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New strong convergence algorithms for general equilibrium and variational inequality problems and resolvent operators in Banach spaces

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Abstract. In this paper, we introduce two new algorithms for solving variational inequalities in Banach spaces. Our aim is finding a common element of the solution set of variational inequalities (for two inverse-strongly monotone operators) and an equilibrium problem and the set of fixed points of two relatively nonexpansive mappings and a family of resolvent operators. Then the strong convergence of the sequences generated by these algorithms to this element will be proved under suitable conditions. Finally, we provide a numerical example to illustrate our main results.

Keywords: relatively nonexpansive mapping, fixed point problem, equilibrium problem, inverse-strongly monotone operator, maximal monotone operator.

1 Introduction

It is well known that variational inequalities are useful and important tools for the study of some branches of applied sciences, and they arise, for example, in optimization problems, equilibrium models, Nash equilibrium problems in noncooperative games, partial differential equation problems, and other problems (see [17, 24]). One of the most important

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methods for solving variational inequalities is the extragradient method introduced by Korpelevich [14] in a finite-dimensional space, which requires two projections onto a closed and convex set and two evaluations of an operator per each iteration. Many authors extended this method to infinite-dimensional spaces (see [7, 10, 24]).

The equilibrium problem is very general because it includes many well-known problems such as variational inequality problems, saddle point problems, etc. (see [6, 11]). Several methods have been proposed to solve the equilibrium problem in Hilbert space (see [4, 15]), and some authors obtained weak and strong convergence algorithms for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space (see [20, 21]). Then the authors proved the strong convergence of the algorithms in a uniformly convex and uniformly smooth Banach space (see [3]).

In this paper, motivated by Cai et al. [4] and Ghadampour et al. [8], using two inverse strongly monotone operators and a family of resolvent operators, we present two new hybrid algorithms. Then we show that our generated sequences are strongly convergent to a common element of the solution set of two variational inequality problems and the fixed point set of two relatively nonexpansion mappings and the fixed point set of a family of resolvent operators and the solution set of the equilibrium problem.

2 Preliminaries

Suppose that C is a nonempty closed convex subset of a real Banach space X with the norm $\|\cdot\|$ and X^* is the dual of X. The variational inequality problem (VI) is as follows:

- Find a point $x \in C$ such that

$$\langle Ax,\, y - x\rangle \geqslant 0 \quad \text{for all } y \in C,$$

where A is a mapping of C into X^*, and $\langle\cdot,\cdot\rangle$ denotes the duality pairing. The solution set of the variational inequality problem is denoted by $VI(C, A)$.

The operator $A : X \to 2^{X^*}$ is said to be

(i) monotone if $\langle x - y,\, x^* - y^*\rangle \geqslant 0$ for all $x, y \in X$ and $x^* \in Ax$, $y^* \in Ay$;

(ii) α-inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle x - y,\, x^* - y^*\rangle \geqslant \alpha\|x^* - y^*\|^2, \quad x, y \in X,\ x^* \in Ax,\ y^* \in Ay;$$

(iii) L-Lipchitz continuous if there exists $L > 0$ such that

$$\|x^* - y^*\| \leqslant L\|x - y\|, \quad x, y \in X,\ x^* \in Ax,\ y^* \in Ay;$$

(iv) demiclosed if for all $\{x_n\} \subset X$ with $x_n \rightharpoonup x$ in X and $y_n \in Ax_n$ with $y_n \to y$ in X^*, it follows that $x \in X$ and $y \in Ax$.

A monotone mapping A is called maximal if its graph $G(A) = \{(x, Ax)\colon x \in D(A)\}$ is not properly contained in the graph of any other monotone mapping. Clearly, the monotone mapping A is maximal if and only if for $(x, x^*) \in X \times X^*$, $\langle x-y,\, x^*-y^*\rangle \geqslant 0$ for each $(y, y^*) \in G(A)$. Then it is implied that $x^* \in Ax$.

Let $f : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem (EP) is as follows:

- Find $x \in C$ such that

$$f(x, y) + \langle Ax,\, y - x\rangle \geqslant 0 \quad \text{for all } y \in C. \tag{1}$$

The solution set of (1) is denoted by $EP(f)$.

Let X be a real smooth Banach space with the norm $\|\cdot\|$, and let X^* be the dual space of X. A function $\delta : [0, 2] \to [0, 1]$ is said to be the modulus of convexity of X if

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|x + y\|}{2}\colon\ \|x\| \leqslant 1,\ \|y\| \leqslant 1,\ \|x - y\| \geqslant \epsilon\right\}$$

for every $\varepsilon \in [0, 2]$. A Banach space X is said to be uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon > 0$. It is well known that a uniformly convex Banach space has the Kadec–Klee property, that is, $x_n \rightharpoonup u$ and $\|x_n\| \to \|u\|$ imply that $x_n \to u$ (see [18]). Let p be a fixed real number with $p \geqslant 2$. A Banach space X is called p-uniformly convex [23] if there exists a constant $c > 0$ such that $\delta \geqslant c\epsilon^p$ for all $\epsilon \in [0, 2]$. The duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \left\{f \in X^*\colon\ \langle x, f\rangle = \|x\|^2 = \|f\|^2\right\}$$

for every $x \in X$. Let $S(X) = \{x \in X\colon\ \|x\| = 1\}$. A Banach space X is said to be smooth if for all $x \in S(X)$, there exists a unique functional $j_x \in X^*$ such that $\langle x, j_x\rangle = \|x\|$ and $\|j_x\| = 1$ (see [1]).

The norm of X is said to be Gâteaux differentiable if for each $x, y \in S(X)$, the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2}$$

exists. In this case, X is said to be smooth, and X is called uniformly smooth if the limit (2) is attained uniformly for all $x, y \in S(X)$ [22]. If a Banach space X is uniformly convex, then X is reflexive and strictly convex, and X^* is uniformly smooth [1]. It is well known that if X is a reflexive, strictly convex, and smooth Banach space and $J^* : X^* \to X$ is the duality mapping on X^*, then $J^{-1} = J^*$. Also, if X is a uniformly smooth Banach space, then J is uniformly norm-to-norm continuous on bounded sets of X, and $J^{-1} = J^*$ is also uniformly norm-to-norm continuous on bounded sets of X^*. Let X be a smooth Banach space, and let J be the duality mapping on X. The function $\phi : X \times X \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \text{for all } x, y \in X. \tag{3}$$

Clearly, from (3) we can conclude that

$$\big(\|x\| - \|y\|\big)^2 \leqslant \phi(x, y) \leqslant \big(\|x\| + \|y\|\big)^2. \tag{4}$$

If X is a reflexive, strictly convex, and smooth Banach space, then, for all $x, y \in X$,

$$\phi(x, y) = 0 \quad \text{if and only if } x = y.$$

Also, it is clear from the definition of the function ϕ that the following condition holds for all $x, y \in X$:

$$\begin{aligned}\phi(x, y) &= \langle x,\, Jx - Jy\rangle + \langle y - x,\, Jy\rangle \\ &\leqslant \|x\|\|Jx - Jy\| + \|y - x\|\|y\|.\end{aligned} \tag{5}$$

Now, the function $V : X \times X^* \to \mathbb{R}$ is defined as

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2$$

for all $x \in X$ and $x^* \in X^*$. Moreover, $V(x, x^*) = \phi(x, J^{-1}x^*)$ for all $x \in X$ and $x^* \in X^*$. If X is a reflexive strictly convex and smooth Banach space with X^* as its dual, then

$$V(x, x^*) + 2\bigl\langle J^{-1}x^* - x,\, y^*\bigr\rangle \leqslant V(x, x^* + y^*) \tag{6}$$

for all $x \in X$ and all $x^*, y^* \in X^*$ [13].

An operator $A : C \to X^*$ is hemicontinuous at $x_0 \in C$ if for any sequence $\{x_n\}$ converging to x_0 along a line, the sequence $\{Ax_n\}$ converges weakly to Ax_0, i.e., $Ax_n = A(x_0 + t_n x) \rightharpoonup Ax_0$ as $t_n \to 0$ for each $x \in C$. The generalized projection $\varPi_C : X \to C$ is the mapping that assigns to each point $x \in X$ the minimizer of the functional $\phi(y, x)$; i.e., $\varPi_C x = x_0$, where x_0 is the solution of the minimization problem

$$\phi(x_0, x) = \min_{y \in C} \phi(y, x).$$

The existence and uniqueness of the operator $\varPi_C$ follow from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J [2]. Let C be a nonempty closed convex subset of X, and let T be a self mapping on C. We denote the set of fixed points of T by $F(T)$, that is, $F(T) = \{x \in C\colon\ x \in Tx\}$. A point $p \in C$ is said to be an asymptotically fixed point of T if C contains a sequence $\{x_n\}$, which converges weakly to p and $Tx_n - x_n \to 0$ [1]. The set of asymptotical fixed points of T will be denoted by $\hat{F}(T)$. A mapping T from C into itself is called relatively nonexpansive if $\hat{F}(T) = F(T)$ and $\phi(p, Tx) \leqslant \phi(p, x)$ for each $x \in C$ and $p \in F(T)$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [5].

We need the following lemmas for proving our main results.

Lemma 1. (*See* [12].) *Let X be a smooth and uniformly convex Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of X. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

Lemma 2. (*See* [2].) *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space X, and let $y \in X$. Then*

$$\phi(x, \varPi_C y) + \phi(\varPi_C y, y) \leqslant \phi(x, y) \quad \text{for all } x \in C.$$

Lemma 3. (*See* [2].) *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space X, and let $x \in X$, $z \in C$. Then*

$$z = \Pi_C x \quad \text{if and only if} \quad \langle y - z,\, Jx - Jz\rangle \leqslant 0 \quad \text{for all } y \in C.$$

Lemma 4. (*See* [27].) *Let X be a 2-uniformly convex and smooth Banach space. Then, for all $x, y \in X$, we have that*

$$\|x - y\| \leqslant \frac{2}{c^2}\|Jx - Jy\|,$$

where $1/c\ (0 < c \leqslant 1)$ is the 2-uniformly convex constant of X.

Lemma 5. (*See* [27].) *Let X be a uniformly convex Banach space and $r > 0$. Then there exists a continuous strictly increasing convex function $g : [0, 2r] \to [0, +\infty)$ such that $g(0) = 0$ and*

$$\big\|tx + (1-t)y\big\|^2 \leqslant t\|x\|^2 + (1-t)\|y\|^2 - t(1-t) \times g\big(\|x - y\|\big) \quad \text{for all } x, y \in B_r(0),\ t \in [0,1]$$

where $B_r(0) = \{z \in E\colon \|z\| \leqslant r\}$.

Lemma 6. (*See* [12].) *Let X be a uniformly convex Banach space and $r > 0$. Then there exists a continuous strictly increasing convex function $g : [0, 2r] \to [0, +\infty)$ such that $g(0) = 0$ and*

$$g\big(\|x - y\|\big) \leqslant \phi(x, y) \quad \text{for all } x, y \in B_r(0),$$

where $B_r(0) = \{z \in X\colon \|z\| \leqslant r\}$.

