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Improved fixed-time consensus of delayed nonlinear leader–follower multi-agent systems: A new stability approach*

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Abstract. This article aims to study the problem of fixed-time consensus for nonlinear leader–follower multi-agent systems (MASs) with time delay. Firstly, a new fixed-time stability lemma is derived, where the condition of inequality contains a time-delayed term. Furthermore, a more precise estimated value of settling time (ST) including delayed parameter is obtained, which is different from those existing fixed-time stability lemmas. Thereby, it provides some options for designers in many practical scenarios when considering the delayed systems. Secondly, for the purpose of explaining its applicability, fixed-time consensus of leader–follower nonlinear MASs is investigated by designing a nonlinear control protocol including constant time delay. The designed protocol not only guarantees fixed-time consensus but also effectively improves the convergence rate. With the new proposed lemma, a novel consensus criterion is designed. This is the first time to obtain the time-delayed dependent fixed-time stability criterion for MASs. Finally, the validity and superiority of the established theoretical results are confirmed by one numerical simulation.

Keywords: multi-agent systems, fixed-time consensus, fixed-time stability, time-delayed systems.

1 Introduction

In recent decades, MASs have been getting increased attention because of their extensive applications. With the development of MASs technology, robots [16], complex networks [21], and unmanned aerial vehicles (UAVs) [29] have all undergone revolutionary changes. For now though, the research of MASs is mainly divided into consensus, formation, flocking and clustering [24], etc. As the fundament of intelligence research, consensus

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problem is one of the most popular topics on the follow-up control of MASs due to its potential engineering applications in many fields, for example, sensor network synchronization, UAVs formation control, underwater vehicles formation, and so on.

Based on the above analysis, many scholars have generated large amounts of instructive research about the consensus issue of MASs. For example, in [1], a novel event-triggered mechanism was presented to eventually achieve the bipartite consensus of linear MASs; in [12], the consensus of linear MASs was researched by using a control scheme combining adaptive control and event triggering; in [32], the problem of leader–follower consensus for MASs was investigated by utilizing reset strategy. Note that lots of the above-mentioned contributions deal with the asymptotic consensus problem. Nevertheless, these research achievements may be unprocurable in real-world applications. As a result, some scholars [8, 15, 20] have extensively explored the finite-time consensus of MASs. Unfortunately, the ST is heavily dependent on the initial conditions. Thus it will severely limit its application range when the systems' initial conditions are hard or impossible to be measured in advance.

Fortunately, a novel concept of fixed-time stability was firstly proposed in [22], where the ST is estimated independently on the initial conditions. Thereafter, this great idea is introduced into the cooperative control issues of MASs. In [14], a distributed observer and an integral sliding mode controller were proposed to study fixed-time leader–follower consensus of heterogeneous MASs; in [18], a novel distributed observer and a sliding surface were designed to investigate fixed-time consensus tracking of second-order MASs; in [20], fixed-time consensus was investigated based on discontinuous inherent dynamics within the framework of leader–follower MASs; paper [28] studied fixed-time tracking consensus of first-order nonlinear MASs with general directed communication topologies.

However, none of the above-mentioned results has taken into account time delay. It is widespread in the process of information transmission, which may result in instability and poor performance of systems. Besides, communication delays are a common phenomenon and frequently encountered in real-world systems. Thus a number of admirable research results have been provided in MASs [9, 19, 25, 30, 31]. In [9], two constructed logical functions were used to handle communication delay to study average consensus of homogeneous MASs; in [19], the fixed-time practical consensus control was investigated with communication delay in MASs; in [31], the leader–follower consensus for nonlinear MASs was explored by designing a new output feedback control approach to deal with the infinite communication delay among agents. In [5], leader–follower formation control of MASs with input time delays was investigated by designing an event-triggered control strategy. The consensus tracking problem of time-delayed MASs was considered in [13, 23]. Nevertheless, to the best of our knowledge, the aforementioned research findings mainly employ the existing fixed-time stability lemmas in [2, 3, 11, 22] to study the consensus of delayed MASs. Therefore, how to design a fixed-time stability lemma including time-delayed term is challenging. This problem is still worth investigating and awaiting breakthrough. Moreover, it is imperative to further study the fixed-time consensus issue of time-delayed MASs in a leader–follower framework coincided with actual demands, for example, UAV swarm, power systems, intelligent traffic control, etc., which is also the motivation of this article.

Under the points discussed above, the contributions of this brief mainly centers on the fixed-time leader–follower consensus of first-order nonlinear delayed MASs. Compared with the current results, the innovativeness is listed as follows:

(i) A new fixed-time stability lemma is established. Unlike the existing lemmas in [3,22], the time-delayed term is introduced into the inequality condition, which improves the traditional inequality condition

$$\dot{V}\big(x(t)\big) \leqslant -aV\big(x(t)\big) - bV^{\xi}\big(x(t)\big) - cV^{\eta}\big(x(t)\big).$$

(ii) An optimization algorithm is used to get a better estimation of the upper bound of ST. It is related to the coefficient of delay term. The precision of ST is enhanced when compared with all the latest research findings that we can find, which demonstrates the theoretical superiority of this work.

(iii) In contrast to the control protocols in [14, 20, 28], a new nonlinear and discontinuous control protocol is presented. It includes the state delay of the agents, an innovative method of setting symbolic function, and two compensation terms. Thus, it is more beneficial to investigate the consensus of delayed MASs.

(iv) Based on the new lemma and the designed control protocol, leader–follower consensus of delayed MASs is discussed, which can improve and extend those existing research results in [13, 23]. Simultaneously, two corollaries are deduced by the lemmas in [3, 22], respectively.

The rest of this article is organized as follows. Section 2 gives a brief review of some preliminaries and problem formulation. In Section 3, first, a new fixed-time stability lemma involving the delay term is established, and the ST is more accurate. Second, with a novel class of nonlinear and discontinuous control protocol, fixed-time leader–follower consensus of nonlinear delayed MASs is investigated, and three different kinds of upper bound of ST are obtained by different stability lemmas. In Section 4, numerical simulation verifies the effectiveness of the obtained theoretical results. Finally, the research findings are summarized in Section 5.

Notations. Throughout this work, some notations are listed as follows. Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}^n$, and $\mathbb{R}^{n\times n}$ represent the real field, the set of positive real numbers, the n-dimensional real vector space, and the space of real $n \times n$ matrix, respectively. $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ individually indicate the smallest and largest eigenvalues of a real symmetric matrix A. For any $\xi > 0$ and $x \in \mathbb{R}$, define $\mathrm{sig}^{\xi}(x) = \mathrm{sign}(x)|x|^{\xi}$, where $\mathrm{sign}(\cdot)$ is the standard sign function. What is more, for any $x = (x_1, x_2, \dots, x_n)^{\mathrm{T}}$, define $\|x\| = (\sum_{i=1}^{n} x_i^2)^{1/2}$, which represents the 2-norm. The symbol $\mathrm{diag}(\cdot)$ means a diagonal matrix.

