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Existence and stability results for triple systems of fractional Sturm–Liouville–Langevin equations with cyclic boundary conditions*

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Abstract. We investigate a triple system of fractional Sturm–Liouville–Langevin equations with cyclic antiperiodic boundary conditions. The fixed point theorem serves as a tool to establish the existence and uniqueness criteria for solutions. By applying the Banach contraction principle, we also obtain the Ulam–Hyers stability of the proposed system. Finally, examples are provided to illustrate main results.

Keywords: fractional Sturm–Liouville–Langevin equation, triple system, existence and uniqueness, Ulam-type stability.

1 Introduction

In this paper, we study a new triple system of nonlinear fractional equations

$$ {}^{C}D^{\beta}\big[\big(p(s)^{C}D^{\alpha}+q(s)\big)x_i(s)+r(s)I^{\theta}g_i\big(s,x_i(s)\big)\big]=f_i\big(s,x_1(s),x_2(s),x_3(s)\big) \quad (1) $$

under cyclic antiperiodic boundary conditions

$$ \begin{aligned} x_1(a)+x_2(b)&=0, & {}^{C}D^{\alpha}x_1(a)+{}^{C}D^{\alpha}x_2(b)&=0,\\ x_2(a)+x_3(b)&=0, & {}^{C}D^{\alpha}x_2(a)+{}^{C}D^{\alpha}x_3(b)&=0,\\ x_3(a)+x_1(b)&=0, & {}^{C}D^{\alpha}x_3(a)+{}^{C}D^{\alpha}x_1(b)&=0, \end{aligned} \quad (2) $$

where $s\in[a,b]$, ${}^{C}D^{\alpha}$ and ${}^{C}D^{\beta}$ denote the Caputo fractional derivative of orders α and β, $0<\alpha,\beta<1$, I^{θ} denotes the Riemann–Liouville fractional integral of order $\theta>0$,

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$p \in C([a,b], \mathbb{R}^+ \backslash \{0\})$, $q, r \in C([a,b], \mathbb{R}^+)$, $f_i : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$, and $g_i : [a,b] \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2, 3$, are given functions.

In recent years, due to the rapid iteration of computer technology and the continuous in-depth research on fractional calculus theory, it has been discovered that fractional calculus is effective in describing physical processes with memory and historical significance. In addition, fractional calculus is widely applied in numerous fields, such as anomalous diffusion [17], fluid mechanics [25], signal processing and control [26], image processing, magnetic resonance imaging [22,23], soft matter research [10], seismic analysis [16], and so on. The self-adjoint property of second-order linear Sturm–Liouville equation (SLE) has been applied in many fields. In quantum mechanics, it ensures that the energy of the Hamiltonian operator is a real number, which is very important to describe the stability of the physical system. The Langevin equation (LE) is a differential equation used to describe stochastic processes. Its inherent randomness has proven to have significant value in describing the dynamic behavior of complex systems. It is worth noting that fractional Sturm–Liouville–Langevin equations (FSLLEs) not only incorporate the self-adjoint properties of SLE and the randomness of LE, but also introduce the nonlocal properties of fractional calculus. These characteristics enable FSLLEs to more accurately describe dynamic behavior in complex systems, especially when dealing with systems with non integer dimensions or nonlocal interactions. In [5], Baleanu et al. studied the coupled FSLEEs with nonlocal boundary conditions and proved the existence result. In [8], the author obtained the existence result of FSLLEs by applying Kuratowski noncompactness measure method. For more detailed information on the existence, stability, and multiplicity of solutions, and on numerical methods for obtaining them, the reader is referred to [2, 6, 7, 9, 13, 20, 21].

On the other hand, cyclic boundary conditions are frequently employed in numerical simulations and computations, particularly when dealing with problems exhibiting periodicity or cyclic characteristics. These conditions require that the values of physical quantities at the boundaries of the simulation region be equal to their corresponding values at the opposite boundaries or satisfy some specific relationship in order to simulate infinite or periodic systems. Under cyclic boundary conditions, the author studied the analytical solution of the mathematical model with relaxation and temperature damping characteristics in [15]. The authors present existence results for solutions to three ordinary differential equations with cyclic boundary conditions, as detailed in [1]. In [19], Matar and Amra discussed fractional triple abstract systems and presented results on existence and uniqueness within cyclic boundary conditions. In a recent work [27], Zhang et al. considered the existence result and Ulam-type stability of triple systems of the fractional Langevin equation with cyclic antiperiodic boundary conditions. For related work on triple system models involving different types of differential equations and numerous boundary conditions, please refer to articles [3, 4, 11, 12, 18, 24].

The novelty of the current work is outlined as follows:

(i) A triple system of FSLEEs with cyclic antiperiodic boundary conditions is proposed. We discuss the existence and uniqueness results, also we analyze Ulam–Hyers stability.

(ii) Problem (1)–(2) is more general than those considered previously. Our work extends the results in [5–9, 13, 21] to a tripled system of FSLLEs, while the problem discussed in [27] appears as a special case.

(iii) FSLLEs are nonlocal equations, which complicates a priori estimation.

(iv) Problem (1)–(2) illustrates coupling relationships among the equations as well as the duality relationships among the boundary conditions.

Section 2 provides background knowledge of fractional integrals and derivatives. In Section 3, the existence and uniqueness results for problem (1)–(2) have been established. Section 4 presents analytical techniques for studying the Ulam–Hyers stability. Finally, numerical examples are presented to illustrate the obtained results.

2 Preliminaries

Definition 1. (See [27].) The Riemann–Liouville fractional integral of order α for a function $f:[a,b]\to\mathbb{R}$ is defined by

$$I^{\beta} f(s) = \frac{1}{\Gamma(\alpha)} \int_a^s (s-\mu)^{\alpha-1} f(\mu)\,\mathrm{d}\mu, \quad s > a,$$

provided the integral exists.

Definition 2. (See [27].) Let $\alpha > 0$, $n = [\alpha] + 1$. The Caputo fractional derivative of order α for a function $f(s) \in AC^n[0,\infty)$ is given by

$${}^{C}D^{\alpha} f(s) = \frac{1}{\Gamma(n-\alpha)} \int_a^s (s-\mu)^{n-\alpha-1} f^{(n)}(\mu)\,\mathrm{d}\mu \quad s > a,$$

where $[\alpha]$ denotes the integer part of the real number α.

Lemma 1. (*See* [27].) *Let* $\alpha > 0$*,* $n = [\alpha] + 1$ *and* $f \in AC^n[a,b]$*. Then*

$$I^{\alpha\,C}D^{\alpha} f(s) = f(s) + \sum_{i=0}^{n-1} c_i s^i, \quad a < s < b,$$

where $c_0, c_1, \dots, c_{n-1} \in \mathbb{R}$.

Lemma 2. (*See* [27].) *Let* $\alpha, \beta > 0$ *and* $f \in C(a,b)$*. Then*

$$\bigl(I^{\alpha} I^{\beta} f\bigr)(s) = (I^{\alpha+\beta} f)(s), \qquad I^{\alpha} s^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} s^{\alpha+\beta-1},$$

$${}^{C}D^{\alpha} I^{\alpha} f(s) = f(s), \qquad {}^{C}D^{\alpha} s^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} s^{\beta-\alpha-1}.$$

Now, we propose an auxiliary lemma to analyze problem (1)–(2).

Lemma 3. *Let* $h_i \in C([a,b],\mathbb{R})$*,* $i = 1,2,3$*,* $1 < \alpha + \beta < 2$*. Then the solution of tripled system*

$${}^{C}D^{\beta}\bigl[\bigl(p(s)^{C}D^{\alpha} + q(s)\bigr)x_i(s) + r(s) I^{\theta} g_i(s)\bigr] = h_i(s), \quad s \in [a,b], \tag{3}$$

subject to BVPs (2)*, is defined by*

$$\begin{aligned}
x_1(s) = {} & \frac{1}{\Gamma(\alpha)} \int_a^s (s-\mu)^{\alpha-1} \frac{1}{p(\mu)} \left(\frac{1}{\Gamma(\beta)} \int_a^\mu (\mu-\nu)^{\beta-1} h_1(\nu)\,\mathrm{d}\nu \right) \mathrm{d}\mu \\
& + \frac{1}{2\Gamma(\beta)} \int_a^b (b-\mu)^{\alpha-1} \bigl[V_1 h_2(\mu) - V_3 h_1(\mu) - V_2 h_3(s)\bigr]\,\mathrm{d}\mu \\
& + \frac{1}{2\Gamma(\alpha)} \int_a^b (b-\mu)^{\alpha-1} \frac{q(\mu)}{p(\mu)} \bigl[x_1(\mu) + x_2(\mu) - x_3(\mu)\bigr]\,\mathrm{d}\mu \\
& - \frac{r(s)}{\Gamma(\alpha)} \int_a^s (s-\mu)^{\alpha-1} \frac{1}{p(\mu)} \left(\frac{1}{\Gamma(\theta)} \int_a^\mu (\mu-\nu)^{\theta-1} g_1(v)\,\mathrm{d}\nu \right) \mathrm{d}\mu \\
& + \frac{r(s)}{2\Gamma(\theta)} \int_a^b (b-\mu)^{\theta-1} \bigl[V_3 g_1(\mu) + V_2 g_3(\mu) - V_1 g_2(\mu)\bigr]\,\mathrm{d}\mu \\
& + \frac{r(s)(s-a)^\alpha}{p(s)\Gamma(\alpha+1)\Gamma(\theta)} \int_a^b (b-\mu)^{\theta-1} \bigl[w_1 g_1(\mu) + w_2 g_2(\mu) - w_3 g_3(\mu)\bigr]\,\mathrm{d}\mu \\
& + \frac{(s-a)^\alpha}{p(s)\Gamma(\alpha+1)\Gamma(\beta)} \int_a^b (b-\mu)^{\beta-1} \bigl[w_3 h_3(\mu) - w_1 h_1(\mu) - w_2 h_2(\mu)\bigr]\,\mathrm{d}\mu \\
& + \frac{r(s)}{2\Gamma(\alpha)} \int_a^b (b-\mu)^{\alpha-1} \frac{1}{p(\mu)} \left(\frac{1}{\Gamma(\theta)} \int_a^b (\mu-\nu)^{\theta-1} \bigl[g_1(\nu)+g_2(\nu)-g_3(\nu)\bigr]\,\mathrm{d}\nu \right) \mathrm{d}\mu \\
& + \frac{1}{2\Gamma(\alpha)} \int_a^b (b-\mu)^{\alpha-1} \frac{1}{p(\mu)} \left(\frac{1}{\Gamma(\beta)} \int_a^\mu (\mu-\nu)^{\beta-1} \bigl[h_3(\nu)-h_1(\nu)-h_2(\nu)\bigr]\,\mathrm{d}\nu \right) \mathrm{d}\mu \\
& - \frac{1}{\Gamma(\alpha)} \int_a^s (s-\mu)^{\alpha-1} \frac{q(\mu)}{p(\mu)} x_1(\mu)\,\mathrm{d}\mu, \qquad (4)
\end{aligned}$$

$$\begin{aligned}
x_2(s) = {} & \frac{1}{\Gamma(\alpha)} \int_a^s (s-\mu)^{\alpha-1} \frac{1}{p(\mu)} \left(\frac{1}{\Gamma(\beta)} \int_a^\mu (\mu-\nu)^{\beta-1} h_2(\nu)\,\mathrm{d}\nu \right) \mathrm{d}\mu \\
& + \frac{1}{2\Gamma(\beta)} \int_a^b (b-\mu)^{\beta-1} \bigl[V_1 h_3(\mu) - V_3 h_2(\mu) - V_2 h_1(\mu)\bigr]\,\mathrm{d}\mu \\
& + \frac{1}{2\Gamma(\alpha)} \int_a^b (b-\mu)^{\alpha-1} \frac{q(\mu)}{p(\mu)} \bigl[x_2(\mu) + x_3(\mu) - x_1(\mu)\bigr]\,\mathrm{d}\mu \\
& - \frac{r(s)}{\Gamma(\alpha)} \int_a^s (s-\mu)^{\alpha-1} \frac{1}{p(s)} \left(\frac{1}{\Gamma(\theta)} \int_a^\mu (\mu-\nu)^{\theta-1} g_2(\nu)\,\mathrm{d}\nu \right) \mathrm{d}\mu
\end{aligned}$$

$$+\frac{r(s)}{2\Gamma(\theta)}\int_a^b(b-\mu)^{\theta-1}\big[V_2g_1(\mu)+V_3g_2(\mu)-V_1g_3(\mu)\big]\,\mathrm{d}\mu$$

$$+\frac{r(s)(s-a)^\alpha}{p(s)\Gamma(\alpha+1)\Gamma(\theta)}\int_a^b(b-\mu)^{\theta-1}\big[w_1g_2(\mu)+w_2g_3(\mu)-w_3g_1(\mu)\big]\,\mathrm{d}\mu$$