Throughout this paper, we assume that $f : C \times C \to \mathbb{R}$ is a bifunction satisfying the following conditions:

- (A1) $f(x, x) = 0$ for all $x \in C$;
- (A2) f is monotone, i.e., $f(x, y) + f(y, x) \leqslant 0$ for all $x, y \in C$;
- (A3) $\lim_{t\downarrow 0} f(tz + (1-t)x, y) \leqslant f(x, y)$ for all $x, y, z \in C$;
- (A4) for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semicontinuous.

Lemma 7. (*See* [16].) *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space X, $A : C \longrightarrow X^*$ be an α-inverse-strongly monotone operator, and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4)*. Then, for all $r > 0$, the following hold*:

(i) *For $x \in X$, there exists $u \in C$ such that*

$$f(u, x) + \langle Au,\, y - u\rangle + \frac{1}{r}\langle y - u,\, Ju - Jx\rangle \geqslant 0 \quad \text{for all } y \in C.$$

(ii) *If X is additionally uniformly smooth and $K_r : X \longrightarrow C$ is defined as*

$$K_r(x) = \left\{u \in C\colon f(u, y) + \langle Au,\, y - u\rangle + \frac{1}{r}\langle y - u,\, Ju - Jx\rangle \geqslant 0,\ y \in C\right\},$$

then the following conditions hold:

1. K_r *is single-valued,*
2. K_r *is firmly nonexpansive, i.e., for all* $x, y \in X$*,*

$$\langle K_r x - K_r y,\ JK_r x - JK_r y\rangle \leqslant \langle K_r x - K_r y,\ Jx - Jy\rangle,$$

3. $F(K_r) = \hat{F}(K_r) = EP(f)$*,*
4. EP *is a closed convex subset of* C*,*
5. $\phi(p, K_r x) + \phi(K_r x, x) \leqslant \phi(p, x)$ *for all* $p \in F(K_r)$*.*

The normal cone for C at a point $v \in C$ is denoted by $N_C(v)$, that is, $N_C(v) := \{x^* \in X^*\colon \langle v - y,\, x^*\rangle \geqslant 0 \text{ for all } y \in C\}$.

Lemma 8. (*See* [19].) *Let* C *be a nonempty closed convex subset of a Banach space* X *and* T *be monotone and hemicontinuous operator of* C *into* X^* *with* $C = D(T)$*, and let* $B \subset X \times X^*$ *be an operator defined as follows*:

$$Bv = \begin{cases} Tv + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then B *is maximal monotone and* $B^{-1}(0) = VI(C, T)$*.*

Definition 1. (See [25].) Let X be a real smooth and uniformly convex Banach space, and let $M : X \to 2^{X^*}$ be a maximal monotone operator. For all $\xi > 0$, define the operator $Q_\xi^M : X \to X$ by $Q_\xi^M x = (J + \xi M)^{-1} Jx$ for all $x \in X$.

Lemma 9. (*See* [22].) *Let* X *be a real smooth and uniformly convex Banach space, and let* $M : X \to 2^{X^*}$ *be a maximal monotone operator. Then* $M^{-1}0$ *is a closed and convex subset of* X*, and the graph* $G(M)$ *of* M *is demiclosed.*

Lemma 10. (*See* [26].) *Let* X *be a real reflexive, strictly convex, and smooth Banach space, and let* $M : X \to 2^{X^*}$ *be a maximal monotone operator with* $M^{-1}0 \neq \emptyset$*. Then, for all* $x \in X$*,* $y \in M^{-1}0$*, and* $\xi > 0$*,* $\phi(y, Q_\xi^M x) + \phi(Q_\xi^M x, x) \leqslant \phi(y, x)$*.*

3 Main results

In this section, we introduce our new iterative algorithms.

Theorem 1. *Let* X *be a real 2-uniformly convex and uniformly smooth Banach space, and let* X^* *be the dual space of* X*. Suppose that* C *is a nonempty closed and convex subset of* X *and the mappings* $A, B : C \to X^*$ *are* α*-inverse strongly monotone and* β*-inverse strongly monotone, respectively. Let* $M_i : X \to 2^{X^*}$ *be a maximal monotone operator with* ${M_i}^{-1}0 \neq \emptyset$ *for each* $i = 1, 2, \dots, m$*. Assume that* f *and* T *are relatively nonexpansive mappings from* C *into itself and* $\Gamma = F(f) \cap F(T) \cap (\cap_{i=1}^m F(Q_\xi^{M_i})) \cap VI(C, A) \cap VI(C, B) \neq \emptyset$*. Let* $\|Ax\| \leqslant \|Ax - Au\|$*,* $\|Bx\| \leqslant \|Bx - Bu\|$ *for each*

$u \in \Gamma$ and $x \in C$. Suppose that $\{x_n\}$ is a sequence generated by $x_1 \in C$ and

$$\begin{aligned}
u_n &= \Pi_C J^{-1}\left(s_{n,0}Jx_n + \sum_{i=1}^m s_{n,i}JQ_\xi^{M_i}x_n\right),\\
z_n &= \Pi_C J^{-1}(Ju_n - \mu Bu_n), \qquad y_n = \Pi_C J^{-1}(Jz_n - \lambda Az_n),\\
w_n &= J^{-1}\left(\sum_{i=1}^m \beta_{n,i}JQ_\xi^{M_i}z_n + \sum_{i=1}^m \gamma_{n,i}JQ_\xi^{M_i}y_n\right),\\
x_{n+1} &= \Pi_C J^{-1}\big[\alpha_{n,1}Jfx_n + \alpha_{n,2}Ju_n + \alpha_{n,3}JTy_n + \alpha_{n,4}Jw_n\big],
\end{aligned}$$

where $\{\beta_{n,i}\}_{i=1}^m$, $\{\gamma_{n,i}\}_{i=1}^m$, and $\{s_{n,i}\}_{i=0}^m$ are real sequences in $[a,b] \subset (0,1)$, $\sum_{i=1}^m(\beta_{n,i}+\gamma_{n,i}) = 1$, and $\sum_{i=0}^m s_{n,i} = 1$. Suppose that μ, λ and $\{\alpha_{n,i}\}_{i=1}^4$ satisfy the following conditions:

(i) $\{\alpha_{n,i}\} \subset (0,1)$, $\sum_{i=1}^4 \alpha_{n,i} = 1$, $\liminf_{n\to+\infty}\alpha_{n,4}, \liminf_{n\to+\infty}\alpha_{n,1}\alpha_{n,2}$, $\liminf_{n\to+\infty}\alpha_{n,2}\alpha_{n,3} > 0$;

(ii) *λ and μ are real numbers such that $0 < \lambda < c^2\alpha/2$ and $0 < \mu < c^2\beta/2$, where $1/c$ is the 2-uniformly convexity constant of X.*

Then $\{x_n\}$ converges strongly to $q = \Pi_{F(T)\cap(\cap_{i=1}^m F(Q_\xi^{M_i}))\cap VI(C,A)\cap VI(C,B)} \circ f(q)$.

Proof. Let $\hat{u} \in \Gamma = F(f) \cap F(T) \cap (\cap_{i=1}^n F(Q_\xi^{M_i})) \cap VI(C,A) \cap VI(C,B)$. By (3), Lemma 2, the convexity of $\|\cdot\|^2$, and our assumptions, we have

$$\begin{aligned}
\phi(\hat{u}, u_n) &\leqslant \phi\left(\hat{u}, J^{-1}\left(s_{n,0}Jx_n + \sum_{i=1}^m s_{n,i}JQ_\xi^{M_i}x_n\right)\right)\\
&= \|\hat{u}\|^2 - 2\left\langle \hat{u},\, s_{n,0}Jx_n + \sum_{i=1}^m s_{n,i}JQ_\xi^{M_i}x_n \right\rangle\\
&\quad + \left\| s_{n,0}Jx_n + \sum_{i=1}^m s_{n,i}JQ_\xi^{M_i}x_n \right\|^2\\
&\leqslant \|\hat{u}\|^2 - 2s_{n,0}\langle \hat{u}, Jx_n\rangle - 2\sum_{i=1}^m s_{n,i}\langle \hat{u}, JQ_\xi^{M_i}x_n\rangle + s_{n,0}\|x_n\|^2\\
&\quad + \sum_{i=1}^m s_{n,i}\big\|Q_\xi^{M_i}x_n\big\|^2\\
&= s_{n,0}\phi(\hat{u}, x_n) + \sum_{i=1}^m s_{n,i}\phi\big(\hat{u}, Q_\xi^{M_i}x_n\big).
\end{aligned} \tag{7}$$

Now, from Lemma 10 and the above it follows that

$$\phi(\hat{u}, u_n) \leqslant s_{n,0}\phi(\hat{u}, x_n) + \sum_{i=1}^m s_{n,i}\phi(\hat{u}, x_n) = \phi(\hat{u}, x_n). \tag{8}$$

From (6) and Lemma 2 it follows that

$$\begin{aligned}\phi(\hat{u}, z_n) &\leqslant \phi\big(\hat{u}, J^{-1}(Ju_n - \mu Bu_n)\big) = V(\hat{u},\, Ju_n - \mu Bu_n)\\ &\leqslant V(\hat{u}, Ju_n) - 2\big\langle J^{-1}(Ju_n - \mu Bu_n) - \hat{u},\, \mu Bu_n\big\rangle\\ &= \phi(\hat{u}, u_n) + 2\big\langle J^{-1}(Ju_n - \mu Bu_n) - J^{-1}(Ju_n),\, -\mu Bu_n\big\rangle\\ &\quad - 2\mu\langle u_n - \hat{u},\, Bu_n\rangle. \end{aligned} \tag{9}$$

By Lemma 4 and our assumptions, we obtain that

$$\begin{aligned}&2\big\langle J^{-1}(Ju_n - \mu Bu_n) - J^{-1}(Ju_n),\, -\mu Bu_n\big\rangle\\ &\quad\leqslant 2\big\|J^{-1}(Ju_n - \mu Bu_n) - J^{-1}(Ju_n)\big\|\|-\mu Bu_n\|\\ &\quad\leqslant \frac{4\mu^2}{c^2}\|Bu_n\|^2 \leqslant \frac{4\mu^2}{c^2}\|Bu_n - B\hat{u}\|^2. \end{aligned} \tag{10}$$

Since $\hat{u} \in VI(C, B)$ and B is β-inverse strongly monotone, it follows that

$$\begin{aligned}&-2\mu\langle u_n - \hat{u},\, Bu_n\rangle\\ &\quad= -2\mu\langle u_n - \hat{u},\, Bu_n - B\hat{u}\rangle - 2\mu\langle u_n - \hat{u},\, B\hat{u}\rangle\\ &\quad\leqslant -2\mu\langle u_n - \hat{u},\, Bu_n - B\hat{u}\rangle \leqslant -2\mu\beta\|Bu_n - B\hat{u}\|^2. \end{aligned} \tag{11}$$