2 Preliminaries and model description

2.1 Graph theory

The interaction topology of N followers is symbolized as an undigraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, of which $\mathcal{V} = \{v_1, v_2, \dots, v_N\}$ denotes the set of nodes, and each node represents a follower, $\mathcal{E} = \{(v_i, v_j),\ v_i, v_j \in \mathcal{V}\}$ is the edge set, and every edge represents the communication link between two nodes, and $A = (a_{ij})_{N\times N}$ is the adjacency matrix with nonnegative elements. Note that $a_{ij} = 1$ if and only if $(v_i, v_j) \in \mathcal{E}$; otherwise $a_{ij} = 0$, $i \neq j$, $i, j = 1, 2, \dots, N$. Furthermore, it is assumed that no self-loop can be found in the graph $\mathcal{G}$, that is to say, $a_{ii} = 0$. Hence, A is symmetric. In the cooperative control, each follower directly interacts with its neighbours, and the neighbor set of v_i is represented as $\mathcal{N}_i = \{v_j \in \mathcal{V},\ (v_i, v_j) \in \mathcal{E}\}$. The Laplacian matrix associated with A is defined as $L = (l_{ij})_{N\times N}$ with elements $l_{ii} = \sum_{j=1,\, j\neq i}^{N} a_{ij}$, $l_{ij} = -a_{ij}$, $i \neq j$, $i, j = 1, 2, \dots, N$, thereby L is also symmetric. In this paper, by considering the communication interaction between N followers and one leader labelled with 0, the diagonal matrix $B = \operatorname{diag}(b_1, b_2, \dots, b_N)$ is defined. More specifically, when the ith follower receives information from the leader, $b_i = 1$; otherwise, $b_i = 0$. Consequently, a new matrix L_B is denoted as $L_B = L + B$.

Assumption 1. The interaction topology of N followers and 1 leader consists at least one directed spanning tree, and it is also connected.

From Assumption 1 it is easy to know that the leader is globally reachable. It can directly transmit its information to at least one follower, that is to say, there exists at least one $b_i > 0$, $i = 1, 2, \dots, N$. Furthermore, when the topology graph is an undirected graph, the matrix L_B is positive definite.

2.2 Basic definition and lemmas

In this paper, we consider a delayed dynamical system modelled as follows:

$$\begin{aligned} \dot{x}(t) &= f\bigl(t,\, x(t),\, x(t-\tau)\bigr), \quad \text{a.e. } t \geqslant 0, \\ x_\varsigma &= \varphi(\varsigma) \in C\bigl([-\tau, 0], \mathbb{R}^n\bigr), \end{aligned} \tag{1}$$

where

$$x(t) = \bigl(x_1(t), x_2(t), \dots, x_n(t)\bigr)^{\mathrm{T}} \in \mathbb{R}^n$$

and

$$x(t-\tau) = \bigl(x_1(t-\tau),\, x_2(t-\tau),\, \dots,\, x_n(t-\tau)\bigr)^{\mathrm{T}} \in \mathbb{R}^n$$

denote the state vector and the delayed state vector, respectively, $\tau > 0$ is a time-delay parameter, which is a finite number; $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear and continuous function, which is Lebesgue measurable as well as local bounded; $\varphi : C[-\tau, 0] \to \mathbb{R}^n$ is the initial condition function.

In order to research the fixed-time stability of system (1), suppose that $f(0) = 0$, and some necessary definition and lemmas are recalled in the following.

Definition 1. (See [4].) A function $V(x) : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ is C-regular if and only if the conditions below are fulfilled:

(i) $V(x)$ is regular, i.e. the derivative of $V(x)$ exists everywhere on $\mathbb{R}^n$;
(ii) $V(x) > 0,\ x \neq 0;\ V(0) = 0$;
(iii) $V(x) \to +\infty$ when $\|x\| \to +\infty$.

Lemma 1. (*See* [6].) *For any real number* $0 < p \leqslant 1$ *and* $q > 1$, *if* $\beta_i \geqslant 0$, $i = 1, 2, \dots, n$, *then*

$$\sum_{i=1}^{n} \beta_i^p \geqslant \left(\sum_{i=1}^{n} \beta_i\right)^p, \qquad \sum_{i=1}^{n} \beta_i^q \geqslant n^{1-q}\left(\sum_{i=1}^{n} \beta_i\right)^q.$$

Lemma 2. (*See* [6].) *If* $x, y \in \mathbb{R}^n$ *and* $P \in \mathbb{R}^{n\times n}$ *is a positive definite matrix, then*

$$2x^{\mathrm{T}}Py \leqslant x^{\mathrm{T}}Px + y^{\mathrm{T}}Py.$$

Lemma 3. (*See* [7].) *For any* $x \in \mathbb{R}^n$ *and any symmetric matrix* $Q \in \mathbb{R}^{n\times n}$, *the following inequality holds*:

$$\lambda_{\min}(Q)x^{\mathrm{T}}x \leqslant x^{\mathrm{T}}Qx \leqslant \lambda_{\max}(Q)x^{\mathrm{T}}x.$$

Lemma 4. (*See* [22].) *Suppose that the function* $V(x(t)) : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ *is* C*-regular. If any solution* $x(t)$ *of system* (1) *fulfills*

$$\dot{V}\big(x(t)\big) \leqslant -aV^{\xi}\big(x(t)\big) - bV^{\eta}\big(x(t)\big), \quad x(t) \neq 0, \tag{2}$$

where $a, b > 0$, $\xi \in (0, 1)$, *and* $\eta > 1$, *then the zero solution of system* (1) *realizes fixed-time stability, and the ST is bounded by*

$$T_{\max}^{1} = \frac{1}{a(1-\xi)} + \frac{1}{b(\eta-1)}.$$

Lemma 5. (*See* [3].) *Suppose that the function* $V(x(t)) : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ *is* C*-regular. If any solution* $x(t)$ *of system* (1) *fulfills*

$$\dot{V}\big(x(t)\big) \leqslant -aV\big(x(t)\big) - bV^{\xi}\big(x(t)\big) - cV^{\eta}\big(x(t)\big), \quad x(t) \neq 0, \tag{3}$$

where $a, b, c > 0$, $\xi \in (0, 1)$, *and* $\eta > 1$, *then the zero solution of system* (1) *can realize fixed-time stability, and the ST is bounded by*

$$T_{\max}^{2} = \frac{1}{a(1-\xi)} \ln\left(1 + \frac{a}{b}\right) + \frac{1}{a(\eta-1)} \ln\left(1 + \frac{a}{c}\right).$$

Remark 1. In the process of proving Lemmas 4 and 5, $V(x(t))$ is always bounded with 1. Only when the relation between $V(x(t))$ and 1 is determined, the method of enlarging and reducing skillfully can be employed to make those inequalities a reality. In addition, this technique has been used in almost all previous research achievements. In this paper, the traditional discussions about $0 < V < 1$ and $V > 1$ are undesired, and a better dividing point will be found to accurately estimate the ST.

Remark 2. Lemma 4 is the former research result on the fixed-time stability without delay. Later, some improved results on the fixed-time stability without delay were established; see, for example, [3, 11]. From inequalities (2) and (3) it is easy to observe that they do not have delayed term. Therefore, when researchers consider the delayed systems, these inequalities cannot be used theoretically in some cases. Moreover, they cannot objectively reflect how much the time delay can influence the systems' stability. Hence, the inequality condition including the time-delayed term will be provided, a novel fixed-time stability lemma will be established, a more accurate estimation of the ST, involving the delayed parameter, will be given in the next section.

2.3 Problem formulation

In this subsection, the considered one-dimensional nonlinear MASs, including one leader labeled as 0 and N followers labeled as $1, 2, \dots, N$, is modelled as

$$\begin{aligned} \dot{x}_0(t) &= f\big(x_0(t),\, x_0(t-\tau),\, t\big), \\ \dot{x}_i(t) &= f\big(x_i(t),\, x_i(t-\tau),\, t\big) + u_i(t), \quad i = 1, 2, \dots, N, \end{aligned} \tag{4}$$

where $x_0(t) \in \mathbb{R}$ and $x_0(t-\tau) \in \mathbb{R}$ denote the state variable and the delayed state variable of the leader, respectively; $x_i(t) \in \mathbb{R}$ and $x_i(t-\tau) \in \mathbb{R}$ represent the state variable and the delayed state variable of the ith follower, respectively; $\tau > 0$ corresponds to the communication delay; $f : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ is the nonlinear and continuous inherent dynamics for agents of the leader and the followers; $u_i(t) = u_i(x_i(t),\, x_i(t-\tau),\, t)$ denotes the ith follower's control input, which will be designed later.