$$+\frac{(s-a)^\alpha}{p(s)\Gamma(\alpha+1)\Gamma(\beta)}\int_a^b(b-\mu)^{\beta-1}\big[w_3h_1(\mu)-w_1h_2(\mu)-w_2h_3(\mu)\big]\,\mathrm{d}\mu$$

$$+\frac{r(s)}{2\Gamma(\alpha)}\int_a^b(b-\mu)^{\alpha-1}\frac{1}{p(\mu)}\left(\frac{1}{\Gamma(\theta)}\int_a^\mu(\mu-\nu)^{\theta-1}\big[g_2(\nu)+g_3(\nu)-g_1(\nu)\big]\,\mathrm{d}\nu\right)\mathrm{d}\mu$$

$$+\frac{1}{2\Gamma(\alpha)}\int_a^b(b-\mu)^{\alpha-1}\frac{1}{p(\mu)}\left(\frac{1}{\Gamma(\beta)}\int_a^\mu(\mu-\nu)^{\beta-1}\big[h_1(\nu)-h_2(\nu)-h_3(\nu)\big]\,\mathrm{d}\nu\right)\mathrm{d}\mu$$

$$-\frac{1}{\Gamma(\alpha)}\int_a^s(s-\mu)^{\alpha-1}\frac{q(\mu)}{p(\mu)}x_2(\mu)\,\mathrm{d}\mu, \tag{5}$$

$$x_3(s)=\frac{1}{\Gamma(\alpha)}\int_a^s(s-\mu)^{\alpha-1}\frac{1}{p(\mu)}\left(\frac{1}{\Gamma(\beta)}\int_a^\mu(\mu-\nu)^{\beta-1}h_3(\nu)\,\mathrm{d}\nu\right)\mathrm{d}\mu$$

$$+\frac{1}{2\Gamma(\beta)}\int_a^b(b-\mu)^{\beta-1}\big[V_1h_1(\mu)-V_2h_2(\mu)-V_3h_3(\mu)\big]\,\mathrm{d}\mu$$

$$+\frac{1}{2\Gamma(\alpha)}\int_a^b(b-\mu)^{\alpha-1}\frac{q(\mu)}{p(\mu)}\big[x_1(\mu)+x_3(\mu)-x_2(\mu)\big]\,\mathrm{d}\mu$$

$$-\frac{r(s)}{\Gamma(\alpha)}\int_a^s(s-\mu)^{\alpha-1}\frac{1}{p(\mu)}\left(\frac{1}{\Gamma(\theta)}\int_a^\mu(\mu-\nu)^{\theta-1}g_3(\nu)\,\mathrm{d}\nu\right)\mathrm{d}\mu$$

$$+\frac{r(s)}{2\Gamma(\theta)}\int_a^b(b-\mu)^{\theta-1}\big[V_2g_2(\mu)+V_3g_3(\mu)-V_1g_1(\mu)\big]\,\mathrm{d}\mu$$

$$+\frac{r(s)(s-a)^\alpha}{p(s)\Gamma(\alpha+1)\Gamma(\theta)}\int_a^b(b-\mu)^{\theta-1}\big[w_1g_3(\mu)+w_2g_1(\mu)-w_3g_2(\mu)\big]\,\mathrm{d}\mu$$

$$+\frac{(s-a)^\alpha}{p(s)\Gamma(\alpha+1)\Gamma(\beta)}\int_a^b(b-\mu)^{\beta-1}\big[w_3h_2(\mu)-w_1h_3(\mu)-w_2h_1(\mu)\big]\,\mathrm{d}\mu$$

$$+\frac{r(s)}{2\Gamma(\alpha)}\int_a^b(b-\mu)^{\alpha-1}\frac{1}{p(\mu)}\left(\frac{1}{\Gamma(\theta)}\int_a^\mu(\mu-\nu)^{\theta-1}\big[g_1(\nu)+g_3(\nu)-g_2(\nu)\big]\,\mathrm{d}\nu\right)\mathrm{d}\mu$$

$$+ \frac{1}{2\Gamma(\alpha)} \int_a^b (b-\mu)^{\alpha-1} \frac{1}{p(\mu)} \left(\frac{1}{\Gamma(\beta)} \int_a^\mu (\mu-\nu)^{\beta-1} \big[h_2(\nu) - h_1(\nu) - h_3(\nu)\big] \,\mathrm{d}\nu \right) \mathrm{d}\mu$$
$$- \frac{1}{\Gamma(\alpha)} \int_a^s (s-\mu)^{\alpha-1} \frac{q(\mu)}{p(\mu)} x_3(\mu) \,\mathrm{d}\mu. \tag{6}$$

Proof. By applying operator I^β to both sides of (3), we get

$$\big[p(s)^C D^\alpha + q(s)\big] x_i(s) + r(s) I^\theta g_i(s) = I^\beta h_i(s) + c_0^i, \quad c_0^i \in \mathbb{R},$$

and

$$^C D^\alpha x_i(s) = \frac{1}{p(s)} I^\beta h_i(s) - \frac{q(s)}{p(s)} x_i(s) - \frac{r(s)}{p(s)} I^\theta g_i(s) + \frac{c_0^i}{p(s)}. \tag{7}$$

Using the operator I^α to (7), then

$$x_i(s) = I^\alpha \frac{1}{p(s)} I^\beta h_i(s) - I^\alpha \frac{q(s)}{p(s)} x_i(s) - I^\alpha \frac{r(s)}{p(s)} I^\alpha g_i(s)$$
$$+ \frac{(s-a)^\alpha}{p(s)\Gamma(\alpha+1)} c_0^i + c_1^i, \quad c_0^i, c_1^i \in \mathbb{R},\ i = 1, 2, 3. \tag{8}$$

From (7) and (8) we get

$$^C D^\alpha x_i(a) = \frac{c_0^i}{p(a)} - \frac{q(a)}{p(a)} x_i(a), \tag{9}$$

$$^C D^\alpha x_i(b) = \frac{1}{p(b)} I^\beta h_i(s)\big|_{s=b} - \frac{q(b)}{p(b)} x_i(b) + \frac{c_0^i}{p(b)} - \frac{r(s)}{p(b)} I^\theta g_i(s)\big|_{s=b}, \tag{10}$$

$$x_i(a) = c_1^i, \tag{11}$$

$$x_i(b) = I^\alpha \left(\frac{1}{p(s)} I^\beta h_i(s) \right)\bigg|_{s=b} - I^\alpha \left(\frac{q(s)}{p(s)} x_i(s) \right)\bigg|_{s=b} \tag{12}$$

$$+ \frac{(b-a)^\alpha}{p(b)\Gamma(\alpha+1) c_0^i} - I^\alpha \left(\frac{r(s)}{p(s)} I^\theta g_i(s) \right)\bigg|_{s=b} + c_1^i. \tag{13}$$

Substituting (9)–(13) into BVPs (2), we obtain

$$\frac{1}{p(a)} c_0^1 + \frac{1}{p(b)} c_0^2 = A, \qquad c_1^1 + \frac{(b-a)^\beta}{p(b)\Gamma(\alpha+1)} c_0^2 + c_1^2 = D,$$
$$\frac{1}{p(a)} c_0^2 + \frac{1}{p(b)} c_0^3 = B, \qquad c_1^2 + \frac{(b-a)^\beta}{p(b)\Gamma(\alpha+1)} c_0^3 + c_1^3 = E,$$
$$\frac{1}{p(a)} c_0^3 + \frac{1}{p(b)} c_0^1 = C, \qquad c_1^3 + \frac{(b-a)^\beta}{p(b)\Gamma(\alpha+1)} c_0^1 + c_1^1 = F,$$

where

$$A = -\frac{1}{p(b)} I^{\beta} h_2(s)\big|_{s=b} + \frac{r(s)}{p(b)} I^{\theta} g_2(s)\big|_{s=b},$$
$$B = -\frac{1}{p(b)} I^{\beta} h_3(s)\big|_{s=b} + \frac{r(s)}{p(b)} I^{\theta} g_3(s)\big|_{s=b},$$
$$C = -\frac{1}{p(b)} I^{\beta} h_1(s)\big|_{s=b} + \frac{r(s)}{p(b)} I^{\theta} g_1(s)\big|_{s=b},$$
$$D = I^{\alpha}\left(\frac{q(s)}{p(s)} x_2(s)\right)\bigg|_{s=b} + I^{\alpha}\left(\frac{r(s)}{p(s)} I^{\theta} g_2(s)\right)\bigg|_{s=b} - I^{\alpha}\left(\frac{1}{p(s)} I^{\beta} h_2(s)\right)\bigg|_{s=b},$$
$$E = I^{\alpha}\left(\frac{q(s)}{p(s)} x_3(s)\right)\bigg|_{s=b} + I^{\alpha}\left(\frac{r(s)}{p(s)} I^{\theta} g_3(s)\right)\bigg|_{s=b} - I^{\alpha}\left(\frac{1}{p(s)} I^{\beta} h_3(s)\right)\bigg|_{s=b},$$
$$F = I^{\alpha}\left(\frac{q(s)}{p(s)} x_1(s)\right)\bigg|_{s=b} + I^{\alpha}\left(\frac{r(s)}{p(s)} I^{\theta} g_1(s)\right)\bigg|_{s=b} - I^{\alpha}\left(\frac{1}{p(s)} I^{\beta} h_1(s)\right)\bigg|_{s=b}.$$

To obtain the values of c_0^i and c_1^i, we solve the following equation:

$$\begin{pmatrix} \frac{1}{p(a)} & \frac{1}{p(b)} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{p(a)} & \frac{1}{p(b)} & 0 & 0 & 0 \\ \frac{1}{p(b)} & 0 & \frac{1}{p(a)} & 0 & 0 & 0 \\ 0 & \frac{(b-a)^{\alpha}}{p(b)\Gamma(\alpha+1)} & 0 & 1 & 1 & 0 \\ 0 & 0 & \frac{(b-a)^{\alpha}}{p(b)\Gamma(\alpha+1)} & 0 & 1 & 1 \\ \frac{(b-a)^{\alpha}}{p(b)\Gamma(\alpha+1)} & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_0^1 \\ c_0^2 \\ c_0^3 \\ c_1^1 \\ c_1^2 \\ c_1^3 \end{pmatrix} = \begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix}.$$

The determinant of the coefficient matrix is nonzero, so we have

$$c_0^1 = \frac{1}{\Gamma(\beta)} \int_a^b (b-\mu)^{\beta-1} \big(w_3 h_3(\mu) - w_1 h_1(\mu) - w_2 h_2(\mu)\big)\, \mathrm{d}\mu$$
$$+ \frac{r(s)}{\Gamma(\theta)} \int_a^b (b-\mu)^{\theta-1} \big[w_1 g_1(\mu) + w_2 g_2(\mu) - w_3 g_3(\mu)\big]\, \mathrm{d}\mu,$$
$$c_0^2 = \frac{1}{\Gamma(\beta)} \int_a^b (b-\mu)^{\beta-1} \big(w_3 h_1(\mu) - w_1 h_2(\mu) - w_2 h_3(\mu)\big)\, \mathrm{d}\mu$$
$$+ \frac{r(s)}{\Gamma(\theta)} \int_a^b (b-\mu)^{\theta-1} \big[w_1 g_2(\mu) + w_2 g_3(\mu) - w_3 g_1(\mu)\big]\, \mathrm{d}\mu,$$
$$c_0^3 = \frac{1}{\Gamma(\beta)} \int_a^b (b-\mu)^{\beta-1} \big(w_3 h_2(\mu) - w_1 h_3(\mu) - w_2 h_1(\mu)\big)\, \mathrm{d}\mu$$
$$+ \frac{r(s)}{\Gamma(\theta)} \int_a^b (b-\mu)^{\theta-1} \big[w_1 g_3(\mu) + w_2 g_1(\mu) - w_3 g_2(\mu)\big]\, \mathrm{d}\mu,$$

$$c_1^1 = \frac{1}{2\Gamma(\alpha)} \int_a^b (b-\mu)^{\alpha-1} \frac{q(\mu)}{p(\mu)} \big(x_1(\mu)+x_2(\mu)-x_3(\mu)\big)\,\mathrm{d}\mu$$
$$+ \frac{1}{2\Gamma(\beta)} \int_a^b (b-\mu)^{\beta-1} \big[V_1h_2(\mu)-V_2h_3(\mu)-V_3h_1(\mu)\big]\,\mathrm{d}\mu$$
$$+ \frac{r(s)}{2\Gamma(\theta)} \int_a^b (b-\mu)^{\theta-1} \big[V_3g_1(\mu)+V_2g_3(\mu)-V_1g_2(\mu)\big]\,\mathrm{d}\mu$$
$$+ \frac{1}{2\Gamma(\alpha)} \int_a^b (b-\mu)^{\alpha-1} \frac{1}{p(\mu)} \left(\frac{1}{\Gamma(\beta)} \int_a^\mu (\mu-\nu)^{\beta-1} \big[h_3(\nu)-h_1(\nu)-h_2(\nu)\big]\,\mathrm{d}\nu\right)\mathrm{d}\mu$$
$$+ \frac{r(s)}{2\Gamma(\alpha)} \int_a^b (b-\mu)^{\alpha-1} \frac{1}{p(\mu)} \left(\frac{1}{\Gamma(\theta)} \int_a^\mu (\mu-\nu)^{\theta-1} \big[g_1(\nu)+g_2(\nu)-g_3(\nu)\big]\,\mathrm{d}\nu\right)\mathrm{d}\mu$$