Now, substituting (10) and (11) into (9) and using our assumptions, we obtain that

$$\phi(\hat{u}, z_n) \leqslant \phi(\hat{u}, u_n) + 2\mu\left(\frac{2\mu}{c^2} - \beta\right)\|Bu_n - B\hat{u}\|^2 \leqslant \phi(\hat{u}, u_n). \tag{12}$$

In a similar way, by repeating the above proof for the sequence $\{y_n\}$, we conclude that

$$\phi(\hat{u}, y_n) \leqslant \phi(\hat{u}, z_n) + 2\lambda\left(\frac{2\lambda}{c^2} - \alpha\right)\|Az_n - A\hat{u}\|^2 \leqslant \phi(\hat{u}, z_n). \tag{13}$$

From (8), (12), and (13) we have

$$\phi(\hat{u}, y_n) \leqslant \phi(\hat{u}, x_n). \tag{14}$$

It follows from (3) and the convexity of $\|\cdot\|^2$ that

$$\begin{aligned}\phi(\hat{u}, w_n) &= \phi\left(\hat{u}, J^{-1}\left(\sum_{i=1}^m \beta_{n,i} JQ_\xi^{M_i} z_n + \sum_{i=1}^m \gamma_{n,i} JQ_\xi^{M_i} y_n\right)\right)\\ &\leqslant \|\hat{u}\|^2 - 2\sum_{i=1}^m \beta_{n,i}\big\langle \hat{u}, JQ_\xi^{M_i} z_n\big\rangle - 2\sum_{i=1}^m \gamma_{n,i}\big\langle \hat{u}, JQ_\xi^{M_i} y_n\big\rangle\\ &\quad + \sum_{i=1}^m \beta_{n,i}\big\|Q_\xi^{M_i} z_n\big\|^2 + \sum_{i=1}^m \gamma_{n,i}\big\|Q_\xi^{M_i} y_n\big\|^2\\ &= \sum_{i=1}^m \beta_{n,i}\phi\big(\hat{u}, Q_\xi^{M_i} z_n\big) + \sum_{i=1}^m \gamma_{n,i}\phi\big(\hat{u}, Q_\xi^{M_i} y_n\big). \end{aligned} \tag{15}$$

Now, by (8), (12), (13), (15), and Lemma 10, we have

$$\phi(\hat{u}, w_n) \leqslant \sum_{i=1}^{m} \beta_{n,i}\phi(\hat{u}, z_n) + \sum_{i=1}^{m} \gamma_{n,i}\phi(\hat{u}, y_n) \leqslant \phi(\hat{u}, x_n). \tag{16}$$

From (3), (16), Lemma 2, the convexity of $\|\cdot\|^2$, and the relatively nonexpansiveness of f and T, we have that

$$\begin{aligned}\phi(\hat{u}, x_{n+1}) &\leqslant \|\hat{u}\|^2 - 2\alpha_{n,1}\langle \hat{u}, Jfx_n\rangle - 2\alpha_{n,2}\langle \hat{u},\ Ju_n\rangle \\ &\quad - 2\alpha_{n,3}\langle \hat{u}, JTy_n\rangle - 2\alpha_{n,4}\langle \hat{u},\ Jw_n\rangle \\ &\quad + \alpha_{n,1}\|fx_n\|^2 + \alpha_{n,2}\|u_n\|^2 + \alpha_{n,3}\|Ty_n\|^2 + \alpha_{n,4}\|w_n\|^2 \\ &= \alpha_{n,1}\phi(\hat{u}, fx_n) + \alpha_{n,2}\phi(\hat{u}, u_n) + \alpha_{n,3}\phi(\hat{u}, Ty_n) + \alpha_{n,4}\phi(\hat{u}, w_n) \\ &\leqslant \alpha_{n,1}\phi(\hat{u}, x_n) + \alpha_{n,2}\phi(\hat{u}, u_n) + \alpha_{n,3}\phi(\hat{u}, y_n) + \alpha_{n,4}\phi(\hat{u}, w_n) \qquad (17) \\ &\leqslant (\alpha_{n,1} + \alpha_{n,4})\phi(\hat{u}, x_n) + \alpha_{n,2}\phi(\hat{u}, u_n) + \alpha_{n,3}\phi(\hat{u}, y_n). \qquad (18)\end{aligned}$$

Therefore, by (8), (14), (18), and condition (i), we have

$$\phi(\hat{u}, x_{n+1}) \leqslant \phi(\hat{u}, x_n).$$

This show that $\{\phi(\hat{u}, x_n)\}$ is bounded and $\lim_{n\to+\infty} \phi(\hat{u}, x_n)$ exists. It follows from (4) that $\{x_n\}$ is bounded. Now, by (8), (12), (13), (16), and relatively nonexpansiveness of f and T, we have that the sequences $\{u_n\}, \{z_n\}, \{y_n\}$, $\{w_n\}$, $\{fx_n\}$, and $\{Ty_n\}$ are bounded.

Next, by (7), (14), (18), and Lemma 10, we conclude that

$$\begin{aligned}\phi(\hat{u}, x_{n+1}) &\leqslant (1 - \alpha_{n,2})\phi(\hat{u}, x_n) + \alpha_{n,2}\phi(\hat{u}, u_n) \\ &\leqslant (1 - \alpha_{n,2})\phi(\hat{u}, x_n) \\ &\quad + \alpha_{n,2}\left(s_{n,0}\phi(\hat{u}, x_n) + \sum_{i=1}^{m} s_{n,i}\phi\big(\hat{u}, Q_\xi^{M_i} x_n\big)\right) \\ &\leqslant (1 - \alpha_{n,2})\phi(\hat{u}, x_n) \\ &\quad + \alpha_{n,2}\left(s_{n,0}\phi(\hat{u}, x_n) + \sum_{i=1}^{m} s_{n,i}\big[\phi(\hat{u}, x_n) - \phi\big(Q_\xi^{M_i} x_n, x_n\big)\big]\right) \\ &= \phi(\hat{u}, x_n) - \alpha_{n,2}\sum_{i=1}^{m} s_{n,i}\phi\big(Q_\xi^{M_i} x_n, x_n\big). \qquad (19)\end{aligned}$$

Now, since $\{\phi(\hat{u}, x_n)\}$ is convergent, it follows from (19), condition (i), and our assumptions that $\lim_{n\to+\infty} \phi(Q_\xi^{M_i} x_n, x_n) = 0$ for each $i = 1, 2, \dots, m$. Hence, from Lemma 1 we have that

$$\lim_{n\to+\infty} \big\|Q_\xi^{M_i} x_n - x_n\big\| = 0 \tag{20}$$

for each $i = 1, 2, \dots, m$. It follows from (5), (20), the boundedness of the sequences $\{x_n\}$ and $\{Q_\xi^{M_i} x_n\}$ for each $i = 1, 2, \dots, m$, and using uniformly norm-to-norm continuity of J on bounded sets, that

$$\lim_{n\to+\infty} \phi\bigl(x_n, Q_\xi^{M_i} x_n\bigr) = 0 \tag{21}$$

for each $i = 1, 2, \dots, m$. By (8), (12), (14), (15), (17), and Lemma 10, we conclude that

$$\begin{aligned}
\phi(\hat{u}, x_{n+1}) &\leqslant (1-\alpha_{n,4})\phi(\hat{u}, x_n) + \alpha_{n,4}\phi(\hat{u}, w_n) \\
&\leqslant (1-\alpha_{n,4})\phi(\hat{u}, x_n) \\
&\quad + \alpha_{n,4}\Biggl(\sum_{i=1}^m \beta_{n,i}\phi(\hat{u}, Q_\xi^{M_i} z_n) + \sum_{i=1}^m \gamma_{n,i}\phi\bigl(\hat{u}, Q_\xi^{M_i} y_n\bigr)\Biggr) \\
&\leqslant (1-\alpha_{n,4})\phi(\hat{u}, x_n) \\
&\quad + \alpha_{n,4}\Biggl(\sum_{i=1}^m \beta_{n,i}\bigl[\phi(\hat{u}, z_n) - \phi\bigl(Q_\xi^{M_i} z_n, z_n\bigr)\bigr] \\
&\quad + \sum_{i=1}^m \gamma_{n,i}\bigl[\phi(\hat{u}, y_n) - \phi\bigl(Q_\xi^{M_i} y_n, y_n\bigr)\bigr]\Biggr) \\
&= \phi(\hat{u}, x_n) - \alpha_{n,4}\sum_{i=1}^m \beta_{n,i}\phi\bigl(Q_\xi^{M_i} z_n, z_n\bigr) \\
&\quad - \alpha_{n,4}\sum_{i=1}^m \gamma_{n,i}\phi\bigl(Q_\xi^{M_i} y_n, y_n\bigr).
\end{aligned}$$

Hence, from (4), the above and our assumptions we obtain

$$\phi(\hat{u}, x_{n+1}) \leqslant \phi(\hat{u}, x_n) - \alpha_{n,4}\sum_{i=1}^m \beta_{n,i}\phi(Q_\xi^{M_i} z_n, z_n) \tag{22}$$

and

$$\phi(\hat{u}, x_{n+1}) \leqslant \phi(\hat{u}, x_n) - \alpha_{n,4}\sum_{i=1}^m \gamma_{n,i}\phi(Q_\xi^{M_i} y_n, y_n). \tag{23}$$

Now, by (22), (23), condition (i), and the same techniques used for proving (20), we conclude that

$$\lim_{n\to+\infty} \bigl\|Q_\xi^{M_i} z_n - z_n\bigr\| = 0, \qquad \lim_{n\to+\infty} \bigl\|Q_\xi^{M_i} y_n - y_n\bigr\| = 0 \tag{24}$$

for each $i = 1, 2, \dots, m$. From (5), (24), and the uniformly norm-to-norm continuity of J on bounded sets we have that

$$\lim_{n\to+\infty} \phi\bigl(z_n, Q_\xi^{M_i} z_n\bigr) = 0, \qquad \lim_{n\to+\infty} \phi\bigl(y_n, Q_\xi^{M_i} y_n\bigr) = 0 \tag{25}$$

for each $i = 1, 2, \dots, m$.