Assumption 2. For any $x(t), y(t) \in \mathbb{R}$, the nonlinear function $f(x(t), t)$ fulfills the inequality

$$\big|f\big(x(t), t\big) - f\big(y(t), t\big)\big| \leqslant l\big|x(t) - y(t)\big|,$$

that is to say, it is Lipschitz continuous.

Definition 2. For some given control protocols $u_i(t)$, $i = 1, 2, \dots, N$, MASs (4) are called fixed-time leader–follower consensus if and only if the conditions below are fulfilled:

$$\begin{aligned} &\lim_{t \to T^{*}-} \big|x_i(t) - x_0(t)\big| = 0, \quad i = 1, 2, \dots, N, \\ &x_i(t) = x_0(t), \quad t \geqslant T^*, \end{aligned}$$

where $T^* > 0$ depends on the initial conditions, and it has an upper bound denoted as $T_{\max}$, which is independent of the initial values.

Before ending this subsection, the following two lemmas will be recalled to develop the new results in the next section.

Lemma 6. (*See* [15].) *If the graph $\mathcal{G}$ is undirected and $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_N)^{\mathrm{T}}$, then*

$$\zeta^{\mathrm{T}} L \zeta = \frac{1}{2} \sum_{i,j=1}^{N} a_{ij} (\zeta_j - \zeta_i)^2.$$

Lemma 7. (*See* [8].) *For the undirected graph $\mathcal{G}$, if $\Phi(\cdot,\cdot)$ satisfies*

$$\Phi(x_i, x_j) = -\Phi(x_j, x_i),$$

where $(x_1, x_2, \dots, x_N)^{\mathrm{T}} \in \mathbb{R}^N$, $i, j = 1, 2, \dots, N$, $i \neq j$, then for any $(y_1, y_2, \dots, y_N)^{\mathrm{T}} \in \mathbb{R}^N$,

$$\sum_{i,j=1}^{N} a_{ij} y_i \Phi(x_j, x_i) = -\frac{1}{2} \sum_{i,j=1}^{N} a_{ij} (y_j - y_i) \Phi(x_j, x_i).$$

3 Main results

In this section, firstly, a new fixed-time stability lemma containing time delay will be given. Then, with a newly designed nonlinear and discontinuous control protocol including state delay, fixed-time leader–follower consensus for delayed MASs will be investigated.

3.1 A new fixed-time stability lemma

Lemma 8. *Suppose that the function $V(x(t)) : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ is C-regular. If any solution $x(t)$ of system* (1) *fulfills*

$$\dot{V}\big(x(t)\big) \leqslant -aV\big(x(t-\tau)\big) - bV^{\xi}\big(x(t)\big) - cV^{\eta}\big(x(t)\big) \tag{5}$$

whenever $V(x(t)) < V(x(t-\tau))$ for any $t \in [-\tau, 0]$ along the trajectory of system (1)*, $a, b, c > 0$, $\xi \in (0,1)$, and $\eta > 1$, then the origin of system* (1) *can realize fixed-time stability, and the upper bound of ST is*

$$T_{\max}^{3} = \left[\frac{1}{a(\eta-1)} + \frac{1}{a(1-\xi)}\right] \ln\left[1 + \frac{a}{b}\left(\frac{b}{c}\right)^{(1-\xi)/(\eta-\xi)}\right].$$

Proof. Firstly, it needs to prove that system (1) is globally stable in finite time.

From inequality (5) one can draw a conclusion that $V(x(t))$ decreases as t increases. Because $V(x(t))$ is C-regular, one has $\lim_{t\to+\infty} V(x(t)) = 0$.

According to system (1), if $t \in [0,\tau]$, then $x(t-\tau) = \varphi(\varsigma)$, where $\varsigma = t-\tau \in [-\tau, 0]$. Therefore, when $t \in [0,\tau]$, we have $V(x(t-\tau)) = V(\varphi(\varsigma))$, which shows that the initial condition $x(\varsigma) = \varphi(\varsigma)$ will determine $V(x(t-\tau))$. Hence, combining it with the decreasing property of $V(x(t))$, it is easy to see that $0 < V(x(t)) < V(x(t-\tau))$ for any $t \in [0,\tau]$. Furthermore, with mathematical deduction, one can draw a conclusion that $0 < V(x(t)) < V(x(t-\tau))$ for any $t > \tau$. Therefore, $V(x(t)) < V(x(t-\tau))$ for any $t > 0$.

According to inequality (5), we define

$$\Psi\big(V\big(x(t)\big)\big) = \int_0^{V(x(t))} \frac{\mathrm{d}v(t)}{av(t-\tau) + bv^{\xi}(t) + cv^{\eta}(t)} \leqslant \int_0^{V(x(t))} \frac{\mathrm{d}v(t)}{av(t) + bv^{\xi}(t) + cv^{\eta}(t)}.$$

Obviously, $\Psi(V(x(t)))$ is nonnegative, and

$$\frac{\mathrm{d}\Psi(V(x(t)))}{\mathrm{d}t} \leqslant -1, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Let $T(x_\tau) - \tau = \Psi(V(x_\tau))$ for all $x_\tau \in \mathbb{R}^n \setminus \{0\}$. Next, we show that $x(t_1) = 0$ for some real number $t_1 \in (0, T(x_\tau)]$. If it is not true, that is to say, $x(t) \neq 0$ for all $t \in (0, T(x_\tau)]$.

Since

$$\frac{\mathrm{d}\Psi(V(x(t)))}{\mathrm{d}t} \leqslant -1, \quad t \in \big(0, T(x_\tau)\big],$$

integrating it from τ to $T(x_\tau)$, it follows

$$0 \leqslant \Psi\big(V\big(x\big(T(x_\tau)\big)\big)\big) \leqslant \Psi\big(V\big(x_\tau\big)\big) - \big(T(x_\tau) - \tau\big) = 0.$$

By using the C-regularity of $V(x(t))$, we obtain that $x(T(x_\tau)) = 0$, which is a contradiction.

In the following, we demonstrate that $x(t) = 0$, where $t \in [t_1, +\infty)$. If it is untrue, namely, there exists t_2 such that $x(t_2) \neq 0$, where $t_2 > t_1$, we define

$$t_3 = \sup\big\{t \in [t_1, t_2)\colon\ x(t) = 0\big\}.$$

Then it is evident that $t_1 \leqslant t_3 < t_2$, $x(t_3) = 0$, $x(t_2) \neq 0$ for any $t \in (t_3, t_2]$. It follows that $\Psi(V(x(t_3))) = 0$ and

$$\frac{\mathrm{d}\Psi(V(x(t)))}{\mathrm{d}t} \leqslant -1, \quad t \in (t_3, t_2].$$

Integrating the above inequality from t_3 to t_2, it follows

$$\Psi\big(V(x(t_2))\big) \leqslant -(t_2 - t_3) < 0.$$

This is in contradiction with the nonnegativity of $\Psi(V(x(t)))$. Therefore, $x(t) = 0$ for all $t \in [t_1, +\infty)$. Correspondingly, the origin of system (1) is globally stable in finite time.

Secondly, the upper bound of ST is estimated.