$$c_1^2 = \frac{1}{2\Gamma(\beta)} \int_a^b (b-\mu)^{\beta-1} \big[V_1h_3(\mu)-V_3h_2(\mu)-V_2h_1(\mu)\big]\,\mathrm{d}\mu$$
$$+ \frac{1}{2\Gamma(\alpha)} \int_a^b (b-\mu)^{\alpha-1} \frac{q(\mu)}{p(\mu)} \big(x_2(\mu)+x_3(\mu)-x_1(\mu)\big)\,\mathrm{d}\mu$$
$$+ \frac{r(s)}{2\Gamma(\theta)} \int_a^b (b-\mu)^{\theta-1} \big[V_2g_1(\mu)+V_3g_2(\mu)-V_1g_3(\mu)\big]\,\mathrm{d}\mu$$
$$+ \frac{1}{2\Gamma(\alpha)} \int_a^b (b-\mu)^{\alpha-1} \frac{1}{p(\mu)} \left(\frac{1}{\Gamma(\beta)} \int_a^\mu (\mu-\nu)^{\beta-1} \big[h_1(\nu)-h_2(\nu)-h_3(\nu)\big]\,\mathrm{d}\nu\right)\mathrm{d}\mu$$
$$+ \frac{r(s)}{2\Gamma(\alpha)} \int_a^b (b-\mu)^{\alpha-1} \frac{1}{p(\mu)} \left(\frac{1}{\Gamma(\theta)} \int_a^\mu (\mu-\nu)^{\theta-1} \big[g_2(\nu)+g_3(\nu)-g_1(\nu)\big]\,\mathrm{d}\nu\right)\mathrm{d}\mu,$$

$$c_1^3 = \frac{1}{2\Gamma(\beta)} \int_a^b (b-\mu)^{\beta-1} \big[V_1h_1(\mu)-V_2h_2(\mu)-V_3h_3(\mu)\big]\,\mathrm{d}\mu$$
$$+ \frac{1}{2\Gamma(\alpha)} \int_a^b (b-\mu)^{\alpha-1} \frac{q(\mu)}{p(\mu)} \big(x_1(\mu)+x_3(\mu)-x_2(\mu)\big)\,\mathrm{d}\mu$$
$$+ \frac{r(s)}{2\Gamma(\theta)} \int_a^b (b-\mu)^{\theta-1} \big[V_2g_2(\mu)+V_3g_3(\mu)-V_1g_1(\mu)\big]\,\mathrm{d}\mu$$
$$+ \frac{1}{2\Gamma(\alpha)} \int_a^b (b-\mu)^{\alpha-1} \frac{1}{p(\mu)} \left(\frac{1}{\Gamma(\beta)} \int_a^\mu (\mu-\nu)^{\beta-1} \big[h_2(\nu)-h_1(\nu)-h_3(\nu)\big]\,\mathrm{d}\nu\right)\mathrm{d}\mu$$
$$+ \frac{r(s)}{2\Gamma(\alpha)} \int_a^b (b-\mu)^{\alpha-1} \frac{1}{p(\mu)} \left(\frac{1}{\Gamma(\theta)} \int_a^\mu (\mu-\nu)^{\theta-1} \big[g_1(\nu)+g_3(\nu)-g_2(\nu)\big]\,\mathrm{d}\nu\right)\mathrm{d}\mu.$$

Finally, we substitute the values of c_0^i and c_1^i $(i = 1, 2, 3)$ into (8), thus obtaining solutions (4)–(6). Conversely, one easily verifies that (x_1, x_2, x_3) given by (4)–(6) satisfies system (3) and BVPs (2). The proof is completed. □

3 Existence and uniqueness results

Let $X = C[(a, b), \mathbb{R}]$ be Banach space endowed with the norm $\|x\|_\infty = \max_{s\in[a,b]} |x(s)|$. Then $\mathbb{X} = X^3$ is a Banach space with the norm

$$\big\|(x_1, x_2, x_3)\big\|_{\mathbb{X}} = \|x_1\|_\infty + \|x_2\|_\infty + \|x_3\|_\infty, \quad (x_1, x_2, x_3) \in \mathbb{X}.$$

In view of Lemma 3, we define the operator $T : \mathbb{X} \to \mathbb{X}$ by

$$T(x_1, x_2, x_3)(s) = \big(T_1(x_1, x_2, x_3)(s), T_2(x_1, x_2, x_3)(s), T_3(x_1, x_2, x_3)(s)\big). \tag{14}$$

Letting

$$\boldsymbol{g}_i(\mu) = g_i\big(x, x_i(\mu)\big), \quad \boldsymbol{f}_i(\mu) = f_i\big(\mu, x_1(\mu), x_2(\mu), x_3(\mu)\big), \quad i = 1, 2, 3,$$

we can express $T_i(x_1, x_2, x_3)(s)$, $i = 1, 2, 3$, as

$$\begin{aligned}
&T_1(x_1, x_2, x_2)(s)\\
&\quad= \frac{1}{\Gamma(\alpha)}\int_a^s (s-\mu)^{\alpha-1}\frac{1}{p(\mu)}\left(\frac{1}{\Gamma(\beta)}\int_a^\mu (\mu-\nu)^{\beta-1}\boldsymbol{f}_1(\nu)\,\mathrm{d}\nu\right)\mathrm{d}\mu\\
&\qquad+ \frac{1}{2\Gamma(\beta)}\int_a^b (b-\mu)^{\alpha-1}\big[V_1\boldsymbol{f}_2(\mu) - V_3\boldsymbol{f}_1(\mu) - V_2\boldsymbol{f}_3(s)\big]\,\mathrm{d}\mu\\
&\qquad+ \frac{1}{2\Gamma(\alpha)}\int_a^b (b-\mu)^{\alpha-1}\frac{q(\mu)}{p(\mu)}\big[x_1(\mu) + x_2(\mu) - x_3(\mu)\big]\,\mathrm{d}\mu\\
&\qquad- \frac{r(s)}{\Gamma(\alpha)}\int_a^s (s-\mu)^{\alpha-1}\frac{1}{p(\mu)}\left(\frac{1}{\Gamma(\theta)}\int_a^\mu (\mu-\nu)^{\theta-1}\boldsymbol{g}_1(v)\,\mathrm{d}\nu\right)\mathrm{d}\mu\\
&\qquad+ \frac{r(s)}{2\Gamma(\theta)}\int_a^b (b-\mu)^{\theta-1}\big[V_3\boldsymbol{g}_1(\mu) + V_2\boldsymbol{g}_3(\mu) - V_1\boldsymbol{g}_2(\mu)\big]\,\mathrm{d}\mu\\
&\qquad+ \frac{r(s)(s-a)^\alpha}{p(s)\Gamma(\alpha+1)\Gamma(\theta)}\int_a^b (b-\mu)^{\theta-1}\big[w_1\boldsymbol{g}_1(\mu) + w_2\boldsymbol{g}_2(\mu) - w_3\boldsymbol{g}_3(\mu)\big]\,\mathrm{d}\mu\\
&\qquad+ \frac{(s-a)^\alpha}{p(s)\Gamma(\alpha+1)\Gamma(\beta)}\int_a^b (b-\mu)^{\beta-1}\big[w_3\boldsymbol{f}_3(\mu) - w_1\boldsymbol{f}_1(\mu) - w_2\boldsymbol{f}_2(\mu)\big]\,\mathrm{d}\mu\\
&\qquad+ \frac{r(s)}{2\Gamma(\alpha)}\int_a^b (b-\mu)^{\alpha-1}\frac{1}{p(\mu)}\left(\frac{1}{\Gamma(\theta)}\int_a^b (\mu-\nu)^{\theta-1}\big[\boldsymbol{g}_1(\nu) + g_2(\nu) - \boldsymbol{g}_3(\nu)\big]\,\mathrm{d}\nu\right)\mathrm{d}\mu
\end{aligned}$$

$$+\frac{1}{2\Gamma(\alpha)}\int_a^b (b-\mu)^{\alpha-1}\frac{1}{p(\mu)}\left(\frac{1}{\Gamma(\beta)}\int_a^\mu (\mu-\nu)^{\beta-1}\big[\boldsymbol{f}_1(\nu)-\boldsymbol{f}_2(\nu)-\boldsymbol{f}_3(\nu)\big]\,\mathrm{d}\nu\right)\mathrm{d}\mu$$
$$-\frac{1}{\Gamma(\alpha)}\int_a^s (s-\mu)^{\alpha-1}\frac{q(\mu)}{p(\mu)}x_1(\mu)\,\mathrm{d}\mu$$

$$T_2(x_1,x_2,x_2)(s)$$
$$=\frac{1}{\Gamma(\alpha)}\int_a^s (s-\mu)^{\alpha-1}\frac{1}{p(\mu)}\left(\frac{1}{\Gamma(\beta)}\int_a^\mu (\mu-\nu)^{\beta-1}\boldsymbol{f}_2(\nu)\,\mathrm{d}\nu\right)\mathrm{d}\mu$$
$$+\frac{1}{2\Gamma(\beta)}\int_a^b (b-\mu)^{\beta-1}\big[V_1\boldsymbol{f}_3(\mu)-V_3\boldsymbol{f}_2(\mu)-V_2\boldsymbol{f}_1(\mu)\big]\,\mathrm{d}\mu$$
$$+\frac{1}{2\Gamma(\alpha)}\int_a^b (b-\mu)^{\alpha-1}\frac{q(\mu)}{p(\mu)}\big[x_2(\mu)+x_3(\mu)-x_1(\mu)\big]\,\mathrm{d}\mu$$
$$-\frac{r(s)}{\Gamma(\alpha)}\int_a^s (s-\mu)^{\alpha-1}\frac{1}{p(s)}\left(\frac{1}{\Gamma(\theta)}\int_a^\mu (\mu-\nu)^{\theta-1}g_2(\nu)\,\mathrm{d}\nu\right)\mathrm{d}\mu$$
$$+\frac{r(s)}{2\Gamma(\theta)}\int_a^b (b-\mu)^{\theta-1}\big[V_2\boldsymbol{g}_1(\mu)+V_3\boldsymbol{g}_2(\mu)-V_1\boldsymbol{g}_3(\mu)\big]\,\mathrm{d}\mu$$
$$+\frac{r(s)(s-a)^\alpha}{p(s)\Gamma(\alpha+1)\Gamma(\theta)}\int_a^b (b-\mu)^{\theta-1}\big[w_1\boldsymbol{g}_2(\mu)+w_2\boldsymbol{g}_3(\mu)-w_3\boldsymbol{g}_1(\mu)\big]\,\mathrm{d}\mu$$
$$+\frac{(s-a)^\alpha}{p(s)\Gamma(\alpha+1)\Gamma(\beta)}\int_a^b (b-\mu)^{\beta-1}\big[w_3\boldsymbol{f}_1(\mu)-w_1\boldsymbol{f}_2(\mu)-w_2\boldsymbol{f}_3(\mu)\big]\,\mathrm{d}\mu$$
$$+\frac{r(s)}{2\Gamma(\alpha)}\int_a^b (b-\mu)^{\alpha-1}\frac{1}{p(\mu)}\left(\frac{1}{\Gamma(\theta)}\int_a^\mu (\mu-\nu)^{\theta-1}\big[\boldsymbol{g}_2(\nu)+\boldsymbol{g}_3(\nu)-\boldsymbol{g}_1(\nu)\big]\,\mathrm{d}\nu\right)\mathrm{d}\mu$$
$$+\frac{1}{2\Gamma(\alpha)}\int_a^b (b-\mu)^{\alpha-1}\frac{1}{p(\mu)}\left(\frac{1}{\Gamma(\beta)}\int_a^\mu (\mu-\nu)^{\beta-1}\big[\boldsymbol{f}_1(\nu)-\boldsymbol{f}_2(\nu)-\boldsymbol{f}_3(\nu)\big]\,\mathrm{d}\nu\right)\mathrm{d}\mu$$
$$-\frac{1}{\Gamma(\alpha)}\int_a^s (s-\mu)^{\alpha-1}\frac{q(\mu)}{p(\mu)}x_2(\mu)\,\mathrm{d}\mu,$$

$$T_3(x_1,x_2,x_2)(s)$$
$$=\frac{1}{\Gamma(\alpha)}\int_a^s (s-\mu)^{\alpha-1}\frac{1}{p(\mu)}\left(\frac{1}{\Gamma(\beta)}\int_a^\mu (\mu-\nu)^{\beta-1}\boldsymbol{f}_3(\nu)\,\mathrm{d}\nu\right)\mathrm{d}\mu$$
$$+\frac{1}{2\Gamma(\beta)}\int_a^b (b-\mu)^{\beta-1}\big[V_1\boldsymbol{f}_1(\mu)-V_2\boldsymbol{f}_2(\mu)-V_3\boldsymbol{f}_3(\mu)\big]\,\mathrm{d}\mu$$