Let $r_1 = \sup_n\{\|fx_n\|, \|u_n\|\}$. Hence, by Lemma 5, there exists a continuous strictly increasing convex function $g_{r_1} : [0, 2r_1] \to [0, +\infty)$ such that $g_{r_1}(0) = 0$, and using (8), (14), (16), the convexity of $\|\cdot\|^2$, and the condition of relatively nonexpansiveness of f and T, we obtain that

$$\begin{aligned}
\phi(\hat{u}, x_{n+1}) &\leqslant \|\hat{u}\|^2 - 2\langle \hat{u}, \alpha_{n,1}Jfx_n + \alpha_{n,2}Ju_n + \alpha_{n,3}JTy_n + \alpha_{n,4}Jw_n\rangle \\
&\quad + \|\alpha_{n,1}Jfx_n + \alpha_{n,2}Ju_n + \alpha_{n,3}JTy_n + \alpha_{n,4}Jw_n\|^2 \\
&\leqslant \|\hat{u}\|^2 - 2\alpha_{n,1}\langle \hat{u}, Jfx_n\rangle - 2\alpha_{n,2}\langle \hat{u}, Ju_n\rangle - 2\alpha_{n,3}\langle \hat{u}, JTy_n\rangle \\
&\quad - 2\alpha_{n,4}\langle \hat{u}, Jw_n\rangle + \alpha_{n,1}\|fx_n\|^2 + \alpha_{n,2}\|u_n\|^2 + \alpha_{n,3}\|Ty_n\|^2 \\
&\quad + \alpha_{n,4}\|w_n\|^2 - \alpha_{n,1}\alpha_{n,2}g_{r_1}\big(\|Jfx_n - Ju_n\|\big) \\
&\leqslant \alpha_{n,1}\phi(\hat{u}, fx_n) + \alpha_{n,2}\phi(\hat{u}, u_n) + \alpha_{n,3}\phi(\hat{u}, Ty_n) + \alpha_{n,4}\phi(\hat{u}, w_n) \\
&\quad - \alpha_{n,1}\alpha_{n,2}g_{r_1}\big(\|Jfx_n - Ju_n\|\big) \\
&\leqslant \alpha_{n,1}\phi(\hat{u}, x_n) + \alpha_{n,2}\phi(\hat{u}, u_n) + \alpha_{n,3}\phi(\hat{u}, y_n) + \alpha_{n,4}\phi(\hat{u}, w_n) \\
&\quad - \alpha_{n,1}\alpha_{n,2}g_{r_1}\big(\|Jfx_n - Ju_n\|\big) \\
&\leqslant \phi(\hat{u}, x_n) - \alpha_{n,1}\alpha_{n,2}g_{r_1}\big(\|Jfx_n - Ju_n\|\big). \qquad (26)
\end{aligned}$$

Let $r_2 = \sup_n\{\|u_n\|, \|Ty_n\|\}$. Then, in a similar way as above, there exists a continuous strictly increasing convex function $g_{r_2} : [0, 2r_2] \to [0, +\infty)$ with $g_{r_2}(0) = 0$ such that

$$\phi(\hat{u}, x_{n+1}) \leqslant \phi(\hat{u}, x_n) - \alpha_{n,2}\alpha_{n,3}g_{r_2}\big(\|Ju_n - JTy_n\|\big). \qquad (27)$$

Therefore, it follows from (26) that

$$\alpha_{n,1}\alpha_{n,2}g_{r_1}\big(\|Jfx_n - Ju_n\|\big) \leqslant \phi(\hat{u}, x_n) - \phi(\hat{u}, x_{n+1}).$$

Hence, from condition (i) we conclude that

$$\lim_{n\to+\infty} g_{r_1}\big(\|Jfx_n - Ju_n\|\big) = 0.$$

Moreover, from the fact that g_{r_1} is a continuous function we have

$$\begin{aligned}
g_{r_1}\Big(\lim_{n\to+\infty} \|Jfx_n - Ju_n\|\Big) &= \lim_{n\to+\infty} g_{r_1}\big(\|Jfx_n - Ju_n\|\big) \\
&= 0 = g_{r_1}(0),
\end{aligned}$$

so

$$\lim_{n\to+\infty} \|Jfx_n - Ju_n\| = 0.$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have that

$$\lim_{n\to+\infty} \|fx_n - u_n\| = 0. \qquad (28)$$

Now, from (27), condition (i), and a similar technique as above we conclude that

$$\lim_{n\to+\infty} \|u_n - Ty_n\| = 0. \qquad (29)$$

Then from (5), (28), (29), the uniformly norm-to-norm continuity of J on bounded sets, and the boundedness of the sequences $\{fx_n\}$, $\{u_n\}$ and $\{Ty_n\}$ we have that

$$\lim_{n\to+\infty}\phi(u_n, fx_n) = 0, \qquad \lim_{n\to+\infty}\phi(u_n, Ty_n) = 0. \tag{30}$$

Using our assumptions, we obtain from (21) and Lemma 2 that

$$\begin{aligned}\phi(x_n, u_n) &\leqslant \phi\left(x_n,\ J^{-1}\left(s_{n,0}Jx_n + \sum_{i=1}^{m} s_{n,i}JQ_\xi^{M_i}x_n\right)\right)\\ &\leqslant \|x_n\|^2 - 2s_{n,0}\langle x_n, Jx_n\rangle - 2\sum_{i=1}^{m} s_{n,i}\langle x_n, JQ_\xi^{M_i}x_n\rangle\\ &\quad + s_{n,0}\|x_n\|^2 + \sum_{i=1}^{m} s_{n,i}\left\|Q_\xi^{M_i}x_n\right\|^2\\ &= s_{n,0}\phi(x_n, x_n) + \sum_{i=1}^{m} s_{n,i}\phi\left(x_n, Q_\xi^{M_i}x_n\right)\\ &= \sum_{i=1}^{m} s_{n,i}\phi\left(x_n, Q_\xi^{M_i}x_n\right) \to 0 \quad as\ n\to+\infty.\end{aligned}$$

Then from Lemma (1) we conclude that

$$\lim_{n\to+\infty}\|x_n - u_n\| = 0. \tag{31}$$

By (8), (12), and (13), we obtain that

$$\phi(\hat{u}, y_n) \leqslant \phi(\hat{u}, x_n) + 2\mu\left(\frac{2\mu}{c^2} - \beta\right)\|Bu_n - B\hat{u}\|^2. \tag{32}$$

Therefore, from (8), (18), and (32) we have

$$\begin{aligned}\phi(\hat{u}, x_{n+1}) &\leqslant (\alpha_{n,1} + \alpha_{n,4})\phi(\hat{u}, x_n) + \alpha_{n,2}\phi(\hat{u}, u_n) + \alpha_{n,3}\phi(\hat{u}, y_n)\\ &\leqslant (\alpha_{n,1} + \alpha_{n,4})\phi(\hat{u}, x_n) + \alpha_{n,2}\phi(\hat{u}, u_n)\\ &\quad + \alpha_{n,3}\left(\phi(\hat{u}, x_n) + 2\mu\left(\frac{2\mu}{c^2} - \beta\right)\|Bu_n - B\hat{u}\|^2\right)\\ &\leqslant \phi(\hat{u}, x_n) + 2\mu\left(\frac{2\mu}{c^2} - \beta\right)\|Bu_n - B\hat{u}\|^2.\end{aligned}$$

Hence,

$$2\mu\left(\beta - \frac{2\mu}{c^2}\right)\|Bu_n - B\hat{u}\|^2 \leqslant \phi(\hat{u}, x_n) - \phi(\hat{u}, x_{n+1}). \tag{33}$$

Since $\{\phi(\hat{u}, x_n)\}$ converges, it follows from (33) and (ii) that

$$\lim_{n\to+\infty}\|Bu_n - B\hat{u}\|^2 = 0. \tag{34}$$

In a similar way as above, from (8), (12), (13), and (18) we have

$$\begin{aligned}\phi(\hat{u}, x_{n+1}) &\leqslant (\alpha_{n,1} + \alpha_{n,4})\phi(\hat{u}, x_n) + \alpha_{n,2}\phi(\hat{u}, u_n) + \alpha_{n,3}\phi(\hat{u}, y_n) \\ &\leqslant (\alpha_{n,1} + \alpha_{n,4})\phi(\hat{u}, x_n) + \alpha_{n,2}\phi(\hat{u}, u_n) \\ &\quad + \alpha_{n,3}\left(\phi(\hat{u}, x_n) + 2\lambda\left(\frac{2\lambda}{c^2} - \alpha\right)\|Az_n - A\hat{u}\|^2\right) \\ &\leqslant \phi(\hat{u}, x_n) + 2\mu\left(\frac{2\lambda}{c^2} - \alpha\right)\|Az_n - A\hat{u}\|^2,\end{aligned}$$

so,

$$\lim_{n\to+\infty} \|Az_n - A\hat{u}\|^2 = 0. \tag{35}$$

Therefore, from (6), (34), and Lemmas 2 and 4 we get

$$\begin{aligned}\phi(u_n, z_n) &\leqslant \phi\bigl(u_n, J^{-1}(Ju_n - \mu Bu_n)\bigr) = V(u_n, Ju_n - \mu Bu_n) \\ &\leqslant V(u_n, Ju_n) - 2\bigl\langle J^{-1}(Ju_n - \mu Bu_n) - u_n,\, \mu Bu_n\bigr\rangle \\ &= \phi(u_n, u_n) - 2\bigl\langle J^{-1}(Ju_n - \mu Bu_n) - J^{-1}(Ju_n),\, \mu Bu_n\bigr\rangle \\ &\leqslant 2\bigl\|J^{-1}(Ju_n - \mu Bu_n) - J^{-1}(Ju_n)\bigr\|\|\mu Bu_n\| \\ &\leqslant \frac{4\mu^2}{c^2}\|Bu_n\|^2 \leqslant \frac{4\mu^2}{c^2}\|Bu_n - B\hat{u}\|^2 \to 0 \quad \text{as } n \to +\infty.\end{aligned}$$

Then, using Lemma 1, we obtain

$$\lim_{n\to+\infty} \|u_n - z_n\| = 0. \tag{36}$$

Also, in a similar way as above, from (6), (35), and Lemmas 2 and 4 we have

$$\begin{aligned}\phi(z_n, y_n) &\leqslant \phi\bigl(z_n, J^{-1}(Jz_n - \lambda Az_n)\bigr) \\ &\leqslant \frac{4\lambda^2}{c^2}\|Az_n - A\hat{u}\|^2 \to 0 \quad \text{as } n \to +\infty.\end{aligned}$$

Then, using Lemma 1, we get

$$\lim_{n\to+\infty} \|z_n - y_n\| = 0. \tag{37}$$

Now, from (24) and (37) we obtain that

$$\lim_{n\to+\infty} \bigl\|z_n - Q_\xi^{M_i} y_n\bigr\| = 0 \tag{38}$$

for each $i = 1, 2, \dots, m$. Therefore, using (5), (38), and the uniformly norm-to-norm continuity of J on bounded sets, we have that

$$\lim_{n\to+\infty} \phi\bigl(z_n, Q_\xi^{M_i} y_n\bigr) = 0 \tag{39}$$

for $i = 1, 2, \ldots, m$. It follows from (3), (25), (39), and the convexity of $\|\cdot\|^2$ that

$$\begin{aligned}\phi(z_n, w_n) &= \phi\Bigg(z_n, J^{-1}\Bigg(\sum_{i=1}^{m}\beta_{n,i} J Q_\xi^{M_i} z_n + \sum_{i=1}^{m}\gamma_{n,i} J Q_\xi^{M_i} y_n\Bigg)\Bigg) \\ &\leqslant \|z_n\|^2 - 2\sum_{i=1}^{m}\beta_{n,i}\big\langle z_n, J Q_\xi^{M_i} z_n\big\rangle - 2\sum_{i=1}^{m}\gamma_{n,i}\big\langle z_n, J Q_\xi^{M_i} y_n\big\rangle \\ &\quad + \sum_{i=1}^{m}\beta_{n,i}\big\|Q_\xi^{M_i} z_n\big\|^2 + \sum_{i=1}^{m}\gamma_{n,i}\big\|Q_\xi^{M_i} y_n\big\|^2 \\ &= \sum_{i=1}^{m}\beta_{n,i}\phi\big(z_n, Q_\xi^{M_i} z_n\big) + \sum_{i=1}^{m}\gamma_{n,i}\phi\big(z_n, Q_\xi^{M_i} y_n\big) \to 0 \quad \text{as } n \to +\infty.\end{aligned}$$

