025$	$e_2(\tau)$ $\delta_2 = 0.000625$	$e_3(\tau)$ $\delta_3 = 0.00015$
40	9.778062	2.3563355	0.3197639	0.2406764
80	10.686678	2.9032137	0.6554001	0.0593598
160	11.177441	3.3694121	0.9078908	0.1935046
320	11.156842	3.5044991	0.9938282	0.2393961
640	11.284466	3.8902388	1.3700367	0.3941284
1280	9.7478361	2.5728394	0.6515743	0.1561563

The method of Tikhonov regularization. For a comparison we apply the general classical variational regularization method, which was proposed by Tikhonov [5]. Perhaps this method is most widely referenced regularization method. We will restrict to a finite-dimensional linear problems framework. Let the linear equation is given

$$A_\tau Q = F^\varepsilon, \quad Q = \{q^0, \dots, q^n\} \in U,$$

on a Hilbert space U, where F^ε is a perturbation of flux function F. The method of Tikhonov regularization defines a regularized solution Q_α solving

$$\min_{Q_\alpha \in U} \|A_\tau Q_\alpha - F^\varepsilon\|^2 + \alpha \|Q_\alpha\|^2, \tag{22}$$

where the norm $\|\cdot\|$ is the classical L_2 norm of a Hilbert space U. The solver is implemented by solving the equivalent normal equation

$$(A_\tau^{\mathrm{T}} A_\tau + \alpha I) Q_\alpha = A_\tau^{\mathrm{T}} F^\varepsilon, \tag{23}$$

where A_τ^{T} is the transposed matrix of A_τ (it defines the adjoint operator associated with A_τ) and I is the identity matrix. We note, that A_τ typically is a lower-triangular matrix and a linear system with this matrix can be solved very efficiently by sequential methods. But $A_\tau^{\mathrm{T}} A_\tau + \alpha I$ is a full matrix and direct solution methods are not efficient.

Since $A_\tau^{\mathrm{T}} A_\tau + \alpha I$ is symmetric and positive definite then in all experiments we solve the linear system (23) using the Conjugate Gradient Method (CGM) which is implemented as an iterative algorithm.

In order to reduce the number of iterations the original system can be replaced with a system with the preconditioned matrix $B^{-1}(A_\tau^{\mathrm{T}} A_\tau + \alpha I)$, where $B = A_\tau^{\mathrm{T}} A_\tau$. The matrix B is defined as a multiplication of two triangular matrices and linear systems with matrix B can be solved very efficiently.

The discrete rectangular method (15) is used in computational experiments. Coefficients of matrix A_τ are defined as

$$a_{nl} = \tau K(t^{n+1}, t^l), \quad \text{if } l \leqslant n,\ 0 \leqslant n,\ l < N.$$

Results of computation experiments are presented in Table 8. The value of the regularization parameter α was selected by using the asymptotical relation $\alpha = O(\delta)$ and the proportionality constant was determined by making few computational experiments.

Table 8. Errors $e(\tau)$ for the discrete solution of the Tikhonov regularization method with rectangular scheme (15) for a sequence of time steps $\tau = T/N$, different regularization parameters α and noise levels δ for the smooth flux function F.

N	$\delta_0 = 0.01$ $\alpha_0 = 0.001$	$\delta_1 = 0.001$ $\alpha_1 = 0.0001$	$\delta_2 = 0.0001$ $\alpha_2 = 0.00001$
100	5.3230868	5.4282109	5.0315919
200	10.9024143	3.3761372	2.6093254
400	19.4906147	3.1472836	1.4510428
800	39.0183178	3.2281995	0.8574195

The number of CGM iterations always was not larger than 50. The obtained results can be compared with results in Table 4 where no regularization techniques were used.