When $V(x(t)) \geqslant r > 0$, combining this with $0 < r \leqslant V(x(t)) < V(x(t-\tau))$ for any $t > 0$ and inequality (5), it follows that

$$\dot{V}\big(x(t)\big) \leqslant -aV\big(x(t)\big) - bV^{\xi}\big(x(t)\big) - cV^{\eta}\big(x(t)\big).$$

Let $v = V(x(t))$. Since $V(x(t)) \geqslant r$, then

$$t_{y=r} = \int\limits_{r}^{v(\tau)} \frac{\mathrm{d}v}{av + bv^{\xi} + cv^{\eta}} \leqslant \int\limits_{r}^{+\infty} \frac{\mathrm{d}v}{av + bv^{\xi} + cv^{\eta}} \leqslant \int\limits_{r}^{+\infty} \frac{\mathrm{d}v}{av + cv^{\eta}}.$$

Letting $u = v^{1-\eta}$, then $\mathrm{d}u = (1-\eta)v^{-\eta}\,\mathrm{d}v$, thus

$$\int\limits_{r}^{+\infty} \frac{\mathrm{d}v}{av + cv^{\eta}} = \frac{1}{\eta - 1}\int\limits_{0}^{r^{1-\eta}} \frac{\mathrm{d}u}{au + c} = \frac{1}{a(\eta - 1)} \ln\!\left(1 + \frac{a}{c}r^{1-\eta}\right). \tag{6}$$

When $0 < V(x(t)) < r$, combining this with $0 < V(x(t)) < V(x(t-\tau))$ for any $t > 0$ and the inequality (5), we obtain

$$\dot{V}(x(t)) \leqslant -aV\big(x(t)\big) - bV^{\xi}\big(x(t)\big) - cV^{\eta}\big(x(t)\big).$$

Let $v = V(x(t))$. Since $0 < V(x(t)) < r$, then

$$T_{\max} - t_{y=r} \leqslant \int_0^r \frac{\mathrm{d}v}{av + bv^\xi + cv^\eta} \leqslant \int_0^r \frac{\mathrm{d}v}{av + bv^\xi}.$$

Letting $u = v^{1-\xi}$, then $\mathrm{d}u = (1-\xi)v^{-\xi}\,\mathrm{d}v$, thus

$$\int_0^r \frac{\mathrm{d}v}{av + bv^\xi} = \frac{1}{1-\xi}\int_0^{r^{1-\xi}} \frac{\mathrm{d}u}{au + b} = \frac{1}{a(1-\xi)}\ln\left(1 + \frac{a}{b}r^{1-\xi}\right). \tag{7}$$

To sum up, based on equalities (6) and (7), the ST is estimated as

$$\begin{aligned} T_{\max} &\leqslant t_{y=r} + \frac{1}{a(1-\xi)}\ln\left(1 + \frac{a}{b}r^{1-\xi}\right) \\ &= \frac{1}{a(\eta-1)}\ln\left(1 + \frac{a}{c}r^{1-\eta}\right) + \frac{1}{a(1-\xi)}\ln\left(1 + \frac{a}{b}r^{1-\xi}\right). \end{aligned}$$

Define

$$T(r) = \frac{1}{a(\eta-1)}\ln\left(1 + \frac{a}{c}r^{1-\eta}\right) + \frac{1}{a(1-\xi)}\ln\left(1 + \frac{a}{b}r^{1-\xi}\right), \quad r > 0.$$

Calculating its derivative, one gives that

$$\begin{aligned} \dot{T}(r) &= \frac{\frac{a}{c}(1-\eta)r^{-\eta}}{a(\eta-1)(1+\frac{a}{c}r^{1-\eta})} + \frac{\frac{a}{b}(1-\xi)r^{-\xi}}{a(1-\xi)(1+\frac{a}{b}r^{1-\xi})} \\ &= -\frac{r^{-\eta}}{c + ar^{1-\eta}} + \frac{r^{-\xi}}{b + ar^{1-\xi}}. \end{aligned}$$

Let $\dot{T}(r) = 0$, one has $(c + ar^{1-\eta})r^{-\xi} = (b + ar^{1-\xi})r^{-\eta}$. Solving this equation gives $r = (b/c)^{1/(\eta-\xi)}$, which is a unique stationary point in $(0, +\infty)$.

Furthermore, we obtain

$$\begin{aligned} \ddot{T}(r) &= -\frac{-\eta r^{-\eta-1}(c + ar^{1-\eta}) - r^{-\eta}a(1-\eta)r^{-\eta}}{(c + ar^{1-\eta})^2} \\ &\quad + \frac{-\xi r^{-\xi-1}(b + ar^{1-\xi}) - r^{-\xi}a(1-\xi)r^{-\xi}}{(b + ar^{1-\xi})^2} \\ &= \frac{\eta c r^{-\eta-1} + ar^{-2\eta}}{(c + ar^{1-\eta})^2} + \frac{-b\xi r^{-\xi-1} - ar^{-2\xi}}{(b + ar^{1-\xi})^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\ddot{T}\left(\left(\frac{b}{c}\right)^{1/(\eta-\xi)}\right) \\ &\quad = \frac{\eta c(\frac{b}{c})^{(-\eta-1)/(\eta-\xi)} + a(\frac{b}{c})^{-2\eta/(\eta-\xi)}}{(c + a(\frac{b}{c})^{(1-\eta)/(\eta-\xi)})^2} - \frac{b\xi(\frac{b}{c})^{(-\xi-1)/(\eta-\xi)} + a(\frac{b}{c})^{-2\xi/(\eta-\xi)}}{(b + a(\frac{b}{c})^{(1-\xi)/(\eta-\xi)})^2} \end{aligned}$$

$$= \big((\eta-\xi)b^{(\eta-2\xi-1)/(\eta-\xi)}c^{(2\eta-\xi+1)/(\eta-\xi)} + 2(\eta-\xi)ab^{-2\xi/(\eta-\xi)}c^{2\eta/(\eta-\xi)} + (\eta-\xi)a^2b^{(-2\xi-\eta+1)/(\eta-\xi)}c^{(2\eta+\xi-1)/(\eta-\xi)}\big) \times \frac{1}{(c+ar^{1-\eta})^2(b+a(\frac{b}{c})^{(1-\xi)/(\eta-\xi)})^2}.$$

Since $\eta > 1$, $0 < \xi < 1$, and $a, b, c > 0$, thus $\ddot{T}((b/c)^{1/(\eta-\xi)}) > 0$. Hence, $T(r)$ has its minimum value at $r = (b/c)^{1/(\eta-\xi)}$. Therefore, the upper bound of the ST can be estimated as

$$T^3_{\max} = \min_{r>0} T(r) = T\left(\left(\frac{b}{c}\right)^{1/(\eta-\xi)}\right) = \left[\frac{1}{a(\eta-1)} + \frac{1}{a(1-\xi)}\right] \ln\left[1 + \frac{a}{b}\left(\frac{b}{c}\right)^{(1-\xi)/(\eta-\xi)}\right].$$

The proof is completed. □

Remark 3. The delay term is included in inequality (5), which can better account for the effect of the systems' delay. Furthermore, it is very easy to conclude that the upper bound of ST is connected with the coefficient of the delay term. It indicates that the delay term of system (1) does affect the process of the fixed-time stability. Therefore, it must be pointed out that the result in this lemma is perfectly new. It is absolutely different from the previous ones.

Remark 4. When $a = 0$, inequality (5) will reduce to inequality (2). Besides, if $\tau = 0$ in inequality (5), then it will become inequality (3). Thus, the result in Lemma 8 could extend the relative content of the fixed-time stability lemmas in [3, 22].

Remark 5. Many researchers have frequently used a fixed positive constant 1 as the cutoff value of $V(x(t))$ to prove the fixed-time stability lemmas. However, in the proof of Lemma 8, an arbitrary positive constant r is selected as the dividing line, which is more flexible. In order to estimate the minimum upper bound of ST, firstly, a function $T(r)$, $r > 0$, is defined. Then its unique stationary point is obtained at $(b/c)^{1/(\eta-\xi)}$. Precisely because of the function $T(r)$ has a unique stationary point in $(0, +\infty)$ and $\ddot{T}((b/c)^{1/(\eta-\xi)}) > 0$, it achieves its minimum at this point, which corresponds to the minimum upper bound of the ST. So the optimal dividing point of analyzing function $V(x(t))$ is reached scientifically and effectively. In summary, Lemma 8 employs this traditional optimization method to derive the least upper bound of ST.