Now, from Lemma 1 we have that $\lim_{n\to+\infty}\|z_n - w_n\| = 0$. Therefore, it is evident from (36) that $\lim_{n\to+\infty}\|u_n - w_n\| = 0$. Then from (4) and the continuity of the mapping J we have

$$\lim_{n\to+\infty}\phi(u_n, w_n) = 0. \tag{40}$$

From (30), (40), and our assumptions we conclude that

$$\begin{aligned}\phi(u_n, x_{n+1}) &\leqslant \|u_n\|^2 - 2\alpha_{n,1}\langle u_n, Jfx_n\rangle - 2\alpha_{n,2}\langle u_n, Ju_n\rangle \\ &\quad - 2\alpha_{n,3}\langle u_n, JTy_n\rangle - 2\alpha_{n,4}\langle u_n, Jw_n\rangle \\ &\quad + \alpha_{n,1}\|fx_n\|^2 + \alpha_{n,2}\|u_n\|^2 + \alpha_{n,3}\|Ty_n\|^2 + \alpha_{n,4}\|w_n\|^2 \\ &= \alpha_{n,1}\phi(u_n, fx_n) + \alpha_{n,2}\phi(u_n, u_n) + \alpha_{n,3}\phi(u_n, Ty_n) \\ &\quad + \alpha_{n,4}\phi(u_n, w_n) \to 0 \quad \text{as } n \to +\infty.\end{aligned}$$

Therefore, by Lemma 1, we have

$$\lim_{n\to+\infty}\|x_{n+1} - u_n\| = 0. \tag{41}$$

From (31) and (41) we obtain that

$$\|x_{n+1} - x_n\| \leqslant \|x_{n+1} - u_n\| + \|u_n - x_n\| \to 0 \quad \text{as } n \to +\infty.$$

Thus $\{x_n\}$ is a Cauchy sequence, hence $\{x_n\}$ converges strongly to a point $q \in C$. Hence, from (31), (36), and (37), we conclude that $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ converge to q.

Next, we prove that $q \in VI(C, B)$. Consider the operator $\tilde{B} \subset X \times X^*$ as follows:

$$\tilde{B}\nu = \begin{cases} B\nu + N_C\nu, & \nu \in C, \\ \emptyset, & \nu \notin C. \end{cases} \tag{42}$$

It is clear from Lemma 8 that $\tilde{B}$ is maximal monotone and $\tilde{B}^{-1}(0) = VI(C, B)$. Now, let $(\nu, \omega) \in G(\tilde{B})$ with $\omega \in \tilde{B}\nu = B\nu + N_C(\nu)$. Hence $\omega - B\nu \in N_C(\nu)$, therefore,

$$\langle \nu - z_n, \omega - B\nu\rangle \geqslant 0. \tag{43}$$

Now, we have from Lemma 3 that $\langle \nu - z_n,\, J(J^{-1}(Ju_n - \mu Bu_n)) - Jz_n\rangle \leqslant 0$. Then

$$\left\langle \nu - z_n,\, Bu_n + \frac{Jz_n - Ju_n}{\mu}\right\rangle \geqslant 0. \tag{44}$$

From (43), (44), and the definition of B we conclude that

$$\begin{aligned}
&\langle \nu - z_n,\, \omega\rangle \\
&\quad\geqslant \langle \nu - z_n,\, B\nu\rangle - \left\langle \nu - z_n,\, Bu_n + \frac{Jz_n - Ju_n}{\mu}\right\rangle \\
&\quad= \langle \nu - z_n,\, B\nu - Bz_n\rangle + \langle \nu - z_n,\, Bz_n\rangle - \left\langle \nu - z_n,\, Bu_n + \frac{Jz_n - Ju_n}{\mu}\right\rangle \\
&\quad\geqslant \langle \nu - z_n,\, Bz_n - Bu_n\rangle - \left\langle \nu - z_n,\, \frac{Jz_n - Ju_n}{\mu}\right\rangle \\
&\quad\geqslant -\|\nu - z_n\|\left(\frac{1}{\beta}\|z_n - u_n\| - \frac{1}{\mu}\|Jz_n - Ju_n\|\right).
\end{aligned} \tag{45}$$

Taking $n \to +\infty$ and using the uniformly norm-to-norm continuity of J on bounded sets and (36), we obtain $\langle \nu - q,\, \omega\rangle \geqslant 0$. Now, from the maximal monotonicity of $\tilde{B}$ we conclude $q \in \tilde{B}^{-1}(0) = VI(C, B)$.

Next, we show that $q \in VI(C, A)$. Let $\tilde{A} \subset X \times X^*$ be an operator defined as follows:

$$\tilde{A}\iota = \begin{cases} A\iota + N_C\iota, & \iota \in C, \\ \emptyset, & \iota \notin C. \end{cases} \tag{46}$$

We know from Lemma 8 that $\tilde{A}$ is maximal monotone and also $\tilde{A}^{-1}(0) = VI(C, A)$. Suppose that $(\iota, \theta) \in G(\tilde{A})$ with $\theta \in \tilde{A}\iota = A\iota + N_C(\iota)$. Then $\theta - A\iota \in N_C(\iota)$. In a similar way as in (45), we obtain that

$$\langle \iota - y_n,\, \theta\rangle \geqslant -\|\iota - y_n\|\left(\frac{1}{\alpha}\|y_n - z_n\| - \frac{1}{\lambda}\|Jy_n - Jz_n\|\right). \tag{47}$$

Taking the limit in the above inequality as $n \to +\infty$, we deduce that $\langle \iota - q,\, \theta\rangle \geqslant 0$. Hence, from the maximal monotonicity of $\tilde{A}$ we imply that $q \in \tilde{A}^{-1}(0) = VI(C, A)$.

Next, we prove that $q \in \cap_{i=1}^m F(Q_\xi^{M_i})$. It follows from (20) and the uniformly continuity of J on bounded subsets of X that

$$\lim_{n\to+\infty} \left\|JQ_\xi^{M_i}x_n - Jx_n\right\| = 0$$

for each $i = 1, 2, \ldots, m$. Hence, by Definition 1, we have $JQ_\xi^{M_i}x_n + \xi M_i Q_\xi^{M_i}x_n = Jx_n$. Therefore, there exists $h_{n,i} \in M_i Q_\xi^{M_i}x_n$ such that $h_{n,i} = (Jx_n - JQ_\xi^{M_i}x_n)/\xi$. So, by the above observation, $h_{n,i} \to 0$ as $n \to +\infty$ for each $i = 1, 2, \ldots, m$. Then from (20) it is clear that $Q_\xi^{M_i}x_n \rightharpoonup q$ as $n \to +\infty$, and using Lemma 9, we have that $0 \in M_i q$ for each $i = 1, 2, \ldots, m$, i.e., $q \in \cap_{i=1}^m M_i^{-1}0 = \cap_{i=1}^m F(Q_\xi^{M_i})$.

Next, we show that $q \in F(f)$. From (28) and (31) we conclude that

$$\lim_{n\to+\infty} \|fx_n - x_n\| = 0.$$

Then q is an asymptotic fixed point of f. Since f is a relatively nonexpansive mapping, $\hat{F}(f) = F(f)$. Therefore, $q \in F(f)$.

Next, we prove that $q \in F(T)$. From (29), (36), and (37) we obtain

$$\lim_{n\to+\infty} \|Ty_n - y_n\| = 0.$$

Hence q is an asymptotic fixed point of T. Now, since T is a relatively nonexpansive mapping, $\hat{F}(T) = F(T)$. So, $q \in F(T)$. Then

$$q = \varPi_{F(T)\cap(\cap_{i=1}^m F(Q_\xi^{M_i}))\cap VI(C,A)\cap VI(C,B)} \circ f(q).$$

This completes the proof. □

Theorem 2. *Let X, X^*, C, A, B, f, T, and M_i for $i = 1, 2, \dots, m$ be as in Theorem* 1. *Suppose that g is a bifunction from $C \times C$ to $\mathbb{R}$, which satisfies conditions* (A1)–(A4). *Let $\varGamma = F(f) \cap F(T) \cap (\cap_{i=1}^m F(Q_\xi^{M_i})) \cap VI(C,A) \cap VI(C,B) \cap EP(g) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence generated by $x_1 \in C$ and*

$$\begin{aligned}
&v_n \in C\colon\ g(v_n, y) + \langle Av_n,\, y - v_n\rangle \\
&\qquad\qquad + \frac{1}{r_n}\langle y - v_n,\, Jv_n - Jx_n\rangle \geqslant 0 \quad \text{for all } y \in C, \\
&k_n = \varPi_C J^{-1}(Jv_n - \mu Bv_n), \\
&u_n = \varPi_C J^{-1}\Biggl(s_{n,0}Jx_n + \sum_{i=1}^m s_{n,i} JQ_\xi^{M_i} x_n\Biggr), \\
&Q_n = \bigl\{\iota \in C\colon\ \phi(\iota, k_n) \leqslant \phi(\iota, x_n)\bigr\}, \\
&z_n = \varPi_{Q_n} J^{-1}(Ju_n - \mu Bu_n), \qquad y_n = \varPi_C J^{-1}(Jz_n - \lambda Az_n), \\
&w_n = J^{-1}\Biggl(\sum_{i=1}^m \beta_{n,i} JQ_\xi^{M_i} z_n + \sum_{i=1}^m \gamma_{n,i} JQ_\xi^{M_i} y_n\Biggr), \\
&h_n = J^{-1}[\alpha_{n,1}Jfx_n + \alpha_{n,2}Ju_n + \alpha_{n,3}Jk_n + \alpha_{n,4}JTy_n + \alpha_{n,5}Jw_n], \\
&x_{n+1} = \varPi_C h_n,
\end{aligned} \tag{48}$$

where $r_n \in [a, +\infty)$ for some $a > 0$, $\{\beta_{n,i}\}_{i=1}^m$, $\{\gamma_{n,i}\}_{i=1}^m$, and $\{s_{n,i}\}_{i=0}^m$ are real sequences in $[a,b] \subset (0,1)$, $\sum_{i=1}^m(\beta_{n,i} + \gamma_{n,i}) = 1$, and $\sum_{i=0}^m s_{n,i} = 1$. Let μ, λ, and $\{\alpha_{n,i}\}_{i=1}^5$ satisfy the following conditions:

(i) $\{\alpha_{n,i}\}_{i=1}^5 \subset (0,1)$, $\sum_{i=1}^5 \alpha_{n,i} = 1$, $\liminf_{n\to+\infty} \alpha_{n,1}\alpha_{n,2} > 0$, *and* $\liminf_{n\to+\infty} \alpha_{n,2}\alpha_{n,4} > 0$.