$$
\begin{aligned}
&+ \frac{1}{2\Gamma(\alpha)} \int_a^b (b-\mu)^{\alpha-1} \frac{q(\mu)}{p(\mu)} \big[x_1(\mu) + x_3(\mu) - x_2(\mu)\big] \, \mathrm{d}\mu \\
&- \frac{r(s)}{\Gamma(\alpha)} \int_a^s (s-\mu)^{\alpha-1} \frac{1}{p(\mu)} \left(\frac{1}{\Gamma(\theta)} \int_a^\mu (\mu-\nu)^{\theta-1} \boldsymbol{g}_3(\nu) \, \mathrm{d}\nu \right) \mathrm{d}\mu \\
&+ \frac{r(s)}{2\Gamma(\theta)} \int_a^b (b-\mu)^{\theta-1} \big[V_2 \boldsymbol{g}_2(\mu) + V_3 \boldsymbol{g}_3(\mu) - V_1 \boldsymbol{g}_1(\mu)\big] \, \mathrm{d}\mu \\
&+ \frac{r(s)(s-a)^\alpha}{p(s)\Gamma(\alpha+1)\Gamma(\theta)} \int_a^b (b-\mu)^{\theta-1} \big[w_1 g_3(\mu) + w_2 \boldsymbol{g}_1(\mu) - w_3 \boldsymbol{g}_2(\mu)\big] \, \mathrm{d}\mu \\
&+ \frac{(s-a)^\alpha}{p(s)\Gamma(\alpha+1)\Gamma(\beta)} \int_a^b (b-\mu)^{\beta-1} \big[w_3 \boldsymbol{f}_2(\mu) - w_1 \boldsymbol{f}_3(\mu) - w_2 \boldsymbol{f}_1(\mu)\big] \, \mathrm{d}\mu \\
&+ \frac{r(s)}{2\Gamma(\alpha)} \int_a^b (b-\mu)^{\alpha-1} \frac{1}{p(\mu)} \left(\frac{1}{\Gamma(\theta)} \int_a^\mu (\mu-\nu)^{\theta-1} \big[\boldsymbol{g}_1(\nu) + \boldsymbol{g}_3(\nu) - \boldsymbol{g}_2(\nu)\big] \, \mathrm{d}\nu \right) \mathrm{d}\mu \\
&+ \frac{1}{2\Gamma(\alpha)} \int_a^b (b-\mu)^{\alpha-1} \frac{1}{p(\mu)} \left(\frac{1}{\Gamma(\beta)} \int_a^\mu (\mu-\nu)^{\beta-1} \big[\boldsymbol{f}_2(\nu) - h_1(\nu) - \boldsymbol{f}_3(\nu)\big] \, \mathrm{d}\nu \right) \mathrm{d}\mu \\
&- \frac{1}{\Gamma(\alpha)} \int_a^s (s-\mu)^{\alpha-1} \frac{q(\mu)}{p(\mu)} x_3(\mu) \, \mathrm{d}\mu.
\end{aligned}
$$

The following constants are introduced to simplify calculations:

$$
\begin{aligned}
\bar{p} &= \min_{s\in[a,b]} \big|p(s)\big|, \qquad \bar{q} = \max_{s\in[a,b]} \big|q(s)\big|, \qquad \bar{r} = \max_{s\in[a,b]} \big|r(s)\big|, \\
\boldsymbol{a} &= \frac{3(b-a)^\beta}{2\Gamma(\alpha+1)\Gamma(\beta+1)\bar{p}} + \frac{(b-a)^{\alpha+\beta} w_1}{\Gamma(\alpha+1)\Gamma(\beta+1)\bar{p}} + \frac{(b-a)^\beta V_3}{2\Gamma(\beta+1)}, \\
\boldsymbol{b} &= \frac{(b-a)^\beta}{2\Gamma(\alpha+1)\Gamma(\beta+1)\bar{p}} + \frac{(b-a)^{\alpha+\beta} w_2}{\Gamma(\alpha+1)\Gamma(\beta+1)\bar{p}} + \frac{(b-a)^\beta V_1}{2\Gamma(\beta+1)}, \\
\boldsymbol{c} &= \frac{(b-a)^\beta}{2\Gamma(\alpha+1)\Gamma(\beta+1)\bar{p}} + \frac{(b-a)^{\alpha+\beta} w_3}{\Gamma(\alpha+1)\Gamma(\beta+1)\bar{p}} + \frac{(b-a)^\beta V_2}{2\Gamma(\beta+1)}, \\
\boldsymbol{d} &= \frac{3\bar{r}(b-a)^{\alpha+\theta}}{2\Gamma(\alpha+1)\Gamma(\theta+1)\bar{p}} + \frac{\bar{r}(b-a)^{\alpha+\theta} w_1}{\Gamma(\alpha+1)\Gamma(\theta+1)\bar{p}} + \frac{\bar{r}(b-a)^\theta V_3}{2\Gamma(\theta+1)}, \\
\boldsymbol{e} &= \frac{\bar{r}(b-a)^{\alpha+\theta}}{2\Gamma(\alpha+1)\Gamma(\theta+1)\bar{p}} + \frac{\bar{r}(b-a)^{\alpha+\theta} w_2}{\Gamma(\alpha+1)\Gamma(\theta+1)\bar{p}} + \frac{\bar{r}(b-a)^\theta V_1}{2\Gamma(\theta+1)}, \\
\boldsymbol{f} &= \frac{\bar{r}(b-a)^{\alpha+\theta}}{2\Gamma(\alpha+1)\Gamma(\theta+1)\bar{p}} + \frac{\bar{r}(b-a)^{\alpha+\theta} w_3}{\Gamma(\alpha+1)\Gamma(\theta+1)\bar{p}} + \frac{\bar{r}(b-a)^\theta V_2}{2\Gamma(\theta+1)}.
\end{aligned}
$$

The following existence result is based on Krasnoselskii's fixed point theorem [14].

Theorem 1. *Assume that*:

(A_1) *The functions* $f_i : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$, $g_i : [a,b] \times \mathbb{R} \to \mathbb{R}$ *are continuous.*

(A_2) *There exist nonnegative functions* $k_i(s), u_i(s), v_i(s), \varpi_i(s) \in C[a,b]$ *such that, for all* $(s, m, n, \Theta) \in [a,b] \times \mathbb{R}^3$,

$$\bigl|f_i(s,m,n,\Theta)\bigr| \leqslant k_i(s) + u_i(s)|m| + v_i(s)|n| + \varpi_i(s)|\Theta|, \quad i = 1,2,3.$$

(A_3) *There exist positive constants* λ_i *such that, for all* $(s,x) \in [a,b] \times \mathbb{R}$, *the following inequality holds*:

$$\bigl|g_i\bigl(s, x_i(s)\bigr)\bigr| \leqslant \lambda_i |x_i|, \quad i = 1,2,3.$$

Then problem (1)–(2) *has at least one solution on* $[a,b]$, *provided that*

$$2\Gamma(\alpha+1)\bar{p} - 5(b-a)^{\alpha}\bar{q} - 2\Gamma(\alpha+1)\bar{p}\nabla > 0, \tag{15}$$

where

$$\begin{aligned}
&\nabla = (\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})(l_1 + l_2 + l_3) + (\boldsymbol{d} + \boldsymbol{e} + \boldsymbol{f})(\lambda_1 + \lambda_2 + \lambda_3),\\
&l_i = u_i + v_i + \varpi_i, \quad i = 1,2,3,\\
&\qquad k_i = \max_{s\in[a,b]} \bigl|k_i(s)\bigr|, \qquad u_i = \max_{s\in[a,b]} \bigl|u_i(s)\bigr|,\\
&\qquad v_i = \max_{s\in[a,b]} \bigl|v_i(s)\bigr|, \qquad \varpi_i = \max_{s\in[a,b]} \bigl|\varpi_i(s)\bigr|.
\end{aligned}$$

Proof. Let $\varepsilon > 0$ be such that

$$\varepsilon > \frac{2\Gamma(\alpha+1)\bar{p}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})\sum_{i=1}^{3} k_i}{2\Gamma(\alpha+1)\bar{p} - 5(b-a)^{\alpha}\bar{q} - 2\Gamma(\alpha+1)\nabla\bar{p}},$$

and consider the subsets

$$B_\varepsilon = \bigl\{x = (x_1, x_2, x_3) \in X^3 \colon \|x\|_{\mathbb{X}} \leqslant \varepsilon\bigr\}.$$

Define two operators F and G on set B_ε by

$$\begin{aligned}
(Fy)(s) &= \bigl(F_1(x_1,x_2,x_3), F_2(x_1,x_2,x_3), F_3(x_1,x_2,x_3)\bigr)(s),\\
(Gy)(s) &= \bigl(G_1(x_1,x_2,x_3), G_2(x_1,x_2,x_3), G_3(x_1,x_2,x_3)\bigr)(s),
\end{aligned}$$

where

$$\begin{aligned}
F_1(x_1,x_2,x_3) = &-\frac{1}{\Gamma(\alpha)}\int_a^s (s-\mu)^{\alpha-1}\frac{q(\mu)}{p(\mu)}x_1(\mu)\,\mathrm{d}\mu\\
&+ \frac{1}{2\Gamma(\alpha)}\int_a^b (b-\mu)^{\alpha-1}\frac{q(\mu)}{p(\mu)}\bigl(x_1(\mu)+x_2(\mu)-x_3(\mu)\bigr)\,\mathrm{d}\mu,
\end{aligned}$$

$$F_2(x_1,x_2,x_3) = -\frac{1}{\Gamma(\alpha)}\int_a^s (s-\mu)^{\alpha-1}\frac{q(\mu)}{p(\mu)}x_2(\mu)\,\mathrm{d}\mu + \frac{1}{2\Gamma(\alpha)}\int_a^b (b-\mu)^{\alpha-1}\frac{q(\mu)}{p(\mu)}\big(x_2(\mu)+x_3(\mu)-x_1(\mu)\big)\,\mathrm{d}\mu,$$

$$F_3(x_1,x_2,x_3) = -\frac{1}{\Gamma(\alpha)}\int_a^s (s-\mu)^{\alpha-1}\frac{q(\mu)}{p(\mu)}x_3(\mu)\,\mathrm{d}\mu + \frac{1}{2\Gamma(\alpha)}\int_a^b (b-\mu)^{\alpha-1}\frac{q(\mu)}{p(\mu)}\big(x_1(\mu)+x_3(\mu)-x_2(\mu)\big)\,\mathrm{d}\mu,$$

$$\begin{aligned}
&G_1(x_1,x_2,x_3)\\
&= \frac{1}{\Gamma(\alpha)}\int_a^s (s-\mu)^{\alpha-1}\frac{1}{p(\mu)}\left(\frac{1}{\Gamma(\beta)}\int_a^\mu (\mu-\nu)^{\alpha-1}\boldsymbol{f}_1(\nu)\,\mathrm{d}\nu\right)\mathrm{d}\mu\\
&\quad + \frac{1}{2\Gamma(\beta)}\int_a^b (b-\mu)^{\beta-1}\big(V_1\boldsymbol{f}_2(\mu)-V_3\boldsymbol{f}_1(\mu)-V_2\boldsymbol{f}_3(\mu)\big)\,\mathrm{d}\mu\\
&\quad + \frac{(s-a)^\alpha}{p(s)\Gamma(\alpha+1)\Gamma(\beta)}\int_a^b (b-\mu)^{\beta-1}\big(w_3\boldsymbol{f}_3(\mu)-w_1\boldsymbol{f}_1(\mu)-w_2\boldsymbol{f}_2(\mu)\big)\,\mathrm{d}\mu\\
&\quad - \frac{r(s)}{\Gamma(\alpha)}\int_a^s (s-\mu)^{\alpha-1}\frac{1}{p(\mu)}\left(\frac{1}{\Gamma(\theta)}\int_a^\mu (\mu-\nu)^{\theta-1}\boldsymbol{g}_1(\nu)\,\mathrm{d}\nu\right)\mathrm{d}\mu\\
&\quad + \frac{r(s)}{2\Gamma(\theta)}\int_a^b (b-\mu)^{\theta-1}\big(V_3\boldsymbol{g}_1(\mu)+V_2\boldsymbol{g}_3(\mu)-V_1\boldsymbol{g}_2(\mu)\big)\,\mathrm{d}\mu\\
&\quad + \frac{r(s)(s-a)^\alpha}{p(s)\Gamma(\alpha+1)\Gamma(\theta)}\int_a^b (b-\mu)^{\theta-1}\big(w_1\boldsymbol{g}_1(\mu)+w_2\boldsymbol{g}_2(\mu)-w_3\boldsymbol{g}_3(\mu)\big)\,\mathrm{d}\mu\\
&\quad + \frac{1}{2\Gamma(\alpha)}\int_a^b (b-\mu)^{\alpha-1}\frac{1}{p(\mu)}\left(\frac{1}{\Gamma(\beta)}\int_a^\mu (\mu-\nu)^{\beta-1}\big(\boldsymbol{f}_3(\nu)-\boldsymbol{f}_1(\nu)-\boldsymbol{f}_2(\nu)\big)\,\mathrm{d}\nu\right)\mathrm{d}\mu\\
&\quad + \frac{r(s)}{2\Gamma(\alpha)}\int_a^b (b-\mu)^{\alpha-1}\frac{1}{p(\mu)}\left(\frac{1}{\Gamma(\theta)}\int_a^\mu (\mu-\nu)^{\theta-1}\big(\boldsymbol{g}_1(\nu)+\boldsymbol{g}_2(\nu)-\boldsymbol{g}_3(\nu)\big)\,\mathrm{d}\nu\right)\mathrm{d}\mu
\end{aligned}$$