The method of Tikhonov regularization for inverse PDE problem (10). This variational method is very general and it can be applied to solve inverse problem (10). We restrict only to formulation of the general framework of this algorithm. Results of computational experiments will be presented in a separate article. Again the regularization of noise perturbations is obtained solving the variational problem:

$$J_\alpha = \sum_{n=0}^{N-1} \left[\left(V^{n+1/2}, 1 \right) - F^{n+1/2} \right]^2 \tau + \alpha \sum_{n=0}^{N-1} \left(q\left(t^{n+1/2} \right) \right)^2 \tau.$$

One of most popular methods to solve this minimization problem is to use the iterative CGM. Note, that we used this method also to implement the iterative algorithm of the Tikhonov method for the Volterra problem (22).

Consider the template for the algorithmic implementation of CGM [6].

1. Choose an initial guess of Q_0 and calculate the residual of the flux function by solving
$$\frac{V_0^{n+1} - V_0^n}{\tau} + A_h V_s^{n+1/2} = Q_0.$$
Calculate the gradient R_0 of functional J_α by solving the adjoint (backward in time) problem
$$-\frac{\Psi_0^{n+1} - \Psi_0^n}{\tau} + A_h \Psi_0^{n+1/2} = \widetilde{R}_0,$$
where the residual $\widetilde{R}_0^{n+1/2} = \left(V_0^{n+1/2}, 1 \right) - F^{n+1/2}$.

Next, for each iteration $s \geqslant 0$:

2. Solve the direct parabolic problem
$$\frac{V_s^{n+1} - V_s^n}{\tau} + A_h V_s^{n+1/2} = D_s,$$
where D_s is a modified gradient of functional J_α. Update the source function Q_s and $\widetilde{R}_{s+1}$.
3. Calculate the gradient R_{s+1} of the functional J_α by solving the adjoint problem and update D_s.
$$-\frac{\Psi_{s+1}^{n+1} - \Psi_{s+1}^n}{\tau} + A_h \Psi_{s+1}^{n+1/2} = \widetilde{R}_{s+1},$$

6 Conclusions

In this paper we have investigated the stability (well-posedness) of the Poiseuille type solution in the nonstationary case of the Navier–Stokes problem. Two mathematical formulations are considered and a general conclusion is that for non-smooth additional flux function the given problem is ill-posed. Thus some regularization techniques should be used to construct robust and efficient solvers.

It is shown that the accuracy of discrete schemes for solving the inverse PDE model is essentially better than the accuracy for similar approximations of the first-kind Volterra integral model. The comparison is done for fixed sizes of time meshes. An opposite conclusion is done when costs of implementations are compared, the CPU time is essentially smaller for solvers based on discrete approximations of the Volterra equation model.

This fact is very important when the Tikhonov method is applied to stabilize discrete schemes for noisy data and CGM is used to minimize the variational functional J_α. In the case of PDE model two discrete parabolic problems should be solved at each iteration.

Conflicts of interest. The author declares no conflicts of interest.