Remark 6. Compared to the reference [10], in Lemma 8, although the time-delayed Lyapunov–Krasovskii functional inequality condition is taken into consideration, only its first-power term is taken into account, which might limit its practical application to a certain extent. In addition, the time-delay parameter's impact on the estimation of the ST has been rigorously analyzed, while the intrinsic consequences of time delay remain outside the scope of this study. As far as the authors know, it is still an open and unanswered question.

Our analysis is confined to the role of the time-delay parameter in the system, leaving the general effects of time delay unexamined.

We have solely addressed the influence of the time-delay parameter on the system, but have not explored the broader implications of time delay itself.

Lemma 9. *If $b \neq c$, then $T^3_{\max} < T^2_{\max}$ holds.*

Proof. Let $t = (\eta - 1)/(1 - \xi)$, then

$$\begin{aligned} &T^2_{\max} - T^3_{\max} \\ &\quad = \frac{1}{a(\eta-1)}\left[t\ln\left(1+\frac{a}{b}\right) + \ln\left(1+\frac{a}{c}\right) - (1+t)\ln\left(1+\frac{a}{b}\left(\frac{b}{c}\right)^{1/(t+1)}\right)\right]. \end{aligned}$$

Define

$$\mathcal{T}(t) = \frac{1}{a(\eta-1)}\left[t\ln\left(1+\frac{a}{b}\right) + \ln\left(1+\frac{a}{c}\right) - (1+t)\ln\left(1+\frac{a}{b}\left(\frac{b}{c}\right)^{1/(t+1)}\right)\right].$$

Calculating its derivative, we obtain that

$$\dot{\mathcal{T}}(t) = \frac{1}{a(\eta-1)}\left[\ln\left(1+\frac{a}{b}\right) - \ln\left(1+\frac{a}{b}\left(\frac{b}{c}\right)^{1/(t+1)}\right) + \frac{1}{1+t}\frac{\frac{a}{b}(\frac{b}{c})^{1/(t+1)}\ln\frac{b}{c}}{1+\frac{a}{b}(\frac{b}{c})^{1/(t+1)}}\right].$$

Furthermore, one has

$$\begin{aligned} \ddot{\mathcal{T}}(t) = {} & \frac{1}{a(\eta-1)}\left[-\frac{\frac{a}{b}(\frac{b}{c})^{1/(t+1)}\ln\frac{b}{c}}{1+\frac{a}{b}(\frac{b}{c})^{1/(t+1)}}\cdot\frac{-1}{(t+1)^2} + \frac{-1}{(t+1)^2}\cdot\frac{\frac{a}{b}(\frac{b}{c})^{1/(t+1)}\ln\frac{b}{c}}{1+\frac{a}{b}(\frac{b}{c})^{1/(t+1)}} + \frac{1}{t+1}\right. \\ & \left.\times\frac{\frac{a}{b}(\frac{b}{c})^{1/(t+1)}\ln^2\frac{b}{c}\cdot\frac{-1}{(t+1)^2}\cdot[1+\frac{a}{b}(\frac{b}{c})^{1/(t+1)}] - (\frac{a}{b})^2(\frac{b}{c})^{2/(t+1)}\ln^2\frac{b}{c}\cdot\frac{-1}{(t+1)^2}}{[1+\frac{a}{b}(\frac{b}{c})^{1/(t+1)}]^2}\right] \\ = {} & \frac{-\frac{a}{b}(\frac{b}{c})^{1/(t+1)}\ln^2\frac{b}{c}}{a(\eta-1)(t+1)^3[1+\frac{a}{b}(\frac{b}{c})^{1/(t+1)}]^2}. \end{aligned}$$

Thus, if $b \neq c$, then $\ddot{\mathcal{T}}(t) < 0$. Hence, $\dot{\mathcal{T}}(t)$ is monotonically decreasing.

Moreover, the mathematical expression of $\dot{\mathcal{T}}(t)$ implies that $\lim_{t\to+\infty}\dot{\mathcal{T}}(t) = 0$, hence $\dot{\mathcal{T}}(t) > 0$ for any $t > 0$. Therefore, $\mathcal{T}(t)$ is monotonically increasing. Thus, one can obtain that $\mathcal{T}(t) > \mathcal{T}(0) = 0$, which indicates that $T^3_{\max} < T^2_{\max}$ holds if $b \neq c$.

The proof of Lemma 9 is completed. □

Remark 7. From Lemma 9 it can be observed that $T^3_{\max} < T^2_{\max}$ if $b \neq c$. Consequently, the result on ST established in Lemma 8 is superior than that stated in Lemma 5. On the other hand, in the reference [3], the authors have proved that $T^2_{\max} < T^1_{\max}$. Thus, $T^3_{\max} < T^1_{\max}$. Therefore, compared with Lemmas 4 and 5, Lemma 8 has more advantages because of its more accurate estimation of the ST.

3.2 Fixed-time leader–follower consensus analysis

In this subsection, the delayed MASs (4) will be proven to reach leader–follower consensus tracking within a fixed time. The following control protocol for ith follower is designed as

$$\begin{aligned} u_i(t) = &\sum_{j\in\mathcal{N}_i} a_{ij}\,\mathrm{sig}^{\xi}\big(e_j(t)-e_i(t)\big) - b_i\,\mathrm{sig}^{\xi}\big(e_i(t)\big) \\ &+ \sum_{j\in\mathcal{N}_i} a_{ij}\,\mathrm{sig}^{\eta}\big(e_j(t)-e_i(t)\big) - b_i\,\mathrm{sig}^{\eta}\big(e_i(t)\big) \\ &- k_1 L_{Bi} e^{\tau}(t) - k_2 L_i e(t) - k_3 L_{Bi} e(t)\bigg(\frac{\|L_B e^{\tau}(t)\|}{\|L_B e(t)\|}\bigg)^2, \end{aligned} \tag{8}$$

where a_{ij} represents the elements of the adjacency matrix A, $e(t) = (e_1(t), e_2(t), \dots, e_N(t))^{\mathrm{T}}$, $e_i(t) = x_i(t) - x_0(t)$ represents the state tracking error, $e^{\tau}(t) = (e_1(t-\tau), e_2(t-\tau), \dots, e_N(t-\tau))^{\mathrm{T}}$, and $e_i(t-\tau) = x_i(t-\tau) - x_0(t-\tau)$ represents the delayed state tracking error, $i = 1, 2, \dots, N$; L_{Bi} denotes the ith row of L_B; $k_1, k_2, k_3 > 0$ denote control gains. The last term in $u_i(t)$ is used to compensate the time-delayed term.