$$\begin{aligned}
&G_2(x_1,x_2,x_3)\\
&= \frac{1}{\Gamma(\alpha)}\int_a^s (s-\mu)^{\alpha-1}\frac{1}{p(\mu)}\left(\frac{1}{\Gamma(\beta)}\int_a^\mu (\mu-\nu)^{\beta-1}\boldsymbol{f}_2(\nu)\,\mathrm{d}\nu\right)\mathrm{d}\mu\\
&\quad + \frac{1}{2\Gamma(\alpha)}\int_a^b (b-\mu)^{\alpha-1}\big(T_1\boldsymbol{f}_3(\mu)-T_3\boldsymbol{f}_2(\mu)-T_2\boldsymbol{f}_1(\mu)\big)\,\mathrm{d}\mu
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{(s-a)^{\alpha}}{p(s)\Gamma(\alpha+1)\Gamma(\beta)} \int\limits_a^b (b-\mu)^{\beta-1}\big(w_3\boldsymbol{f}_1(\mu) - w_1\boldsymbol{f}_2(\mu) - w_2\boldsymbol{f}_3(\mu)\big)\,\mathrm{d}\mu \\
&- \frac{r(s)}{\Gamma(\alpha)} \int\limits_a^s (s-\mu)^{\alpha-1}\frac{1}{p(\mu)}\left(\frac{1}{\Gamma(\theta)}\int\limits_a^{\mu}(\mu-\nu)^{\theta-1}\boldsymbol{g}_2(\nu)\,\mathrm{d}\nu\right)\mathrm{d}\mu \\
&+ \frac{r(s)}{2p(b)\Gamma(\theta)} \int\limits_a^b (b-\mu)^{\theta-1}\big(T_2\boldsymbol{g}_1(\mu) + T_3\boldsymbol{g}_2(\mu) - T_1\boldsymbol{g}_3(\mu)\big)\,\mathrm{d}\mu \\
&+ \frac{r(s)(s-a)^{\alpha}}{p(s)\Gamma(\alpha+1)\Gamma(\theta)} \int\limits_a^b (b-\mu)^{\theta-1}\big(w_1\boldsymbol{g}_2(\mu) + w_2\boldsymbol{g}_3(\mu) - w_3\boldsymbol{g}_1(\mu)\big)\,\mathrm{d}\mu \\
&+ \frac{1}{2\Gamma(\alpha)} \int\limits_a^b (b-\mu)^{\alpha-1}\frac{1}{p(\mu)}\left(\frac{1}{\Gamma(\beta)}\int\limits_a^{\mu}(\mu-\nu)^{\beta-1}\big(\boldsymbol{f}_1(\nu) - \boldsymbol{f}_2(\nu) - \boldsymbol{f}_3(\nu)\big)\,\mathrm{d}\nu\right)\mathrm{d}\mu \\
&+ \frac{r(s)}{2\Gamma(\alpha)} \int\limits_a^b (b-\mu)^{\alpha-1}\frac{1}{p(\mu)}\left(\frac{1}{\Gamma(\theta)}\int\limits_a^{\mu}(\mu-\nu)^{\theta-1}\big(\boldsymbol{g}_2(\nu) + \boldsymbol{g}_3(\nu) - \boldsymbol{g}_1(\nu)\big)\,\mathrm{d}\nu\right)\mathrm{d}\mu
\end{aligned}
$$

$$
\begin{aligned}
&G_3(x_1,x_2,x_3) \\
&\quad= \frac{1}{\Gamma(\alpha)} \int\limits_a^s (s-\mu)^{\alpha-1}\frac{1}{p(\mu)}\left(\frac{1}{\Gamma(\beta)}\int\limits_a^{\mu}(\mu-\nu)^{\beta-1}\boldsymbol{f}_3(\nu)\,\mathrm{d}\nu\right)\mathrm{d}\mu \\
&\quad+ \frac{1}{2\Gamma(\beta)} \int\limits_a^b (b-\mu)^{\beta-1}\big(T_1\boldsymbol{f}_1(\mu) - T_2\boldsymbol{f}_2(\mu) - T_3\boldsymbol{f}_3(\mu)\big)\,\mathrm{d}\mu \\
&\quad+ \frac{(s-a)^{\alpha}}{p(s)\Gamma(\alpha+1)\Gamma(\beta)} \int\limits_a^b (b-\mu)^{\beta-1}\big(w_3\boldsymbol{f}_2(\mu) - w_1\boldsymbol{f}_3(\mu) - w_2\boldsymbol{f}_1(\mu)\big)\,\mathrm{d}\mu \\
&\quad- \frac{r(s)}{\Gamma(\alpha)} \int\limits_a^s (s-\mu)^{\alpha-1}\frac{1}{p(\mu)}\left(\frac{1}{\Gamma(\theta)}\int\limits_a^{\mu}(\mu-\nu)^{\theta-1}\boldsymbol{g}_3(\nu)\,\mathrm{d}\nu\right)\mathrm{d}\mu \\
&\quad+ \frac{r(s)}{2\Gamma(\theta)} \int\limits_a^b (b-\mu)^{\theta-1}\big(T_2\boldsymbol{g}_2(\mu) + T_3\boldsymbol{g}_3(\mu) - T_1\boldsymbol{g}_1(\mu)\big)\,\mathrm{d}\mu \\
&\quad+ \frac{r(s)(s-a)^{\alpha}}{p(s)\Gamma(\alpha+1)\Gamma(\theta)} \int\limits_a^b (b-\mu)^{\theta-1}\big(w_1\boldsymbol{g}_3(\mu) + w_2\boldsymbol{g}_1(\mu) - w_3\boldsymbol{g}_2(\mu)\big)\,\mathrm{d}\mu \\
&\quad+ \frac{1}{2\Gamma(\alpha)} \int\limits_a^b (b-\mu)^{\alpha-1}\frac{1}{p(\mu)}\left(\frac{1}{\Gamma(\beta)}\int\limits_a^{\mu}(\mu-\nu)^{\beta-1}\big(\boldsymbol{f}_2(\nu) - \boldsymbol{f}_1(\nu) - \boldsymbol{f}_3(\nu)\big)\,\mathrm{d}\nu\right)\mathrm{d}\mu \\
&\quad+ \frac{r(s)}{2\Gamma(\alpha)} \int\limits_a^b (b-\mu)^{\alpha-1}\frac{1}{p(\mu)}\left(\frac{1}{\Gamma(\theta)}\int\limits_a^{\mu}(\mu-\nu)^{\theta-1}\big(\boldsymbol{g}_1(\nu) + \boldsymbol{g}_3(\nu) - \boldsymbol{g}_2(\nu)\big)\,\mathrm{d}\nu\right)\mathrm{d}\mu.
\end{aligned}
$$

The proof will now be presented in three steps.

(i) In fact, for all $x, y \in B_\epsilon$ such that $\|x\|_{\mathbb{X}} \leqslant, \|y\|_{\mathbb{X}} \leqslant \varepsilon$, by condition ($A_2$), then

$$\begin{aligned}
\left|G_1(x)\right| &\leqslant \frac{(b-a)^\beta}{2\Gamma(\beta+1)}\big(V_3\big(k_1+l_1\|x\|_{\mathbb{X}}\big)+V_1\big(k_2+l_2\|x\|_{\mathbb{X}}\big)+V_2\big(k_3+l_3\|x\|_{\mathbb{X}}\big)\big)\\
&\quad+\frac{(b-a)^{\alpha+\theta}\sum_{i=1}^3\lambda_i\|x\|_{\mathbb{X}}}{2\Gamma(\alpha+1)\Gamma(\theta+1)\bar{p}}+\frac{(b-a)^{\alpha+\beta}(k_1+l_1\|x\|_{\mathbb{X}})}{\Gamma(\alpha+1)\Gamma(\beta+1)\bar{p}}\\
&\quad+\frac{(b-a)^{\alpha+\beta}\sum_{i=1}^3 w_i(k_i+l_i\|x\|_{\mathbb{X}})}{\Gamma(\alpha+1)\Gamma(\beta+1)\bar{p}}+\frac{\bar{r}(b-a)^{\alpha+\theta}\sum_{i=1}^3\lambda_i w_i\|x\|_{\mathbb{X}}}{\Gamma(\alpha+1)\Gamma(\theta+1)\bar{p}}\\
&\quad+\frac{\bar{r}(b-a)^{\alpha+\beta}\sum_{i=1}^3(k_i+l_i\|x\|_{\mathbb{X}})}{2\Gamma(\alpha+1)\Gamma(\beta+1)\bar{p}}+\frac{\bar{r}(b-a)^{\alpha+\theta}\lambda_1\|x\|_{\mathbb{X}}}{\Gamma(\alpha+1)\Gamma(\theta+1)\bar{p}}\\
&\quad+\frac{\bar{r}(b-a)^\theta(V_3\lambda_1+V_1\lambda_2+V_2\lambda_3)\|x\|_{\mathbb{X}}}{2\Gamma(\theta+1)}\\
&\leqslant ak_1+bk_2+ck_3+(al_1+bl_2+cl_3+d\lambda_1+e\lambda_2+f\lambda_3)\varepsilon,\\
|F_1 y| &\leqslant \frac{1}{\Gamma(\alpha)}\int_a^s(s-\mu)^{\alpha-1}\frac{|q(\mu)|}{|p(\mu)|}\left|y_1(\mu)\right|\mathrm{d}\mu\\
&\quad+\frac{1}{2\Gamma(\alpha)}\int_a^b(b-\mu)^{\alpha-1}\frac{|q(\mu)|}{|p(\mu)|}\big(\left|y_1(\mu)\right|+\left|y_2(\mu)\right|+\left|y_3(\mu)\right|\big)\mathrm{d}\mu\\
&\leqslant \frac{(b-a)^\alpha\bar{q}}{\Gamma(\alpha+1)\bar{p}}\|y_1\|_\infty+\frac{(b-a)^\alpha\bar{q}}{2\Gamma(\alpha+1)\bar{p}}\big(\|y_1\|_\infty+\|y_2\|_\infty+\|y_3\|_\infty\big)\\
&\leqslant \frac{(2\|y_1\|_\infty+\varepsilon)\bar{q}}{2\Gamma(\alpha+1)\bar{p}}(b-a)^\alpha.
\end{aligned}$$

Therefore, the following estimates are provided:

$$\begin{aligned}
|G_1x+F_1y| \leqslant{}& \boldsymbol{a}k_1+\boldsymbol{b}k_2+\boldsymbol{c}k_3+(\boldsymbol{a}l_1+\boldsymbol{b}l_2+\boldsymbol{c}l_3+\boldsymbol{d}\lambda_1+\boldsymbol{e}\lambda_2+\boldsymbol{f}\lambda_3)\varepsilon\\
&+\frac{(2\|y_1\|_\infty+\varepsilon)\bar{q}}{2\Gamma(\alpha+1)\bar{p}}(b-a)^\alpha.
\end{aligned}$$

Through analogous calculations, we also determine

$$\begin{aligned}
|G_2x+F_2y| \leqslant{}& \boldsymbol{c}k_1+\boldsymbol{a}k_2+\boldsymbol{b}k_3+(\boldsymbol{c}l_1+\boldsymbol{a}l_2+\boldsymbol{b}l_3+\boldsymbol{f}\lambda_1+\boldsymbol{d}\lambda_2+\boldsymbol{e}\lambda_3)\varepsilon\\
&+\frac{(2\|y_2\|_\infty+\varepsilon)\bar{q}}{2\Gamma(\alpha+1)\bar{p}}(b-a)^\alpha,\\
|G_3x+F_3y| \leqslant{}& \boldsymbol{b}k_1+\boldsymbol{c}k_2+\boldsymbol{a}k_3+(\boldsymbol{b}l_1+\boldsymbol{c}l_2+\boldsymbol{a}l_3+\boldsymbol{e}\lambda_1+\boldsymbol{f}\lambda_2+\boldsymbol{d}\lambda_3)\varepsilon\\
&+\frac{(2\|y_3\|_\infty+\varepsilon)\bar{q}}{2\Gamma(\alpha+1)\bar{p}}(b-a)^\alpha.
\end{aligned}$$

Consequently,

$$\begin{aligned}\|Gx+Fy\|_{\mathbb{X}} &= \sum_{i=1}^{3}\|G_ix+F_iy\|_\infty \\ &\leqslant \left[(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})\sum_{i=1}^{3} l_i + (\boldsymbol{d}+\boldsymbol{e}+\boldsymbol{f})\sum_{i=1}^{3}\lambda_i\right]\varepsilon \\ &\quad + (\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})\sum_{i=1}^{3} k_i + \frac{5(b-a)^\alpha\bar{q}\varepsilon}{2\Gamma(\alpha+1)\bar{p}} \\ &\leqslant \varepsilon.\end{aligned}$$

Thus, $Gx+Fy\in B_\varepsilon$.