(ii) λ *and* μ *are real numbers such that* $0 < \lambda < c^2\alpha/2$ *and* $0 < \mu < c^2\beta/2$*, where* $1/c$ *is the 2-uniformly convexity constant of* X.

Then $\{x_n\}$ *converges strongly to*

$$q = \Pi_{F(T)\cap(\cap_{i=1}^{n} F(Q_\xi^{M_i}))\cap VI(C,A)\cap VI(C,B)\cap EP(g)} \circ f(q).$$

Proof. First, we prove that $\{x_n\}$ is well defined. Let $\kappa \in \Gamma$. It is clear from Lemma 7 and algorithm (48) that $v_n = K_{r_n} x_n$, and hence,

$$\phi(\kappa, v_n) \leqslant \phi(\kappa, x_n). \tag{49}$$

From (6) and Lemma 2 we have that

$$\begin{aligned}\phi(\kappa, k_n) &\leqslant \phi\big(\kappa,\ J^{-1}(Jv_n - \mu B v_n)\big) = V(\kappa,\ Jv_n - \mu B v_n)\\ &\leqslant V(\kappa, Jv_n) - 2\big\langle J^{-1}(Jv_n - \mu B v_n) - \kappa,\ \mu B v_n\big\rangle\\ &= \phi(\kappa, v_n) - 2\mu\langle v_n - \kappa,\ B v_n\rangle\\ &\quad + 2\big\langle J^{-1}(Jv_n - \mu B v_n) - J^{-1}(Jv_n),\ -\mu B v_n\big\rangle. \end{aligned}\tag{50}$$

From the β-inverse strongly monotonicity of B and the fact that $\kappa \in VI(C, B)$ we have that

$$\begin{aligned}-2\mu\langle v_n - \kappa,\ Bv_n\rangle &= -2\mu\langle v_n - \kappa,\ Bv_n - B\kappa\rangle - 2\mu\langle v_n - \kappa,\ B\kappa\rangle\\ &\leqslant -2\mu\beta\|Bv_n - B\kappa\|^2.\end{aligned}\tag{51}$$

By Lemma 4 and the condition $\|Bx\| \leqslant \|Bx - B\kappa\|$ for all $x \in C$, it follows that

$$\begin{aligned}&2\big\langle J^{-1}(Jv_n - \mu Bv_n) - J^{-1}(Jv_n),\ -\mu Bv_n\big\rangle\\ &\quad\leqslant 2\big\|J^{-1}(Jv_n - \mu Bv_n) - J^{-1}(Jv_n)\big\|\|\mu Bv_n\|\\ &\quad\leqslant \frac{4\mu^2}{c^2}\|Bv_n\|^2 \leqslant \frac{4\mu^2}{c^2}\|Bv_n - B\kappa\|^2.\end{aligned}\tag{52}$$

Hence, substituting (51) and (52) into (50), we have that

$$\phi(\kappa, k_n) \leqslant \phi(\kappa, v_n) + 2\mu\left(\frac{2\mu}{c^2} - \beta\right)\|Bv_n - B\kappa\|^2 \leqslant \phi(\kappa, v_n). \tag{53}$$

Therefore, it follows from (49) and (53) that

$$\phi(\kappa, k_n) \leqslant \phi(\kappa, x_n). \tag{54}$$

This shows that $\kappa \in Q_n$, hence $\{x_n\}$ is well defined.

Next, we show that Q_n is a closed and convex subset of C for all $n \in \mathbb{N}$. To this end, using the definition ϕ, it is clear that the inequality $\phi(\iota, k_n) \leqslant \phi(\iota, x_n)$ is equivalent to

$$2\langle \iota,\ Jx_n - Jk_n\rangle \leqslant \|x_n\|^2 - \|k_n\|^2. \tag{55}$$

Hence, it is clear from (55) that Q_n is closed and convex for each $n \in \mathbb{N}$.

Let $\hat{u} \in \Gamma$. Note that using Lemma 2, the inequalities (8), (13), (14), and (16) hold for the algorithm (48). Now, from (3), (14), (16), (49), Lemma 2, the convexity of $\|\cdot\|^2$, and the relatively nonexpansiveness of f and T it follows that

$$\begin{aligned}\phi(\hat{u}, x_{n+1}) &\leqslant \|\hat{u}\|^2 - 2\alpha_{n,1}\langle \hat{u}, Jfx_n\rangle - 2\alpha_{n,2}\langle \hat{u}, Ju_n\rangle \\ &\quad - 2\alpha_{n,3}\langle \hat{u}, Jk_n\rangle - 2\alpha_{n,4}\langle \hat{u}, JTy_n\rangle - 2\alpha_{n,5}\langle \hat{u}, Jw_n\rangle \\ &\quad + \alpha_{n,1}\|fx_n\|^2 + \alpha_{n,2}\|u_n\|^2 + \alpha_{n,3}\|k_n\|^2 \\ &\quad + \alpha_{n,4}\|Ty_n\|^2 + \alpha_{n,5}\|w_n\|^2 \\ &= \alpha_{n,1}\phi(\hat{u}, fx_n) + \alpha_{n,2}\phi(\hat{u}, u_n) + \alpha_{n,3}\phi(\hat{u}, k_n) \\ &\quad + \alpha_{n,4}\phi(\hat{u}, Ty_n) + \alpha_{n,5}\phi(\hat{u}, w_n) \\ &\leqslant \alpha_{1,n}\phi(\hat{u}, x_n) + \alpha_{n,2}\phi(\hat{u}, u_n) + \alpha_{n,3}\phi(\hat{u}, k_n) \\ &\quad + \alpha_{n,4}\phi(\hat{u}, y_n) + \alpha_{n,5}\phi(\hat{u}, w_n) \\ &\leqslant (1 - \alpha_{n,2})\phi(\hat{u}, x_n) + \alpha_{n,2}\phi(\hat{u}, u_n). \end{aligned} \tag{56}$$

By (8) and (56), we have

$$\phi(\hat{u}, x_{n+1}) \leqslant \phi(\hat{u}, x_n).$$

This demonstrates that $\{\phi(\hat{u}, x_n)\}$ is bounded and $\lim_{n\to+\infty} \phi(\hat{u}, x_n)$ exists. It follows from (4) that $\{x_n\}$ is bounded. Therefore, by (8), (12), (13), (49), (53), and the relatively nonexpansiveness of f and T, we conclude that $\{u_n\}$, $\{z_n\}$, $\{y_n\}$, $\{v_n\}$, $\{k_n\}$, $\{fx_n\}$, and $\{Ty_n\}$ are bounded.

Let $r_3 = \sup_n\{\|fx_n\|, \|u_n\|\}$. Hence, by Lemma 5, there exists a continuous strictly increasing convex function $g_{r_3} : [0, 2r_3] \to [0, +\infty)$ such that $g_{r_3}(0) = 0$, and using (8), (14), (16), (54), the convexity of $\|\cdot\|^2$, and the condition relatively nonexpansiveness of f and T, we have

$$\begin{aligned}\phi(\hat{u}, x_{n+1}) &\leqslant \|\hat{u}\|^2 - 2\alpha_{n,1}\langle \hat{u}, Jfx_n\rangle - 2\alpha_{n,2}\langle \hat{u}, Ju_n\rangle - 2\alpha_{n,3}\langle \hat{u}, Jk_n\rangle \\ &\quad - 2\alpha_{n,4}\langle \hat{u}, JTy_n\rangle - 2\alpha_{n,5}\langle \hat{u}, Jw_n\rangle + \alpha_{n,1}\|fx_n\|^2 \\ &\quad + \alpha_{n,2}\|u_n\|^2 + \alpha_{n,3}\|k_n\|^2 + \alpha_{n,4}\|Ty_n\|^2 + \alpha_{n,5}\|w_n\|^2 \\ &\quad - \alpha_{n,1}\alpha_{n,2}g_{r_3}\bigl(\|Jfx_n - Ju_n\|\bigr) \\ &= \alpha_{n,1}\phi(\hat{u}, fx_n) + \alpha_{n,2}\phi(\hat{u}, u_n) + \alpha_{n,3}\phi(\hat{u}, k_n) + \alpha_{n,4}\phi(\hat{u}, Ty_n) \\ &\quad + \alpha_{n,5}\phi(\hat{u}, Jw_n) - \alpha_{n,1}\alpha_{n,2}g_{r_3}\bigl(\|Jfx_n - Ju_n\|\bigr) \\ &\leqslant \alpha_{n,1}\phi(\hat{u}, x_n) + \alpha_{n,2}\phi(\hat{u}, u_n) + \alpha_{n,3}\phi(\hat{u}, k_n) + \alpha_{n,4}\phi(\hat{u}, y_n) \\ &\quad + \alpha_{n,5}\phi(\hat{u}, w_n) - \alpha_{n,1}\alpha_{n,2}g_{r_3}\bigl(\|Jfx_n - Ju_n\|\bigr) \\ &\leqslant \phi(\hat{u}, x_n) - \alpha_{n,1}\alpha_{n,2}g_{r_3}\bigl(\|Jfx_n - Ju_n\|\bigr). \end{aligned} \tag{57}$$

Let $r_4 = \sup_n\{\|u_n\|, \|Ty_n\|\}$. Then, in a similar way as above, there exists a continuous strictly increasing convex function $g_{r_4} : [0, 2r_4] \to [0, +\infty)$ with $g_{r_4}(0) = 0$ such that

$$\phi(\hat{u}, x_{n+1}) \leqslant \phi(\hat{u}, x_n) - \alpha_{n,2}\alpha_{n,4}g_{r_4}\bigl(\|Ju_n - JTy_n\|\bigr). \tag{58}$$

Hence, by (57), we have that

$$\alpha_{n,1}\alpha_{n,2}g_{r_3}\big(\|Jfx_n - Ju_n\|\big) \leqslant \phi(\hat{u}, x_n) - \phi(\hat{u}, x_{n+1}).$$

Now, by condition (i), we have that

$$\lim_{n\to+\infty} g_{r_3}\big(\|Jfx_n - Ju_n\|\big) = 0.$$

Therefore,

$$\begin{aligned} g_{r_3}\Big(\lim_{n\to+\infty}\|Jfx_n - Ju_n\|\Big) &= \lim_{n\to+\infty} g_{r_3}\big(\|Jfx_n - Ju_n\|\big) \\ &= 0 = g_{r_3}(0) \end{aligned}$$

because g_{r_3} is a continuous function. Then

$$\lim_{n\to+\infty}\|fx_n - u_n\| = 0. \tag{59}$$

Similarly, from (58) we obtain that

$$\lim_{n\to+\infty}\|u_n - Ty_n\| = 0. \tag{60}$$

It follows from (5), (59), and (60) that

$$\lim_{n\to+\infty}\phi(u_n, fx_n) = 0, \qquad \lim_{n\to+\infty}\phi(u_n, Ty_n) = 0. \tag{61}$$