Theorem 1. *Consider the MASs* (4) *with the control protocol* (8). *Let Assumptions* 1 *and* 2 *hold, and let positive constants* k_1, k_2, k_3, *and* l *satisfy the following conditions*:

$$2k_3\frac{\lambda_{\min}^3(L_B)}{\lambda_{\max}^2(L_B)} - k_1 > 0, \tag{9}$$

$$2k_2\lambda_{\min}(L_B) - 2l - k_1\lambda_{\max}^2(L_B) > 0. \tag{10}$$

Then the MASs can realize fixed-time leader–follower consensus, the ST can be obtained, and its upper bound is evaluated as

$$T_{\max}^4 = \bigg[\frac{2}{\mathcal{A}(\eta-1)} + \frac{2}{\mathcal{A}(1-\xi)}\bigg]\ln\bigg[1 + \frac{\mathcal{A}}{\mathcal{B}}\bigg(\frac{\mathcal{B}}{\mathcal{C}}\bigg)^{(1-\xi)/(\eta-\xi)}\bigg],$$

where

$$\mathcal{A} = 2k_3\frac{\lambda_{\min}^3(L_B)}{\lambda_{\max}^2(L_B)} - k_1, \qquad \mathcal{B} = 2^{(\xi-1)/2}\lambda_{\min}^{(\xi+1)/2}(2L_{\hat{A}} + D_{\hat{b}}),$$
$$\mathcal{C} = 2^{(\eta-1)/2}\big[N(N+1)\big]^{(1-\eta)/2}\lambda_{\min}^{(\eta+1)/2}(2L_{\tilde{A}} + D_{\tilde{b}}).$$

Proof. Choose the Lyapunov function $V(t)$ as

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} e_i^2(t).$$

Differentiating $V(t)$ along (4) yields

$$\begin{aligned}
\dot{V}(t) &= \sum_{i=1}^{N} e_i(t)\dot{e}_i(t) = \sum_{i=1}^{N} e_i(t)\big[u_i(t) + f\big(x_i(t), t\big) - f\big(x_0(t), t\big)\big] \\
&= \sum_{i=1}^{N} e_i(t) \sum_{j \in \mathcal{N}_i} a_{ij} \operatorname{sig}^{\xi}\big(e_j(t) - e_i(t)\big) - \sum_{i=1}^{N} e_i(t) b_i \operatorname{sig}^{\xi}\big(e_i(t)\big) \\
&\quad + \sum_{i=1}^{N} e_i(t) \sum_{j \in \mathcal{N}_i} a_{ij} \operatorname{sig}^{\eta}\big(e_j(t) - e_i(t)\big) - \sum_{i=1}^{N} e_i(t) b_i \operatorname{sig}^{\eta}\big(e_i(t)\big) \\
&\quad - k_1 \sum_{i=1}^{N} e_i(t) L_{Bi} e^{\tau}(t) - k_2 \sum_{i=1}^{N} e_i(t) L_{Bi} e(t) \\
&\quad - k_3 \sum_{i=1}^{N} e_i(t) L_{Bi} e(t) \left(\frac{\|L_B e^{\tau}(t)\|}{\|L_B e(t)\|} \right)^2 \\
&\quad + \sum_{i=1}^{N} e_i(t)\big[f\big(x_i(t), t\big) - f\big(x_0(t), t\big)\big].
\end{aligned}$$

According to Lemma 7, we obtain

$$\begin{aligned}
&\sum_{i=1}^{N} e_i(t) \sum_{j \in \mathcal{N}_i} a_{ij} \operatorname{sig}^{\xi}\big(e_j(t) - e_i(t)\big) \\
&\quad = -\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \big(e_j(t) - e_i(t)\big) \operatorname{sig}^{\xi}\big(e_j(t) - e_i(t)\big) \\
&\quad = -\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \big|e_j(t) - e_i(t)\big|^{\xi+1}.
\end{aligned} \tag{11}$$

In addition,

$$-\sum_{i=1}^{N} e_i(t) b_i \operatorname{sig}^{\xi}\big(e_i(t)\big) = -\sum_{i=1}^{N} b_i \big|e_i(t)\big|^{\xi+1}. \tag{12}$$

Combining (11) and (12) with Lemma 1, one obtains

$$\begin{aligned}
&\sum_{i=1}^{N} e_i(t) \sum_{j \in \mathcal{N}_i} a_{ij} \operatorname{sig}^{\xi}\big(e_j(t) - e_i(t)\big) - \sum_{i=1}^{N} e_i(t) b_i \operatorname{sig}^{\xi}\big(e_i(t)\big) \\
&\quad = -\frac{1}{2} \sum_{i=1}^{N} \left[\sum_{j=1}^{N} a_{ij} \big|e_j(t) - e_i(t)\big|^{\xi+1} + 2 b_i \big|e_i(t)\big|^{\xi+1} \right]
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\sum_{i=1}^{N}\left[\sum_{j=1}^{N}\left(a_{ij}^{1/(\xi+1)}\left|e_j(t)-e_i(t)\right|\right)^{\xi+1} + \left((2b_i)^{1/(\xi+1)}\left|e_i(t)\right|\right)^{\xi+1}\right]\\
&\leqslant -\frac{1}{2}\left[\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}^{2/(\xi+1)}\left|e_j(t)-e_i(t)\right|^2 + \sum_{i=1}^{N}(2b_i)^{2/(\xi+1)}\left|e_i(t)\right|^2\right]^{(\xi+1)/2}. \qquad (13)
\end{aligned}
$$

Define a new matrix $\hat{A} = (\hat{a}_{ij})_{N\times N}$ with $\hat{a}_{ij} = a_{ij}^{2/(\xi+1)}$, where a_{ij} represents the elements of the adjacency matrix A, and $2/(\xi+1)$ represents its power. Then $\hat{A}$ can be regarded as a nonnegative adjacency matrix of the undigraph $\mathcal{G}(\hat{A})$. According to Assumption 1, $\mathcal{G}(\hat{A})$ is also connected. Let $L_{\hat{A}}$ be the Laplacian matrix of $\mathcal{G}(\hat{A})$. We can obtain from Lemma 6 that

$$
\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}^{2/(\xi+1)}\left|e_j(t)-e_i(t)\right|^2 = 2e^{\mathrm{T}}(t)L_{\hat{A}}e(t). \qquad (14)
$$

Let

$$
D_{\hat{b}} = \operatorname{diag}\left((2b_1)^{2/(\xi+1)}, (2b_2)^{2/(\xi+1)}, \dots, (2b_N)^{2/(\xi+1)}\right),
$$

where b_i, $i = 1, 2, \dots, N$, denotes the communication interaction between ith follower and the leader. Hence, the diagonal matrix $D_{\hat{b}}$ is the matrix transformation of the diagonal matrix B. Therefore,

$$
\sum_{i=1}^{N}(2b_i)^{2/(\xi+1)}\left|e_i(t)\right|^2 = e^{\mathrm{T}}(t)D_{\hat{b}}e(t). \qquad (15)
$$

Therefore, substituting (14) and (15) into inequality (13) yields that

$$
\begin{aligned}
&\sum_{i=1}^{N}e_i(t)\sum_{j\in\mathcal{N}_i}a_{ij}\operatorname{sig}^{\xi}\left(e_j(t)-e_i(t)\right) - \sum_{i=1}^{N}e_i(t)b_i\operatorname{sig}^{\xi}\left(e_i(t)\right)\\
&\quad\leqslant -\frac{1}{2}\left[2e^{\mathrm{T}}(t)L_{\hat{A}}e(t) + e^{\mathrm{T}}(t)D_{\hat{b}}e(t)\right]^{(\xi+1)/2}. \qquad (16)
\end{aligned}
$$

Since $2L_{\hat{A}} + D_{\hat{b}}$ is positive definite, from the above inequality (16) it follows that

$$
\begin{aligned}
&\sum_{i=1}^{N}e_i(t)\sum_{j\in\mathcal{N}_i}a_{ij}\operatorname{sig}^{\xi}\left(e_j(t)-e_i(t)\right) - \sum_{i=1}^{N}e_i(t)b_i\operatorname{sig}^{\xi}\left(e_i(t)\right)\\
&\quad\leqslant -\frac{1}{2}\left[\lambda_{\min}(2L_{\hat{A}} + D_{\hat{b}})e^{\mathrm{T}}(t)e(t)\right]^{(\xi+1)/2}\\
&\quad= -2^{(\xi-1)/2}\lambda_{\min}^{(\xi+1)/2}\left(2L_{\hat{A}} + D_{\hat{b}}\right)V^{(\xi+1)/2}(t). \qquad (17)
\end{aligned}
$$