(ii) This step proves that F is contraction. For all $x,y\in B_\varepsilon$, we have

$$\begin{aligned}|F_1x-F_1y| &\leqslant \frac{(b-a)^\alpha\bar{q}}{2\Gamma(\alpha+1)\bar{p}}\left(\sum_{i=1}^{3}\|x_i-y_i\|_\infty\right) + \frac{(b-a)^\alpha\bar{q}}{\Gamma(\alpha+1)\bar{p}}\|x_1-y_1\|_\infty \\ &= \frac{(\|x-y\|_{\mathbf{X}}+2\|x_1-y_1\|_\infty)\bar{q}}{2\Gamma(\alpha+1)\bar{p}}(b-a)^\alpha.\end{aligned}$$

Similarly, we have

$$|F_ix-F_iy| \leqslant \frac{(\|x-y\|_{\mathbb{X}}+2\|x_i-y_i\|_\infty)\bar{q}}{2\Gamma(\alpha+1)\bar{p}}(b-a)^\alpha,\quad i=2,3.$$

So,

$$\begin{aligned}\|Fx-Fy\|_\infty &= \|F_1x-F_1z\|_\infty+\|F_2x-F_2z\|_\infty+\|F_3x-F_3z\|_\infty \\ &\leqslant \frac{(3\|x-y\|_{\mathbb{X}}+2\|x-y\|_\infty)\bar{q}}{2\Gamma(\alpha+1)\bar{p}}(b-a)^\alpha \\ &\leqslant \frac{5(b-a)^\alpha\bar{q}}{2\bar{p}\Gamma(\alpha+1)}\|x-y\|_{\mathbb{X}}.\end{aligned}$$

Since condition (15) is satisfied, F is contraction operator.

(iii) Continuity of the functions f_i implies that operators G_i is continuous. By condition (i), we conclude that G_i are uniformly bounded on the set B_ε. Then, for $a<s_1<s_2\leqslant b$, we have

$$\begin{aligned}&\bigl|G_1x(s_2)-G_1x(s_1)\bigr| \\ &\quad\leqslant \frac{(b-a)^\beta(k_1+l_1\|x\|_{\mathbb{X}})}{\Gamma(\alpha)\Gamma(\beta+1)\bar{p}} \\ &\qquad\times\left\{\int_a^{s_1}\bigl((s_2-\mu)^{\alpha-1}-(s_1-\mu)^{\alpha-1}\bigr)\,\mathrm{d}\mu + \int_{s_1}^{s_2}(s_2-\mu)^{\alpha-1}\,\mathrm{d}\mu\right\} \\ &\qquad+\frac{\bar{r}(b-a)^\theta\lambda_1}{\Gamma(\beta)\Gamma(\theta+1)\bar{p}}\left\{\int_a^{s_1}\bigl((s_2-\mu)^{\alpha-1}-(s_1-\mu)^{\alpha-1}\bigr)\,\mathrm{d}\mu + \int_{s_1}^{s_2}(s_2-\mu)^{\alpha-1}\,\mathrm{d}\mu\right\}\end{aligned}$$

$$
\begin{aligned}
&+\frac{\bar r(b-a)^{\theta}((s_2-a)^{\alpha}-(s_1-a)^{\alpha})}{\Gamma(\alpha+1)\Gamma(\theta+1)\bar p}\sum_{i=1}^{3}w_i\lambda_i\\
&+\frac{(b-a)^{\beta}((s_2-a)^{\alpha}-(s_1-a)^{\alpha})}{\Gamma(\alpha+1)\Gamma(\beta+1)\bar p}\sum_{i=1}^{3}w_1(k_1+l_1\|x\|_{\mathbb X})\\
&\leqslant\frac{(s_2-a)^{\alpha}-(s_1-a)^{\alpha}}{\Gamma(\alpha+1)\bar p}\biggl\{\frac{(b-a)^{\beta}}{\Gamma(\beta+1)}\bigl[w_1(k_1+l_1\|x\|_{\mathbb X})+w_2\bigl(k_2+l_2\|x\|_{\mathbb X}\bigr)\\
&+w_3\bigl(k_3+l_3\|x\|_{\mathbb X}\bigr)+\bigl(k_1+l_1\|x\|_{\mathbb X}\bigr)\bigr]+\frac{\bar r(b-a)^{\theta}(\lambda_1+w_1\lambda_1+w_2\lambda_2+w_3\lambda_3)}{\Gamma(\theta+1)}\biggr\}.
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
&\bigl|G_2x(s_2)-G_2x(s_1)\bigr|\\
&\leqslant\frac{(s_2-a)^{\alpha}-(s_1-a)^{\alpha}}{\Gamma(\alpha+1)\bar p}\biggl\{\frac{(b-a)^{\beta}}{\Gamma(\beta+1)}\bigl[w_3\bigl(k_1+l_1\|x\|_{\mathbb X}\bigr)+w_1\bigl(k_2+l_2\|x\|_{\mathbb X}\bigr)\\
&+w_2(k_3+l_3\|x\|_{\mathbb X})+\bigl(k_2+l_2\|x\|_{\mathbb X}\bigr)\bigr]+\frac{\bar r(b-a)^{\theta}(\lambda_1+w_1\lambda_1+w_2\lambda_2+w_3\lambda_3)}{\Gamma(\theta+1)}\biggr\},\\
&\bigl|G_3x(s_2)-G_3x(s_1)\bigr|\\
&\leqslant\frac{(s_2-a)^{\alpha}-(s_1-a)^{\alpha}}{\Gamma(\alpha+1)\bar p}\biggl\{\frac{(b-a)^{\beta}}{\Gamma(\beta+1)}\bigl[w_2\bigl(k_1+l_1\|x\|_{\mathbb X}\bigr)+w_3\bigl(k_2+l_2\|x\|_{\mathbb X}\bigr)\\
&+w_1(k_3+l_3\|x\|_{\mathbb X})+\bigl(k_3+l_3\|x\|_{\mathbb X}\bigr)\bigr]+\frac{\bar r(b-a)^{\theta}(\lambda_1+w_1\lambda_1+w_2\lambda_2+w_3\lambda_3)}{\Gamma(\theta+1)}\biggr\}.
\end{aligned}
$$

$|G_ix(s_2)-G_ix(s_1)|\to 0$, $i=1,2,3$, as $s_2\to s_1$ independent of x. Thus, G_i is equicontinuous on B_ε. According to the Arzelà–Ascoli theorem, G_i is a compact on B_ε. Therefore, problem (1)–(2) has at least one solution on $[a,b]$. □

The following uniqueness result relies on the Banach contraction principle [14].

Theorem 2. *Assume that:*

- (A_1) *The functions $f_i:[a,b]\times\mathbb{R}^3\to\mathbb{R}$ and $g_i:[a,b]\times\mathbb{R}\to\mathbb{R}$, $i=1,2,3$, are continuous.*
- (A_4) *There exist constants $\eta_i, L_i>0$ such that, for all $s\in[a,b]$, $x_i,y_i\in\mathbb{R}$, $i=1,2,3$, the following inequalities hold*:

$$
\begin{aligned}
\bigl|g_i\bigl(s,x_i(s)\bigr)-g_i\bigl(s,y_i(s)\bigr)\bigr|&\leqslant\eta_i|x_i-y_i|,\\
\bigl|f_i(s,x_1,x_2,x_3)-f_i(s,y_1,y_2,y_3)\bigr|&\leqslant L_i\bigl(|x_1-y_1|+|x_2-y_2|+|x_3-y_3|\bigr).
\end{aligned}
$$

Then problem (1)–(2) *has unique solution on* $[a,b]$, *provided that*

$$2\Gamma(\alpha+1)\bar p\varDelta+9(b-a)^{\alpha}\bar q<2\Gamma(\alpha+1)\bar p, \tag{16}$$

where

$$\varDelta=(\boldsymbol a+\boldsymbol b+\boldsymbol c)(L_1+L_2+L_3)+(\boldsymbol e+\boldsymbol f+\boldsymbol d)(\eta_1+\eta_2+\eta_3).$$

Proof. Let $\rho > 0$ be such that

$$\rho \geqslant \frac{(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})\sum_{i=1}^{3} M_i + (\boldsymbol{d}+\boldsymbol{e}+\boldsymbol{f})\sum_{i=1}^{3} N_i}{2\Gamma(\alpha+1)\bar{p} - 9(b-a)^{\alpha}\bar{q} - \Delta 2\Gamma(\alpha+1)\bar{p}},$$

where

$$M_i = \max_{s\in[a,b]} \bigl|f_i(s,0,0,0)\bigr|, \quad N_i = \max_{s\in[a,b]} \bigl|g_i(s,0)\bigr|, \quad i=1,2,3.$$

We prove $TB_\rho \subset B_\rho$, where $B_\rho = \{x \in \mathbb{X}\colon \|x\|_{\mathbb{X}} \leqslant \rho\}$. From ($A_4$), for all $x \in B_\rho$,

$$\begin{aligned}
\bigl|g_i\bigl(s,x_i(s)\bigr)\bigr| &\leqslant \bigl|g_i\bigl(x,x_i(s)\bigr) - g_i(s,0)\bigr| + \bigl|g_i(s,0)\bigr| \\
&\leqslant \eta_i\bigl\|x_i\|_\infty + N_i \leqslant \eta_i\|x_i\|_{\mathbb{X}} + N_i \leqslant \eta_i\rho + N_i, \\
\bigl|f_i(s,x_1,x_2,x_3)\bigr| &\leqslant \bigl|f_i(s,x_1,x_2,x_3) - f_i(s,0,0,0)\bigr| + \bigl|f_i(s,0,0,0)\bigr| \\
&\leqslant L_i\bigl(\|x_1\|_\infty + \|x_2\|_\infty + \|x_3\|_\infty\bigr) + M_i \\
&\leqslant L_i\|x\|_{\mathbb{X}} + M_i \leqslant L_i\rho + M_i, \quad i=1,2,3,
\end{aligned}$$

thus, we have

$$\begin{aligned}
&\bigl|T_1(x_1,x_2,x_3)(s)(x_1,x_2,x_3)\bigr| \\
&\quad\leqslant \frac{3(b-a)^{\alpha+\beta}}{2\Gamma(\alpha+1)\Gamma(\beta+1)\bar{p}}\left(\sum_{i=1}^{3} w_i(L_i\rho+M_i) + \sum_{i=1}^{3}(L_i\rho+M_i) + (L_1\rho+M_1)\right) \\
&\qquad + \frac{3\bar{r}(b-a)^{\alpha+\theta}}{2\Gamma(\alpha+1)\Gamma(\theta+1)\bar{p}}\left(\sum_{i=1}^{3} w_i(\eta_i\rho+N_i) + \sum_{i=1}^{3}(\eta_i\rho+N_i) + (\eta_1\rho+N_1)\right) \\
&\qquad + \frac{\bar{r}(b-a)^{\theta}}{2\Gamma(\theta+1)\bar{p}}\bigl(V_3(\eta_1\rho+N_1) + V_1(\eta_2\rho+N_2) + V_2(\eta_3\rho+N_3)\bigr) + \frac{3(b-a)^{\alpha}\bar{q}\rho}{2\Gamma(\alpha+1)\bar{p}} \\
&\qquad + \frac{(b-a)^{\beta}}{2\Gamma(\beta+1)\bar{p}}\bigl(V_3(L_1\rho+M_1) + V_1(L_2\rho+M_2) + V_2(L_3\rho+M_3)\bigr) \\
&\quad\leqslant \frac{3(b-a)^{\alpha}\bar{q}\rho}{2\Gamma(\alpha+1)\bar{p}} + \boldsymbol{a}(L_1\rho+M_1) + \boldsymbol{b}(L_2\rho+M_2) + \boldsymbol{c}(L_3\rho+M_3) + \boldsymbol{d}(\eta_1\rho+N_1) \\
&\qquad + \boldsymbol{e}(\eta_2\rho+N_2) + \boldsymbol{f}(\eta_3\rho+N_3).
\end{aligned}$$