References

1. E. Canon, F. Chardard, G. Panasenko, O. Štikonienė, Numerical solution of the viscous flows in a network of thin tubes: Equations on the graph, *J. Comput. Phys.*, **435**:110262, 2021, https://doi.org/10.1016/j.jcp.2021.110262.
2. E. Canon, F. Chardard, G. Panasenko, O. Štikonienė, Asymptotic and discretization of a weakly singular kernel: Application to viscous flows in a network of thin tubes, *J. Comput. Phys.*, **491**: 112327, 2023, https://doi.org/10.1016/j.jcp.2023.112327.
3. C.J. Gladwin, On optimal integration methods for Volterra integral equations of the first kind, *Math. Comput.*, **39**(160):511–518, 1982, https://doi.org/10.2307/2007328.
4. D. Glotov, W.E. Hames, A.J. Meir, S. Ngoma, An integral constrained parabolic problem with applications in thermochronology, *Comp. Math. Appl.*, **71**(11):2301–2312, 2016, https://doi.org/10.1016/j.camwa.2016.01.017.
5. G.H. Golub, P.C. Hansen, D.P. Oleary, Tikhonov regularization and total least squares, *SIAM J. Matrix Anal. Appl.*, **21**(1):185–194, 1999, https://doi.org/10.1137/S0895479897326432.
6. D.N. Hao, P.X. Thanh, D. Lesnic, N. Ivanchov, Determination of a source in the heat equation from integral observations, *J. Comput. Appl. Math.*, **264**:82–98, 2014, https://doi.org/10.1016/j.cam.2014.01.005.
7. W. Hundsdorfer, J.G. Verwer, *Numerical Solution of Time-Dependent Advection-Diffusion-Reaction Equations*, Springer Ser. Comput. Math., Vol. 33, Springer, Berlin, Heidelberg, 2003, https://doi.org/10.1007/978-3-662-09017-6.
8. K. Kaulakytė, N. Kozulinas, K. Pileckas, Time-periodic Poiseuille-type solution with minimally regular flow-rate, *Nonlinear Anal. Model. Control*, **26**(5):947–968, 2021, https://doi.org/10.15388/namc.2021.26.24502.
9. P.K. Lamm, A survey of regularization methods for first-kind Volterra equations, in D. Colton, H.W. Engl, A.K. Louis, J.R. McLaughlin, W. Rundell (Eds.), *Surveys on Solution Methods for Inverse Problems*, Springer, Vienna, 2000, pp. 53–82, https://doi.org/10.1007/978-3-7091-6296-5_4.
10. M. Luskin, R. Rannacher, Wendland, On the smoothing property of the Crank-Nicolson scheme, *Appl. Anal.*, **14**(2):117–135, 1982, https://doi.org/10.1080/00036818208839415.

11. J. Merker, A. Matas, Estimation of discontinuous parameters in linear elliptic equations by a regularized inverse problem, *Partial Differ. Equations Appl. Math.*, **5**:100384, 2022, `https://doi.org/10.1016/j.padiff.2022.100384`.
12. M.S. Olufsen, C.S. Peskin, W.Y. Kim, E.M. Pedersen, A. Nadim, J. Larsen, Numerical simulation and experimental validation of blood flow in arteries with structured-tree outflow conditions, *Ann. Biomed. Eng.*, **28**(11):1281–1299, 2000, `https://doi.org/10.1114/1.1326031`.
13. O. Osterby, Five ways of reducing the Crank–Nicolson oscillations, *BIT*, **43**(4):811–822, 2003, `https://doi.org/10.1023/B:BITN.0000009942.00540.94`.
14. K. Pileckas, V. Keblikas, On the existence of nonstationary Poiseulle solution, *Sib. Math. J.*, **46**(3):514–526, 2005, `https://doi.org/10.1007/s11202-005-0053-7`.
15. K. Pileckas, R. Čiegis, Existence of nonstationary Poiseuille type solutions under minimal regularity assumptions, *Z. Angew. Math. Phys.*, **71**(192), 2020, `https://doi.org/10.1007/s00033-020-01422-5`.
16. M.U. Qureshi, G.D.A. Vaughan, C. Sainsbury, C.S. Peskin M. Johnson, M.S. Olufsen, N.A. Hill, Numerical simulation of blood flow and pressure drop in the pulmonary arterial and venous circulation, *Biomech. Model. Mechanobiol.*, **13**(5):1137–1154, 2014, `https://doi.org/10.1007/s10237-014-0563-y`.
17. C.A. Taylor, T.J.R. Hughes, C.K. Zarins, Finite element modeling of blood flow in arteries, *Comput. Methods Appl. Mech. Eng.*, **158**(1–2):155–196, 1998, `https://doi.org/10.1016/S0045-7825(98)80008-X`.
18. J.R. Womersley, Method for the calculation of velocity, rate of flow and viscous drag in arteries when the pressure gradient is known, *J. Physiol.*, **127**(3):553–563, 1955, `https://doi.org/10.1113/jphysiol.1955.sp005276`.