Note that equalities (31), (36), and (40) hold for the algorithm (48). From (31) and (36) we have that

$$\lim_{n\to+\infty}\|z_n - x_n\| = 0. \tag{62}$$

Then, using (5), we have that $\lim_{n\to+\infty}\phi(z_n, x_n) = 0$. Since $z_n \in Q_n$, we have $\lim_{n\to}\phi(z_n, k_n) = 0$. Hence, from Lemma 1 we have

$$\lim_{n\to+\infty}\|z_n - k_n\| = 0. \tag{63}$$

Then it follows from (36) that $\lim_{n\to+\infty}\|u_n - k_n\| = 0$. So, by (5), we obtain that

$$\lim_{n\to+\infty}\phi(u_n, k_n) = 0. \tag{64}$$

Now, we conclude from (40), (61), and (64) that

$$\begin{aligned} \phi(u_n, x_{n+1}) \leqslant{}& \|u_n\|^2 - 2\alpha_{n,1}\langle u_n, Jfx_n\rangle - 2\alpha_{n,2}\langle u_n, Ju_n\rangle - 2\alpha_{n,3}\langle u_n, Jk_n\rangle \\ & - 2\alpha_{n,4}\langle u_n, JTy_n\rangle - 2\alpha_{n,5}\langle u_n, Jw_n\rangle + \alpha_{n,1}\|fx_n\|^2 \\ & + \alpha_{n,2}\|u_n\|^2 + \alpha_{n,3}\|k_n\|^2 + \alpha_{n,4}\|Ty_n\|^2 + \alpha_{n,5}\|w_n\|^2 \\ ={}& \alpha_{n,1}\phi(u_n, fx_n) + \alpha_{n,2}\phi(u_n, u_n) + \alpha_{n,3}\phi(u_n, k_n) \\ & + \alpha_{n,4}\phi(u_n, Ty_n) + \alpha_{n,5}\phi(u_n, w_n) \to 0 \quad \text{as } n\to+\infty. \end{aligned}$$

Hence, by Lemma 1, we have

$$\lim_{n\to+\infty}\|u_n - x_{n+1}\| = 0. \tag{65}$$

Now, we obtain from (31) and (65) that

$$\lim_{n\to+\infty} \|x_{n+1} - x_n\| = 0.$$

Therefore, $\{x_n\}$ is a Cauchy sequence. So, $\{x_n\}$ converges strongly to a point $q \in C$. Moreover, by (31), (36), (37), (59), and (60), we conclude that $\{u_n\}$, $\{z_n\}$, $\{y_n\}$, $\{fx_n\}$, and $\{Ty_n\}$ strongly converge to q. It is clear that relations (42)–(47) are valid for algorithm (48). Hence, as in the proof of Theorem 1, we conclude that $q \in VI(C, A) \cap VI(C, B)$.

Next, we show that $q \in F(Q_\xi^{M_i})$ for each $i = 1, 2, \dots, m$. From (7), (14), (56), Lemma 10 and similar to (19), we have that

$$\phi(\hat{u}, x_{n+1}) \leqslant \phi(\hat{u}, x_n) - \alpha_{n,2} \sum_{i=1}^{m} s_{n,i} \phi\big(Q_\xi^{M_i} x_n, x_n\big).$$

Therefore, equality (20) is valid for algorithm (48). So, as in the proof of Theorem 1, we see that $q \in \cap_{i=1}^m F(Q_\xi^{M_i})$.

Next, we prove that $q \in F(f)$. It follows from (31) and (59) that

$$\lim_{n\to+\infty} \|fx_n - x_n\| = 0. \tag{66}$$

Moreover, $x_n \rightharpoonup q$, hence, by (66), we conclude that q is an asymptotically fixed point of f. On the other hand, $q \in \hat{F}(f) = F(f)$ because f is a relatively nonexpansive mapping.

Next, we show that $q \in F(T)$. From (36), (37), and (60) we have

$$\lim_{n\to+\infty} \|Ty_n - y_n\| = 0.$$

Hence q is an asymptotic fixed point of T. Now, since T is a relatively nonexpansive mapping, $\hat{F}(T) = F(T)$. So, $q \in F(T)$.

Finally, we prove that $q \in EP(g)$. From (62) and (63) we obtain that

$$\lim_{n\to+\infty} \|x_n - k_n\| = 0. \tag{67}$$

Let $r_5 = \sup_n\{\|v_n\|, \|x_n\|\}$. Hence, from Lemma 6 there exists a continuous, convex, and strictly increasing function $g_{r_5} : [0, 2r_5] \longrightarrow [0, +\infty)$ such that $g_{r_5}(0) = 0$ and

$$g_{r_5}\big(\|v_n - x_n\|\big) \leqslant \phi(v_n, x_n). \tag{68}$$

Now, by (53), (67), (68), Lemma 7, and the fact that $v_n = K_{r_n} x_n$, we conclude that

$$\begin{aligned}
g_{r_5}\big(\|v_n - x_n\|\big) &\leqslant \phi(v_n, x_n) \leqslant \phi(\hat{u}, x_n) - \phi(\hat{u}, v_n)\\
&\leqslant \phi(\hat{u}, x_n) - \phi(\hat{u}, k_n) = \|x_n\|^2 - \|k_n\|^2 - 2\langle \hat{u},\, Jx_n - Jk_n\rangle\\
&\leqslant \big(\|x_n - k_n\| + \|k_n\|\big)^2 - \|k_n\|^2 - 2\langle \hat{u},\, Jx_n - Jk_n\rangle\\
&\leqslant \|x_n - k_n\|^2 + 2\|k_n\|\|x_n - k_n\| + 2\|\hat{u}\|\|Jx_n - Jk_n\|\\
&\to 0 \quad \text{as } n \to +\infty.
\end{aligned}$$

Since g_{r_5} is a continuous strictly increasing convex function, $\|v_n - x_n\| \to 0$ as $n \to +\infty$. So,

$$\lim_{n\to+\infty} \|Jv_n - Jx_n\| = 0. \tag{69}$$

From $v_n = K_{r_n} x_n$ we have that $g(v_n, y) + \langle Av_n,\, y - v_n\rangle + \langle y - v_n,\, Jv_n - Jx_n\rangle / r_n \geqslant 0$ for all $y \in C$. Moreover, by condition (A2), $g(y, v_n) \leqslant -g(v_n, y)$ for all $y \in C$. Therefore,

$$g(y, v_n) \leqslant \langle Av_n,\, y - v_n\rangle + \frac{1}{r_n}\langle y - v_n,\, Jv_n - Jx_n\rangle$$

for all $y \in C$. Letting $n \to +\infty$ and using (69) with condition (A4), we obtain

$$g(y, q) \leqslant \langle Aq,\, y - q\rangle \tag{70}$$

for all $y \in C$. Assume that $y_t = ty + (1 - t)q$ for all $y \in C$ and $t \in (0, 1)$. By (70), conditions (A1), (A4), the convexity of g, and the monotonicity of A, we have that

$$\begin{aligned}
0 &= g(y_t, y_t) + \langle Ay_t,\, y_t - y_t\rangle \\
&\leqslant tg(y_t, y) + (1 - t)g(y_t, q) + \big\langle Ay_t,\, ty + (1 - t)q - y_t\big\rangle \\
&= tg(y_t, y) + (1 - t)g(y_t, q) + t\langle Ay_t,\, y - y_t\rangle + (1 - t)\langle Ay_t,\, q - y_t\rangle \\
&= tg(y_t, y) + (1 - t)g(y_t, q) + t\langle Ay_t,\, y - y_t\rangle \\
&\quad + (1 - t)\langle Ay_t - Aq,\, q - y_t\rangle + (1 - t)\langle Aq,\, q - y_t\rangle \\
&\leqslant tg(y_t, y) + t\langle Ay_t,\, y - y_t\rangle
\end{aligned}$$

for all $y \in C$. Hence $0 \leqslant g(y_t, y) + \langle Ay_t,\, y - y_t\rangle$. Letting $t \to 0$ and using condition (A3), we conclude that $0 \leqslant g(q, y) + \langle Aq,\, y - q\rangle$ for all $y \in C$. Then $q \in EP(g)$. Therefore, $q = \Pi_{F(T)\cap(\cap_{i=1}^m F(Q_\xi^{M_i}))\cap VI(C,A)\cap VI(C,B)\cap EP(g)} \circ f(q)$, and this completes the proof. □

4 Numerical example and remark

Remark. If $A = B = kI$ for a real number $k \geqslant 0$, $X = \mathbb{R}$, and C is a nonempty closed and convex subset of X, then $\Gamma = \{0\}$ is the only case that $|Ax| \leqslant |Ax - Au|$ and $|Bx| \leqslant |Bx - Bu|$ for all $x \in C$ and $u \in \Gamma$. If also, $A = B = 0$ and Γ is an arbitrary subset of C, then, obviously, the above conditions hold. We also refer the readers to [9, p. 3686, Remark 3.4].

The following example illustrates the behavior of algorithm (48) of Theorem 2.

Example 1. Let $X = \mathbb{R}$, $C = [-5, 5]$, $A = B = I$, $\mu = \lambda = 1/3$, $c = 1$, $\alpha = \beta = 1$. Suppose that f and T are self-mappings on C defined by $f(x) = T(x) = x/3$ for all $x \in C$. Consider the function $g : C \times C \to \mathbb{R}$ defined by

$$g(u, y) := 12y^2 + 9uy - 21u^2$$

for all $u, y \in C$. It is clear that conditions (A1)–(A4) are satisfied. Suppose that $x \in X$, $r > 0$, and $v \in K_r x$. Then, by Lemma 7, it follows that

$$g(v, y) + \langle Av,\, y - v\rangle + \frac{1}{r}\langle y - v,\, Jv - Jx\rangle \geqslant 0$$

for all $y \in C$, i.e.,

$$\begin{aligned} 0 &\leqslant 12ry^2 + 9rvy - 21rv^2 + rvy - rv^2 + vy - v^2 + vx - xy \\ &= 12ry^2 + (10rv + v - x)y - 22rv^2 - v^2 + vx. \end{aligned}$$

Let $a = 12r$, $b = 10rv + v - x$ and $c = -22rv^2 - v^2 + vx$. Then we have that $\Delta = b^2 - 4ac \leqslant 0$, i.e.,

$$\begin{aligned} 0 &\geqslant (10rv + v - x)^2 - 48r\bigl(-22rv^2 - v^2 + vx\bigr) \\ &= 1156r^2v^2 + 68rv^2 + v^2 - 68rvx - 2vx + x^2 \\ &= \bigl((34r + 1)v - x\bigr)^2. \end{aligned}$$

It follows that $v = x/(34r + 1)$. Hence $K_r x = x/(34r + 1)$. Now, by Theorem 2, we obtain that $v_n = x_n/(34r_n + 1)$. Since $F(K_{r_n}) = \{0\}$, from Lemma 7 we have $EP(g) = \{0\}$.