In a similar way, one has

$$
\sum_{i=1}^{N}e_i(t)\sum_{j\in\mathcal{N}_i}a_{ij}\operatorname{sig}^{\eta}\left(e_j(t)-e_i(t)\right) - \sum_{i=1}^{N}e_i(t)b_i\operatorname{sig}^{\eta}\left(e_i(t)\right)
$$

$$\leqslant -\frac{1}{2}\big[N(N+1)\big]^{(1-\eta)/2}\big[\lambda_{\min}(2L_{\tilde{A}}+D_{\tilde{b}})e^{\mathrm{T}}(t)e(t)\big]^{(\xi+1)/2}$$
$$= -2^{(\eta-1)/2}\big[N(N+1)\big]^{(1-\eta)/2}\lambda_{\min}^{(\eta+1)/2}(2L_{\tilde{A}}+D_{\tilde{b}})V^{(\eta+1)/2}(t), \quad (18)$$

where $\tilde{A} = (a_{ij}^{2/(\eta+1)})_{N\times N}$ can be regarded as a nonnegative adjacency matrix of the undigraph $\mathcal{G}(\tilde{A})$, $L_{\tilde{A}}$ represents the Laplacian matrix of $\mathcal{G}(\tilde{A})$, and

$$D_{\tilde{b}} = \operatorname{diag}\big((2b_1)^{2/(\eta+1)}, (2b_2)^{2/(\eta+1)}, \dots, (2b_N)^{2/(\eta+1)}\big)$$

is the matrix transformation of the diagonal matrix B.

According to the positivity of matrix L_B and Lemmas 2 and 3, one obtains that

$$-k_1\sum_{i=1}^{N} e_i(t)L_{Bi}e^{\tau}(t) = -k_1 e^{\mathrm{T}}(t)L_B e^{\tau}(t) = -k_1\big(e(t)L_B\big)^{\mathrm{T}}e^{\tau}(t)$$
$$\leqslant \frac{1}{2}k_1 e^{\mathrm{T}}(t)L_B^2 e(t) + \frac{1}{2}k_1\big(e^{\tau}(t)\big)^{\mathrm{T}}e^{\tau}(t)$$
$$\leqslant k_1\lambda_{\max}^2(L_B)V(t) + k_1 V(t-\tau), \quad (19)$$

$$-k_2\sum_{i=1}^{N} e_i(t)L_{Bi}e(t) = -k_2 e^{\mathrm{T}}(t)L_B e(t) \leqslant -k_2\lambda_{\min}(L_B)e^{\mathrm{T}}(t)e(t)$$
$$= -2k_2\lambda_{\min}(L_B)V(t). \quad (20)$$

From Lemma 3 the following holds:

$$-k_3\sum_{i=1}^{N} e_i(t)L_{Bi}e(t)\left(\frac{\|L_B e^{\tau}(t)\|}{\|L_B e(t)\|}\right)^2$$
$$= -k_3 e^{\mathrm{T}}(t)L_B e(t)\frac{(e^{\tau}(t))^{\mathrm{T}}L_B^{\mathrm{T}}L_B e^{\tau}(t)}{e^{\mathrm{T}}(t)L_B^{\mathrm{T}}L_B e(t)}$$
$$\leqslant -k_3\lambda_{\min}(L_B)e^{\mathrm{T}}(t)e(t)\frac{\lambda_{\min}(L_B^{\mathrm{T}}L_B)(e^{\tau}(t))^{\mathrm{T}}e^{\tau}(t)}{\lambda_{\max}(L_B^{\mathrm{T}}L_B)e^{\mathrm{T}}(t)e(t)}$$
$$= -2k_3\frac{\lambda_{\min}^3(L_B)}{\lambda_{\max}^2(L_B)}V(t-\tau). \quad (21)$$

Under Assumption 2, one has

$$\sum_{i=1}^{N} e_i(t)\big[f\big(x_i(t),t\big) - f\big(x_0(t),t\big)\big]$$
$$\leqslant \sum_{i=1}^{N} e_i(t)l\big|x_i(t)-x_0(t)\big| = l\sum_{i=1}^{N} e_i(t)\big|e_i(t)\big| \leqslant 2lV(t). \quad (22)$$

Conditions (9) and (10), together with (17)–(22), lead to

$$\begin{aligned}\dot{V}(t) \leqslant& -\biggl(2k_3\frac{\lambda_{\min}^3(L_B)}{\lambda_{\max}^2(L_B)} - k_1\biggr)V\bigl(x(t-\tau)\bigr)\\&- 2^{(\xi-1)/2}\lambda_{\min}^{(\xi+1)/2}(2L_{\hat{A}} + D_{\hat{b}})V^{(\xi+1)/2}(t)\\&- 2^{(\eta-1)/2}\bigl[N(N+1)\bigr]^{(1-\eta)/2}\lambda_{\min}^{(\eta+1)/2}(2L_{\tilde{A}} + D_{\tilde{b}})V^{(\eta+1)/2}(t)\\&- \bigl(2k_2\lambda_{\min}(L_B) - 2l - k_1\lambda_{\max}^2(L_B)\bigr)V(t)\\\leqslant& -\mathcal{A}V(t-\tau) - \mathcal{B}V^{(\xi+1)/2}(t) - \mathcal{C}V^{(\eta+1)/2}(t).\end{aligned}$$

Therefore, by means of Lemma 8, MASs (4) can achieve fixed-time leader–follower consensus, and ST is evaluated as

$$T_{\max}^4 = \biggl[\frac{2}{\mathcal{A}(\eta-1)} + \frac{2}{\mathcal{A}(1-\xi)}\biggr]\ln\biggl[1 + \frac{\mathcal{A}}{\mathcal{B}}\biggl(\frac{\mathcal{B}}{\mathcal{C}}\biggr)^{(1-\xi)/(\eta-\xi)}\biggr].$$

The proof is completed. □

Remark 8. By using the new provided inequality condition (5) in Lemma 8, it can be concluded that the fixed-time consensus of delayed leader–follower MASs has been obtained in Theorem 1. From the estimation of ST $T_{\max}^4$ it further explains that the time delay assuredly impacts the process of fixed-time leader–follower consensus. For the fixed-time consensus of MASs in [13, 23], those existing research achievements by using the former fixed-time stability lemmas can be improved and extended.

Remark 9. Fixed-time leader–follower consensus of delayed MASs has many practical potential applications in cooperative control such as UAVs formation flight control, unmanned vessel formation control, and multiple robots cooperative control. Due to the limited communication capacity and the communication bandwidth, there is a communication delay in information transmission at some agents. The presence of communication delays may degrade consensus performance and even compromise consistency and stability of agents. Therefore, it is important and necessary to explore the problem of time delays in the fixed-time consensus of MASs to improve consensus performance and maintain the stability of the formation.

Remark 10. In Theorem 1, fixed-time consensus of system (4) has been solely addressed, while omitting the stability analysis of its solutions. The presence of time delay complicates the dynamical properties of this system, rendering it analytically intractable. Certain algebraic methods can be employed to analyze its dynamical properties [17, 26], and finite-element method [27] can be used to analyze its numerical solutions.