Similarly, we can deduce that

$$\begin{aligned}
&\bigl|T_2(x_1,x_2,x_3)(s)\bigr| \\
&\quad\leqslant \frac{3(b-a)^{\alpha}\bar{q}\rho}{2\Gamma(\alpha+1)\bar{p}} + \boldsymbol{c}(L_1\rho+M_1) + \boldsymbol{a}(L_2\rho+M_2) + \boldsymbol{b}(L_3\rho+M_3) \\
&\qquad + \boldsymbol{f}(\eta_1\rho+N_1) + \boldsymbol{d}(\eta_2\rho+N_2) + \boldsymbol{e}(\eta_3\rho+N_3), \\
&\bigl|T_3(x_1,x_2,x_3)(s)\bigr| \\
&\quad\leqslant \frac{3(b-a)^{\alpha}\bar{q}\rho}{2\Gamma(\alpha+1)\bar{p}} + \boldsymbol{b}(L_1\rho+M_1) + \boldsymbol{c}(L_2\rho+M_2) + \boldsymbol{a}(L_3\rho+M_3) \\
&\qquad + \boldsymbol{e}(\eta_1\rho+N_1) + \boldsymbol{f}(\eta_2\rho+N_2) + \boldsymbol{d}(\eta_3\rho+N_3).
\end{aligned}$$

As a result, we have

$$\left\|T(x_1,x_2,x_3)(s)\right\|_{\mathbb{X}} \leqslant \frac{9(b-a)^{\alpha}\bar{q}\rho}{2\Gamma(\alpha+1)\bar{p}} + (\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})\sum_{i=1}^{3} M_i + (\boldsymbol{d}+\boldsymbol{e}+\boldsymbol{f})\sum_{i=1}^{3} N_i + \left((\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})\sum_{i=1}^{3} L_i + (\boldsymbol{d}+\boldsymbol{e}+\boldsymbol{f})\sum_{i=1}^{3}\eta_i\right)\rho \leqslant \rho.$$

This implies $TB_\rho \subset B_\rho$. Now we prove that T is a contraction mapping on B_ρ. For any $x=(x_1,x_2,x_3)$, $y=(y_1,y_2,y_3)\in\mathbb{X}$, we have

$$\begin{aligned}
&\left|T_1x(s)-T_1y(s)\right| \\
&\leqslant \frac{(b-a)^{\alpha}\bar{q}}{\Gamma(\alpha+1)\bar{p}}\|x_1-y_1\|_{\mathbb{X}} + \frac{(b-a)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)\bar{p}}\left(\sum_{i=1}^{3} w_iL_i\|x-y\|_{\mathbb{X}}\right) \\
&\quad + \frac{\bar{r}(b-a)^{\alpha+\theta}}{\Gamma(\alpha+1)\Gamma(\theta+1)\bar{p}}\left(\sum_{i=1}^{3} w_i\mu_i\|x-y\|_{\mathbb{X}}\right) + \frac{(b-a)^{\alpha}\bar{q}}{2\Gamma(\alpha+1)\bar{p}}\|x-y\|_{\mathbb{X}} \\
&\quad + \frac{3(b-a)^{\alpha+\beta}}{2\Gamma(\alpha+1)\Gamma(\beta+1)\bar{p}}\left(\sum_{i=1}^{3} L_i\|x-y\|_{\mathbb{X}} + L_1\|x-y\|_{\mathbb{X}}\right) \\
&\quad + \frac{\bar{r}(b-a)^{\alpha+\theta}}{2\Gamma(\alpha+1)\Gamma(\theta+1)\bar{p}}\left(\sum_{i=1}^{3} \mu_i\|x-y\|_{\mathbb{X}} + 2\mu_1\|x-y\|_{\mathbb{X}}\right) \\
&\quad + \frac{(b-a)^{\beta}}{2\Gamma(\beta+1)\bar{p}}\left(V_3L_1\|x-y\|_{\mathbb{X}} + V_1L_2\|x-y\|_{\mathbb{X}} + V_2L_3\|x-y\|_{\mathbb{X}}\right) \\
&\quad + \frac{\bar{r}(b-a)^{\theta}}{2\Gamma(\theta+1)\bar{p}}\left(V_3\mu_1\|x-y\|_{\mathbb{X}} + V_1\mu_2\|x-y\|_{\mathbb{X}} + V_2\mu_3\|x-y\|_{\mathbb{X}}\right) \\
&\leqslant (\boldsymbol{a}L_1+\boldsymbol{b}L_2+\boldsymbol{c}L_3+\boldsymbol{d}\mu_1+\boldsymbol{e}\mu_2+\boldsymbol{f}\mu_3)\|x-y\|_{\mathbb{X}} \\
&\quad + \frac{\left(\|x-y\|_{\mathbb{X}}+2\|x_1-y_1\|_{\mathbb{X}}\right)\bar{q}}{2\Gamma(\alpha+1)\bar{p}}(b-a)^{\alpha}.
\end{aligned}$$

Similarly, we can get

$$\begin{aligned}
\left|T_2x(s)-T_2y(s)\right| &\leqslant (\boldsymbol{c}L_1+\boldsymbol{a}L_2+\boldsymbol{b}L_3+\boldsymbol{f}\eta_1+\boldsymbol{d}\eta_2+\boldsymbol{e}\eta_3)\|x-y\|_{\mathbb{X}} \\
&\quad + \frac{\left(\|x-y\|_{\mathbb{X}}+2\|x_2-y_2\|_{\mathbb{X}}\right)\bar{q}}{2\Gamma(\alpha+1)\bar{p}}(b-a)^{\alpha}, \\
\left|T_3x(s)-T_3y(s)\right| &\leqslant (\boldsymbol{b}L_1+\boldsymbol{c}L_2+\boldsymbol{a}L_3+\boldsymbol{e}\eta_1+\boldsymbol{f}\eta_2+\boldsymbol{d}\eta_3)\|x-y\|_{\mathbb{X}} \\
&\quad + \frac{\left(\|x-y\|_{\mathbb{X}}+2\|x_3-y_3\|_{\mathbb{X}}\right)\bar{q}}{2\Gamma(\alpha+1)\bar{p}}(b-a)^{\alpha}.
\end{aligned}$$

Thus, we obtain

$$\|Tx-Ty\|_{\mathbb{X}} \leqslant \left(\Delta + \frac{9\bar{q}(b-a)^{\alpha}}{2\bar{p}\Gamma(\alpha+1)}\right)\|x-y\|_{\mathbb{X}}.$$

According to condition (16), T has a unique fixed point $x \in B_\rho$, which is the unique solution for problem (1)–(2). The proof is completed. □

4 Ulam stability

In this section, we prove the Ulam–Hyers stability of system (1)–(2). To this end, we first present the relevant stability concepts related to our problem. Let $f_i : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$ be continuous, $\varepsilon_i > 0$, $i = 1,2,3$, and $s \in [a,b]$. Suppose

$$\begin{aligned}&\bigl|{}^C D^\beta \bigl[\bigl(p(s)^C D^\alpha + q(s)\bigr)x_i(s) + r(s) I^\theta g_i\bigl(s, x_i(s)\bigr)\bigr]\\ &\quad - f_i\bigl(s, x_1(s), x_2(s), x_3(s)\bigr)\bigr| \leqslant \varepsilon_i.\end{aligned} \tag{17}$$

Definition 3. Problem (1)–(2) is said to be Ulam–Hyers stable if there exist a constant $c_{f_1,f_2,f_3} > 0$ and a function $\varepsilon = \varepsilon(\varepsilon_1, \varepsilon_2, \varepsilon_3) > 0$ such that, for each $z = (z_1, z_2, z_3) \in \mathbb{X}$ satisfying (17) and BVPs (2), there exists a solution $x = (x_1, x_2, x_3) \in \mathbb{X}$ of problem (1)–(2) with

$$\|x - z\|_{\mathbb{X}} \leqslant c_{f_1,f_2,f_3}\varepsilon.$$

Theorem 3. *If hypotheses* (A_1)*,* (A_4) *and condition* (15) *are satisfied, then problem* (1)–(2) *is Ulam–Hyers stable.*

Proof. Let z_i satisfy (17) and BVPs (2), and let z_i, $i = 1,2,3$, be solution of

$$\begin{aligned}&{}^C D^\beta \bigl[\bigl(p(s)^C D^\alpha + q(s)\bigr)z_i(s) + r(s) I^\theta g_i\bigl(s, z_i(s)\bigr)\bigr]\\ &\quad = f_i\bigl(s, z_1(s), z_2(s), z_3(s)\bigr) + \varphi_i(s).\end{aligned}$$

By Lemma 3, we have

$$\begin{aligned}z_1(s) &= \frac{1}{\Gamma(\alpha)}\int_a^s (s-\mu)^{\alpha-1}\frac{1}{p(s)}\left(\frac{1}{\Gamma(\beta)}\int_a^\mu (\mu-\nu)^{\beta-1}|\varphi_1(\nu)|\,\mathrm{d}\nu\right)\mathrm{d}\mu\\ &\quad + \frac{1}{2\Gamma(\beta)}\int_a^b (b-\mu)^{\beta-1}|V_1\varphi_2(\mu) - V_3\varphi_1(\mu) - V_2\varphi_3(\mu)|\,\mathrm{d}\mu\\ &\quad + \frac{1}{2\Gamma(\alpha)}\int_a^b (b-\mu)^{\alpha-1}\frac{1}{p(\mu)}\left(\frac{1}{\Gamma(\beta)}\int_a^\mu (\mu-\nu)^{\beta-1}|\varphi_3(\nu) - \varphi_1(\nu) - \varphi_2(\nu)|\,\mathrm{d}\nu\right)\mathrm{d}\mu\\ &\quad + \frac{(s-a)^\alpha}{p(s)\Gamma(\alpha+1)\Gamma(\beta)}\int_a^b (b-\mu)^{\beta-1}|w_3\varphi_3(\mu) - w_1\varphi_1(\mu) - w_2\varphi_2(\mu)|\,\mathrm{d}\mu + x_1(s),\\ z_2(s) &= \frac{1}{\Gamma(\alpha)}\int_a^s (s-\mu)^{\alpha-1}\frac{1}{p(\mu)}\left(\frac{1}{\Gamma(\beta)}\int_a^\mu (\mu-\nu)^{\beta-1}|\varphi_2(\nu)|\,\mathrm{d}\nu\right)\mathrm{d}\mu\\ &\quad + \frac{1}{2\Gamma(\beta)}\int_a^b (b-\mu)^{\beta-1}\bigl|V_1\varphi_3(\mu) - V_2\varphi_1(\mu) - V_3\varphi_2(\mu)\bigr|\,\mathrm{d}\mu\end{aligned}$$

$$+ \frac{1}{2\Gamma(\alpha)} \int_a^b (b-\mu)^{\alpha-1} \frac{1}{p(\mu)} \left(\frac{1}{\Gamma(\beta)} \int_a^\mu (\mu-\nu)^{\beta-1} \left|\varphi_1(\nu) - \varphi_2(\nu) - \varphi_3(\nu)\right| \mathrm{d}\nu \right) \mathrm{d}\mu$$

$$+ \frac{(s-a)^\alpha}{p(s)\Gamma(\alpha+1)\Gamma(\beta)} \int_a^b (b-\mu)^{\beta-1} \left|w_3\varphi_1(\mu) - w_1\varphi_2(\mu) - w_2\varphi_3(\mu)\right| \mathrm{d}\mu + x_2(s),$$

$$z_3(s) = \frac{1}{\Gamma(\alpha)} \int_a^s (s-\mu)^{\alpha-1} \frac{1}{p(\mu)} \left(\frac{1}{\Gamma(\beta)} \int_a^\mu (\mu-\nu)^{\beta-1} \varphi_3(\nu) \,\mathrm{d}\nu \right) \mathrm{d}\mu$$

$$+ \frac{1}{2\Gamma(\beta)} \int_a^b (b-\mu)^{\beta-1} \big(V_1\varphi_1(\mu) - V_2\varphi_2(\mu) - V_3\varphi_3(\mu)\big) \,\mathrm{d}\mu$$

$$+ \frac{1}{2\Gamma(\alpha)} \int_a^b (b-\mu)^{\alpha-1} \frac{1}{p(\mu)} \left(\frac{1}{\Gamma(\beta)} \int_a^\mu (\mu-\nu)^{\alpha-1} \big(\varphi_2(\nu) - \varphi_1(\nu) - \varphi_3(\nu)\big) \,\mathrm{d}\nu \right) \mathrm{d}\mu$$

$$+ \frac{(s-a)^\alpha}{p(s)\Gamma(\alpha+1)\Gamma(\beta)} \int_a^b (b-\mu)^{\beta-1} \big(w_3\varphi_2(\mu) - w_1\varphi_3(\mu) - w_2\varphi_1(\mu)\big) \,\mathrm{d}\mu + x_3(s).$$

Under the current conditions, T defined in (15) is a contraction operator, and hence problem (1)–(2) has a unique solution $x = (x_1, x_2, x_3) \in \mathbb{X}$, which is the fixed point of T. From (16) we have

$$\|Tx - Tz\|_{\mathbb{X}} = \|x - Tz\|_{\mathbb{X}} \leqslant \left[\varDelta + \frac{5(b-a)^\alpha \bar{q}}{2\Gamma(\alpha+1)\bar{p}} \right] \|x - z\|_{\mathbb{X}}.$$