Obviously, $F(f) = \{0\}$ and $\phi(0, f(x)) \leqslant \phi(0, x)$ for all $x \in C$. It is clear that $\hat{F}(f) = \{0\} = F(f)$. Therefore, f is a relatively nonexpansive mapping. Similarly, T is a relatively nonexpansive mapping. Moreover, it is obvious that $0 \in VI(C, I)$. Now, we define $M_i : \mathbb{R} \to 2^{\mathbb{R}}$ by $M_i x = \{2x\}$ for each $i = 1, 2, \dots, m$ and $\xi = 1/2$, hence $Q_\xi^{M_i} = x/2$ for each $i = 1, 2, \dots, m$. Clearly, $0 \in F(Q_\xi^{M_i})$ for each $i = 1, 2, \dots, m$. Therefore,

$$0 = \Pi_{\{0\}} \circ f(0) = \Pi_{F(T)\cap VI(C,A)\cap VI(C,B)\cap(\cap_{i=1}^m F(Q_\xi^{M_i})\cap EP(g)} \circ f(0).$$

Next, we assume that $m = 3$. For each $x \in X$, define the mapping $M_i : \mathbb{R} \to 2^{\mathbb{R}}$ by $M_i x = \{2x\}$ and let $\xi = 1/2$, hence $Q_\xi^{M_i} = x/2$. We choose $\alpha_{n,1} = 1/5 + 1/(8n)$, $\alpha_{n,2} = 1/5 - 1/(6n)$, $\alpha_{n,3} = 1/5 + 1/(12n)$, $\alpha_{n,4} = 1/5 - 1/(6n)$, $\alpha_{n,5} = 1/5 + 1/(8n)$ and $s_{n,0} = s_{n,1} = s_{n,2} = s_{n,3} = 1/4$, $\beta_{n,1} = \beta_{n,2} = \beta_{n,3} = \gamma_{n,1} = \gamma_{n,2} = \gamma_{n,3} = 1/6$, $r_n = 1/34$ for all $n \in \mathbb{N}$ and $v_0 = 0$. Therefore, $\{\alpha_{n,i}\}_{i=1}^5$ satisfies the conditions of Theorem 2. We know that $x_n \in C$, hence

$$\begin{aligned} k_n &= \frac{1}{3}x_n, \qquad u_n = \frac{5}{8}x_n, \qquad Q_n = \bigl\{\iota \in C\colon\, |\iota - k_n| \leqslant |\iota - x_n|\bigr\}, \\ z_n &= \frac{5}{12}x_n, \qquad y_n = \frac{5}{18}x_n, \qquad w_n = \frac{25}{144}x_n, \\ x_{n+1} &= \left(\frac{1}{5} + \frac{1}{8n}\right)\frac{1}{3}x_n + \left(\frac{1}{5} - \frac{1}{6n}\right)\frac{5}{8}x_n + \left(\frac{1}{5} + \frac{1}{12n}\right)\frac{5}{8}x_n \\ &\quad + \left(\frac{1}{5} - \frac{1}{6n}\right)\left(\frac{1}{3}\right)\frac{5}{18}x_n + \left(\frac{1}{5} + \frac{1}{8n}\right)\frac{25}{144}x_n. \end{aligned}$$

See Fig. 1 for the value $x_1 = 3$.

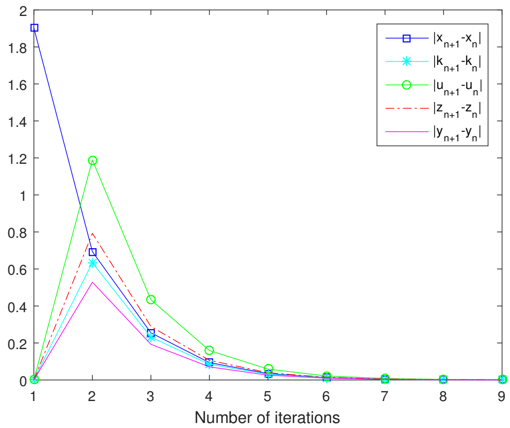

Figure 1. Convergence behavior of generated sequences by Example 1.

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References

1. R.P. Agarwal, D. O'Regan, D.R. Sahu, *Fixed Point Theory for Lipschitzian-type Mappings with Applications*, Springer, New York, 2009, `https://doi.org/10.1007/978-0-387-75818-3`.
2. Y.I. Alber, Metric and generalized projection operators in Banach spaces: Properties and applications, in *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, Lect. Notes Pure Appl. Math., Vol. 178, Marcel Dekker, New York, 1996, pp. 15–50, `https://doi.org/10.48550/arXiv.funct-an/9311001`.
3. G. Cai, S. Bu, Weak convergence theorems for general equilibrium problems and variational inequality problems and fixed point problems in Banach spaces, *Acta Math. Sci., Ser. B, Engl. Ed.*, **33**(1):302–320, 2013, `https://doi.org/10.1016/S0252-9602(12)60212-4`.
4. G. Cai, Y. Shehu, O.S. Iyiola, Strong convergence results for variational inequalities and fixed point problems using modified viscosity implicit rules, *Numer. Algorithms*, **77**(2):535–558, 2018, `https://doi.org/10.1007/s11075-017-0327-8`.
5. Y. Cens, S. Reich, Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization, *Optimization*, **37**(4):323–339, 1996, `https://doi.org/10.1080/02331939608844225`.
6. G.Z. Eskandani, M. Raeisi, T.M. Rassias, A hybrid extragradient method for solving pseudo-monotone equilibrium problems using Bregman distance, *J. Fixed Point Theory Appl.*, **20**(3): 132, 2018, `https://doi.org/10.1007/s11784-018-0611-9`.

7. M. Ghadampour, D. O'Regan, E. Soori, R.P. Agarwal, A generalized strong convergence algorithm in the presence of errors for variational inequality problems in Hilbert spaces, *J. Funct. Spaces*, **2021**:9911241, 2021, https://doi.org/10.1155/2021/9911241.
8. M. Ghadampour, E. Soori, R.P. Agarwal, D. O'Regan, Two generalized strong convergence algorithms for variational inequality problems in Banach spaces, *Fixed Point Theory*, **25**(1): 143–162, 2024, https://doi.org/10.24193/fpt-ro.2024.1.09.
9. H. Iiduka, W. Takahashi, Strong convergence by hybrid type method for monotone operators in a Banach space, *Nonlinear Anal., Theory Methods Appl.*, **68**(12):3679–3688, 2008, https://doi.org/10.1016/j.na.2007.04.010.
10. L.O. Jolaoso, A. Taiwo, T.O. Alakoya, O.T. Mewomo, A strong convergence theorem for solving pseudo-monotone variational inequalities using projection methods, *J. Optim. Theory Appl.*, **185**:744–766, 2020, https://doi.org/10.1007/s10957-020-01672-3.
11. Z. Jouymandi, F. Moradlou, Extragradient methods for split feasibility problems and generalized equilibrium problems in Banach spaces, *Math. Methods Appl. Sci.*, **41**(2):826–838, 2018, https://doi.org/10.1002/mma.4647.
12. S. Kamimura, W. Takahashi, Strong convergence of a proximal-type algorithm in a Banach space, *SIAM J. Optim.*, **13**(3):938–945, 2003, https://doi.org/10.1137/S105262340139611X.
13. F. Kohsaka, W. Takahashi, Strong convergence of an iterative sequence for maximal monotone operators in a Banach space, *Abstr. Appl. Anal.*, **2004**(3):239–249, 2004, https://doi.org/10.1155/S1085337504309036.
14. G.M. Korpelevich, The extragradient method for finding saddle points and other problems, *Ekon. Mat. Metody*, **12**(4):747–756, 1976.
15. B.M. Lashkarizadeh, E. Soori, Strong convergence of a general implicit algorithm for variational inequality problems and equilibrium problems and a continuous representation of nonexpansive mappings, *Bull. Iran. Math. Soc.*, **40**(4):977–1001, 2014.
16. Y.-C. Liou, Shrinking projection method of proximal-type for a generalized equilibrium problem, a maximal monotone operator and a pair of relatively nonexpansive mappings, *Taiwanese J. Math.*, **14**(2):517–540, 2010, https://doi.org/10.11650/twjm/1500405805.
17. B. Orouji, D. O'Regan, E. Soori, R.P. Agarwal, The split common null point problem for generalized resolvents and nonexpansive mappings in Banach spaces, *Taiwanese J. Math.*, **23**(1): 165–183, 2022.
18. S. Reich, Book review: Geometry of Banach spaces, duality mappings and nonlinear problems, *Bull. Am. Math. Soc.*, **26**(2):367–370, 1992, https://doi.org/10.1090/S0273-0979-1992-00287-2.
19. R.T. Rockafellar, Monotone operators and the proximal point algorithm, *Bull. Am. Math. Soc.*, **14**(5):877–898, 1976, https://doi.org/10.1137/0314056.
20. A. Tada, W. Takahashi, Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem, *J. Optim. Theory Appl.*, **133**(3):359–370, 2007, https://doi.org/10.1007/s10957-007-9187-z.
21. S. Takahashi, W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, *J. Math. Anal. Appl.*, **331**(1):506–515, 2007, https://doi.org/10.1016/j.jmaa.2006.08.036.

22. W. Takahashi, *Nonlinear Functional Analysis: Fixed Point Theory and Its Applications*, Yokohama Publishers, Yokohama, 2000.
23. Y. Takahashi, K. Hashimoto, M. Kato, On sharp uniform convexity, smoothness, and strong type, cotype inegualities, *J. Nonlinear Convex Anal.*, **3**(2):267–281, 2002.
24. D.V. Thong, D.V. Hieu, Some extragradient-viscosity algorithms for solving variational inequality problems and fixed point problems, *Numer. Algorithms*, **82**(3):761–789, 2019, https://doi.org/10.1007/s11075-018-0626-8.
25. L. Wei, Y.F. Su, H.Y. Zhou, New iterative schemes for strongly relatively nonexpansive mappings and maximal monotone operators, *Appl. Math., Ser. B (Engl. Ed.)*, **25**(2):199–208, 2010, https://doi.org/10.1007/s11766-010-2195-z.
26. L. Wei, H.Y. Zhou, The new iterative scheme with errors of zero point for maximal monotone operator in Banach space, *Math. Appl.*, **19**(1):101–105, 2006.
27. H.-K. Xu, Inequalities in banach spaces with applications, *Nonlinear Anal., Theory Methods Appl.*, **16**(12):1127–1138, 2009, https://doi.org/10.1016/0362-546X(91)90200-K.