This implies

$$\|x - z\|_{\mathbb{X}} \leqslant \frac{2\Gamma(\alpha+1)\bar{p}}{2\Gamma(\alpha+1)\bar{p}(1-\varDelta) - 5(b-a)^\alpha \bar{q}} \|Tz - x\|_{\mathbb{X}}.$$

On the other hand, we have the following estimate:

$$\begin{aligned}
&\left|T_1(z_1, z_2, z_2)(s) - x_1(s)\right| \\
&\quad \leqslant \frac{1}{\Gamma(\alpha)} \int_a^s (s-\mu)^{\alpha-1} \frac{1}{p(\mu)} \left(\frac{1}{\Gamma(\beta)} \int_a^\mu (\mu-\nu)^{\beta-1} |\varphi_1(\nu)| \,\mathrm{d}\nu \right) \mathrm{d}\mu \\
&\qquad + \frac{1}{2\Gamma(\beta)} \int_a^b (b-\mu)^{\beta-1} \left|V_1\varphi_2(\mu) - V_3\varphi_1(\mu) - V_2\varphi_3(\mu)\right| \mathrm{d}\mu \\
&\qquad + \frac{(s-a)^\alpha}{p(s)\Gamma(\alpha+1)\Gamma(\beta)} \int_a^b (b-\mu)^{\beta-1} \left|w_3\varphi_3(\mu) - w_1\varphi_1(\mu) - w_2\varphi_2(\mu)\right| \mathrm{d}\mu \\
&\qquad + \frac{1}{2\Gamma(\alpha)} \int_a^b (b-\mu)^{\alpha-1} \frac{1}{p(\mu)} \left(\frac{1}{\Gamma(\beta)} \int_a^\mu (\mu-\nu)^{\beta-1} |\varphi_3(\nu) - \varphi_1(\nu) - \varphi_2(\nu)| \,\mathrm{d}\nu \right) \mathrm{d}\mu
\end{aligned}$$

$$\leqslant \frac{(b-a)^{\alpha+\beta}}{2\Gamma(\beta+1)}\left\{(V_3\varepsilon_1+V_1\varepsilon_2+V_2\varepsilon_3)+\frac{(3\varepsilon_1+\varepsilon_2+\varepsilon_3+2\sum_{i=1}^{3}w_i\varepsilon_i)}{\bar{p}\Gamma(\alpha+1)}\right\}.$$

Using a similar approach, we get

$$\begin{aligned}\left|T_2(z_1,z_2,z_3)(s)-x_2(s)\right| &\leqslant \frac{(b-a)^{\alpha+\beta}}{2\Gamma(\beta+1)}\left\{(V_2\varepsilon_1+V_3\varepsilon_2+V_1\varepsilon_3)\right.\\ &\quad \left.+\frac{[(1+2w_3)\varepsilon_1+(2w_1+3)\varepsilon_2+(2w_2+1)\varepsilon_3]}{2\bar{p}\Gamma(\alpha+1)}\right\},\\ \left|T_3(z_1,z_2,z_3)(s)-x_3(s)\right| &\leqslant \frac{(b-a)^{\alpha+\beta}}{2\Gamma(\beta+1)}\left\{(V_1\varepsilon_1+V_2\varepsilon_2+V_3\varepsilon_3)\right.\\ &\quad \left.+\frac{[(1+2w_2)\varepsilon_1+(2w_3+1)\varepsilon_2+(2w_1+3)\varepsilon_3]}{2\bar{p}\Gamma(\alpha+1)}\right\}.\end{aligned}$$

Based on the above inequality, we obtain

$$\|Tz-x\|_{\mathbb{X}} \leqslant \frac{(b-a)^{\alpha+\beta}\sum_{i=1}^{3}\varepsilon_i}{2\Gamma(\beta+1)}\left\{\frac{5+\sum_{i=1}^{3}w_i}{\bar{p}\Gamma(\alpha+1)}+\sum_{i=1}^{3}V_i\right\}.$$

Let $\varepsilon=\max\{\varepsilon_1,\varepsilon_2,\varepsilon_3\}$, then we have

$$\|x-z\|_{\mathbb{X}} \leqslant \frac{3\Gamma(\alpha+1)\bar{p}(b-a)^{\alpha+\beta}}{[2\Gamma(\alpha+1)\bar{p}-5(b-a)^{\alpha}\bar{q}]\Gamma(\beta+1)}\left\{\frac{5+\sum_{i=1}^{3}w_i}{\bar{p}\Gamma(\alpha+1)}+\sum_{i=1}^{3}V_i\right\}.$$

Hence, problem (1)–(2) is Ulam–Hyers stable. □

5 Examples

Example 1. Let $\beta=4/5$, $\alpha=1/4$, $\theta=1/2$, $i=1,2,3$. Consider

$$\begin{aligned}&{}^{C}D^{1/4}\left[p(s){}^{C}D^{4/5}+q(s)x_i(s)+r(s)I^{1/2}g_i(s,x_i(s)\right]\\ &\quad =f_i\big(s,x_1(s),x_2(s),x_3(s)\big),\\ &x_1(0)+x_2(1)=0, \qquad {}^{C}D^{4/5}x_1(0)+{}^{C}D^{4/5}x_2(1)=0,\\ &x_2(0)+x_3(1)=0, \qquad {}^{C}D^{4/5}x_2(0)+{}^{C}D^{4/5}x_3(1)=0,\\ &x_3(0)+x_1(1)=0, \qquad {}^{C}D^{4/5}x_3(0)+{}^{C}D^{4/5}x_1(1)=0,\\ &g_i\big(s,x_i(s)\big)=\frac{|x_i(s)|}{3(49+i+s)(1+|x_i(s)|)}, \quad i=1,2,\\ &f_1\big(s,x_1(s),x_2(s),x_3(s)\big)\\ &\quad =\mathrm{e}^s+\frac{\cos x_1(s)}{25(5+s)}+\frac{x_2(s)}{15(3+5\mathrm{e}^s)}+\frac{s}{120}\frac{x_3^2(s)}{(1+|x_3(s)|)},\end{aligned} \tag{18_1}$$

$$\begin{aligned} &f_2\bigl(s, x_1(s), x_2(s), x_3(s)\bigr) \\ &\quad = 1 + \frac{x_1(s)}{15\sqrt{2s^2+81}} + \frac{x_2(s)}{9+(5+4\mathrm{e}^s)^2} + \frac{s}{100}\frac{|x_3(s)|}{(1+|x_3(s)|)}, \\ &f_3\bigl(s, x_1(s), x_2(s), x_3(s)\bigr) \\ &\quad = 3 + \ln(1+s) + \frac{x_1(s)}{80+s^2} + \frac{(s+2)x_2(s)}{8(5+s)^2} + \frac{x_3(s)}{2(\mathrm{e}^s+47)}. \end{aligned} \tag{18$_2$}$$

We choose

$$\begin{aligned} &p(s) = \left(s - \frac{1}{2}\right)^2 + \frac{3}{4}, \quad q(s) = \frac{s}{50}, \quad r(s) = 1, \\ &k_1(s) = \mathrm{e}^s, \quad k_2(s) = 1 + \sin s, \quad k_3(s) = 3 + \ln(1+s), \\ &u_1(s) = \frac{1}{25(5+s)}, \quad u_2(s) = \frac{1}{15\sqrt{2s^2+81}}, \quad u_3(s) = \frac{x_1(s)}{80+s^2}, \\ &v_1(s) = \frac{1}{15(3+5\mathrm{e}^s)}, \quad v_2(s) = \frac{1}{9+(5+4\mathrm{e}^s)^2}, \quad v_3(s) = \frac{(s+2)}{8(5+s)^2}, \\ &\varpi_1(s) = \frac{s}{120}, \quad \varpi_2(s) = \frac{1}{(10+s)^2}, \quad \varpi_3(s) = \frac{1}{2(\mathrm{e}^s+47)}. \end{aligned}$$

For $s \in [0,1]$, we can calculate $\bar{p} = 3/4$, $\bar{q} = 1/50$, $\bar{r} = 1$, $\lambda_1 = 1/150$, $\lambda_2 = 1/153$, $\lambda_3 = 1/156$, $u_1 = 1/125$, $u_2 = 1/135$, $u_3 = 1/80$, $v_1 = 1/120$, $v_2 = 1/90$, $v_3 = 1/100$, $\varpi_1 = 1/120$, $\varpi_2 = 1/100$, $\varpi_3 = 1/96$,

$$\sum_{i=1}^{3} l_i = \sum_{i=1}^{3} (u_i + v_i + \varpi_i) \approx 0.08610, \qquad \sum_{i=1}^{3} \lambda_{\mathrm{i}} \approx 0.01962,$$

$$\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = \frac{79}{12\Gamma(5/4)\Gamma(9/5)} \approx 8.1951, \qquad \boldsymbol{d} + \boldsymbol{e} + \boldsymbol{f} = \frac{47}{6\Gamma(5/4)\Gamma(3/2)} \approx 9.7520.$$

Hence, we obtain

$$\frac{5(b-a)^\alpha \bar{q}}{2\Gamma(\alpha+1)\bar{p}} + (\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}) \sum_{i=1}^{3} l_i + (\boldsymbol{d} + \boldsymbol{e} + \boldsymbol{f}) \sum_{i=1}^{3} \lambda_i \approx 0.9705 < 1.$$

Therefore, condition (15) is satisfied, and problem (18) has at least one solution on $[0,1]$.

Example 2. Let $\alpha = 7/8$, $\beta = \theta = 4/5$, $i = 1,2,3$. Consider

$$\begin{aligned} &{}^C D^{4/5}\bigl[p(s)^C D^{7/8} + q(s) z_i(s) + r(s) I^{4/5} g_i\bigl(s, x_i(s)\bigr)\bigr] \\ &\quad = f_i\bigl(s, x_1(s), x_2(s), x_3(s)\bigr), \\ &x_1(0) + x_2(1) = 0, \qquad {}^C D^{7/8} x_1(0) + {}^C D^{7/8} x_2(1) = 0, \\ &x_2(0) + x_3(1) = 0, \qquad {}^C D^{7/8} x_2(0) + {}^C D^{7/8} x_3(1) = 0, \\ &x_3(0) + x_1(1) = 0, \qquad {}^C D^{7/8} x_3(0) + {}^C D^{7/8} x_1(1) = 0, \end{aligned} \tag{19}$$

where

$$p(s) = \frac{4(s-\frac{1}{2})^2 + 7}{8}, \quad q(s) = \frac{s}{25}, \quad r(s) = 1,$$

$$g_i\bigl(s, x_i(s)\bigr) = \frac{|x_i(s)|}{3(24+i+s)(1+|x_i(s)|)}, \quad i = 1, 2,$$

$$f_1\bigl(s, x_1(s), x_2(s), x_3(s)\bigr) = \frac{2 + |x_1(s)| + |x_2(s)| + |x_3(s)|}{(s+50)(1+|x_1(s)| + |x_2(s)| + |x_3(s)|)},$$

$$f_2\bigl(s, x_1(s), x_2(s), x_3(s)\bigr) = \frac{1}{5(s^2+8)}\left[\mathrm{e}^s x_1(s) + \sin(x_2(s)) + \frac{|x_3(s)|}{1+|x_3(s)|}\right],$$

$$f_3\bigl(s, x_1(s), x_2(s), x_3(s)\bigr) = \frac{\cos(s)x_1(s)}{30(\mathrm{e}^s+1)} + \frac{\sin(\pi x_2(s))}{60\pi} + \frac{\cos x_1(s)}{30\sqrt{4+s^2}}$$

for $s \in [0,1]$. We take $\bar{p} = 7/8$, $\bar{q} = 1/25$, $\mu_1 = 1/75$, $\mu_2 = 1/78$, $\mu_3 = 1/81$, $L_1 = 1/50$, $L_2 = 1/40$, $L_3 = 1/60$,

$$\Delta = (\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})(L_1 + L_2 + L_3) + (\boldsymbol{d} + \boldsymbol{e} + \boldsymbol{f})(\mu_1 + \mu_2 + \mu_3) \approx 0.6665.$$

Therefore, we find

$$\Delta + \frac{9(b-a)^\alpha \bar{q}}{2\Gamma(\alpha+1)\bar{p}} \approx 0.8854 < 1,$$

that is, condition (16) is satisfied, so problem (19) has a unique solution. Moreover one can easily obtain that (19) is Ulam–Hyers stable by Theorem 3.

6 Conclusions

We consider a new triple nonlinear fractional system with cyclic boundary conditions and delve into the study of the qualitative behaviors of its solutions, including existence, uniqueness, and stability. By using fixed point theorem, the existence result are obtained. In addition, the Ulam–Hyers stability results of the given system were analyzed. Based on the above research, this paper can be extended to study triple nonlinear fractional pantograph differential equation with p-Laplacian operator.

Conflicts of interest. The authors declare no conflicts of interest.

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