Corollary 1. *Consider the MASs* (4) *with the control protocol* (8). *Let Assumptions* 1 *and* 2 *hold, and let positive constants* k_1, k_2, k_3, *and* l *satisfy the following conditions*:

$$\begin{gathered}2k_3\frac{\lambda_{\min}^3(L_B)}{\lambda_{\max}^2(L_B)} - k_1 > 0,\\2k_2\lambda_{\min}(L_B) - 2l - k_1\lambda_{\max}^2(L_B) > 0.\end{gathered}$$

Then the MASs can realize fixed-time leader–follower consensus, the ST can be obtained, and its upper bound is evaluated as

$$T_{\max}^{5} = \frac{2}{\mathcal{B}(1-\xi)} + \frac{2}{\mathcal{C}(\eta-1)},$$

where

$$\mathcal{B} = 2^{(\xi-1)/2}\lambda_{\min}^{(\xi+1)/2}(2L_{\hat{A}} + D_{\hat{b}}),$$
$$\mathcal{C} = 2^{(\eta-1)/2}[N(N+1)]^{(1-\eta)/2}\lambda_{\min}^{(\eta+1)/2}(2L_{\tilde{A}} + D_{\tilde{b}}).$$

Proof. In a similar approach, one can prove that

$$\begin{aligned}\dot{V}(t) &\leqslant -\mathcal{A}V(t-\tau) - \mathcal{B}V^{(\xi+1)/2}(t) - \mathcal{C}V^{(\eta+1)/2}(t)\\ &\leqslant -\mathcal{B}V^{(\xi+1)/2}(t) - \mathcal{C}V^{(\eta+1)/2}(t),\end{aligned}$$

where $\mathcal{A} = 2k_3\lambda_{\min}^3(L_B)/\lambda_{\max}^2(L_B) - k_1$.

By using Lemma 4, MASs (4) can achieve fixed-time leader–follower consensus. In addition, it is very easy to compute the upper bound of ST $T_{\max}^5$. □

In the control protocol (8), if $\tau = 0$, then the following control protocol, a special case of (8), can be designed as

$$\begin{aligned}u_i(t) = {}&\sum_{j\in\mathcal{N}_i} a_{ij}\,\mathrm{sig}^{\xi}\big(x_j(t) - x_i(t)\big) - b_i\,\mathrm{sig}^{\xi}\,\big(x_i(t) - x_0(t)\big)\\ &+ \sum_{j\in\mathcal{N}_i} a_{ij}\,\mathrm{sig}^{\eta}\big(x_j(t) - x_i(t)\big) - b_i\,\mathrm{sig}^{\eta}\,\big(x_i(t) - x_0(t)\big)\\ &+ kL_{Bi}e(t), \quad i = 1,2,\dots,N,\end{aligned} \tag{23}$$

where $k = k_1 + k_2 + k_3$, and they have the same meanings with those in (8).

Therefore, according to Lemma 5, the following corollary can be derived.

Corollary 2. *Consider the MASs* (4) *with the control protocol* (23)*. Let Assumptions* 1 *and* 2 *hold, and let positive constants k and l satisfy the following condition*:

$$2k\lambda_{\min}(L_B) - 2l > 0.$$

Then the MASs can realize fixed-time leader–follower consensus, the ST can be obtained and its upper bound is evaluated as

$$T_{\max}^{6} = \frac{2}{\mathcal{A}(1-\xi)}\ln\biggl(1 + \frac{\mathcal{A}}{\mathcal{B}}\biggr) + \frac{2}{\mathcal{A}(\eta-1)}\ln\biggl(1 + \frac{\mathcal{A}}{\mathcal{C}}\biggr),$$

where

$$\mathcal{A} = 2k\lambda_{\min}(L_B) - 2l, \qquad \mathcal{B} = 2^{(\xi-1)/2}\lambda_{\min}^{(\xi+1)/2}(2L_{\hat{A}} + D_{\hat{b}}),$$
$$\mathcal{C} = 2^{(\eta-1)/2}\big[N(N+1)\big]^{(1-\eta)/2}\lambda_{\min}^{(\eta+1)/2}(2L_{\tilde{A}} + D_{\tilde{b}}).$$

Proof. In a similar approach, we construct the Lyapunov function $V(t)$ as

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} e_i^2(t).$$

Differentiating $V(t)$ along (4) yields

$$\begin{aligned}
\dot{V}(t) &= \sum_{i=1}^{N} e_i(t)\dot{e}_i(t) = \sum_{i=1}^{N} e_i(t)\big[u_i(t) + f\big(x_i(t), t\big) - f\big(x_0(t), t\big)\big] \\
&= \sum_{i=1}^{N} e_i(t) \sum_{j\in\mathcal{N}_i} a_{ij} \operatorname{sig}^{\xi}\big(e_j(t) - e_i(t)\big) - \sum_{i=1}^{N} e_i(t) b_i \operatorname{sig}^{\xi}\big(e_i(t)\big) \\
&\quad + \sum_{i=1}^{N} e_i(t) \sum_{j\in\mathcal{N}_i} a_{ij} \operatorname{sig}^{\eta}\big(e_j(t) - e_i(t)\big) - \sum_{i=1}^{N} e_i(t) b_i \operatorname{sig}^{\eta}\big(e_i(t)\big) \\
&\quad + k\sum_{i=1}^{N} e_i(t) L_{Bi} e(t) + \sum_{i=1}^{N} e_i(t)\big[f\big(x_i(t), t\big) - f\big(x_0(t), t\big)\big].
\end{aligned}$$

From (17), (18), (20), and (22) one can prove that

$$\begin{aligned}
\dot{V}(t) &\leqslant -\big(2k\lambda_{\min}(L_B) - 2l\big)V(t) \\
&\quad - 2^{(\xi-1)/2}\lambda_{\min}^{(\xi+1)/2}(2L_{\hat{A}} + D_{\hat{b}})V^{(\xi+1)/2}(t) \\
&\quad - 2^{(\eta-1)/2}\big[N(N+1)\big]^{(1-\eta)/2}\lambda_{\min}^{(\eta+1)/2}(2L_{\tilde{A}} + D_{\tilde{b}})V^{(\eta+1)/2}(t) \\
&\leqslant -\mathcal{A}V(t) - \mathcal{B}V^{(\xi+1)/2}(t) - \mathcal{C}V^{(\eta+1)/2}(t).
\end{aligned}$$

By means of Lemma 5, MASs (4) can reach fixed-time leader–follower consensus. Moreover, it is very easy to calculate the upper bound of ST $T_{\max}^6$. □

4 Numerical simulation

In order to confirm the effectiveness and superiority of the corresponding results obtained above, one numerical simulation example is provided in this section.

Example. Considering a class of first-order nonlinear isomorphic MAS with one leader, index 0, and 4 followers labeled as $1, 2, 3, 4$. Figure 1 exhibits the interaction topology. It is a connected undigraph satisfying Assumption 1. It is evident that the adjacency matrix A, diagonal matrix B, and matrix L_B are respectively denoted as

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad L_B = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & 0 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & -1 & -1 & 3 \end{pmatrix}.$$

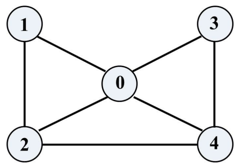


Figure 1. The interaction topology of MAS.

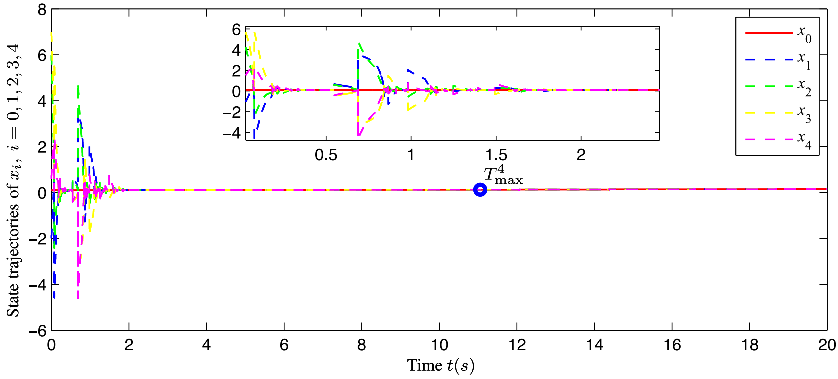
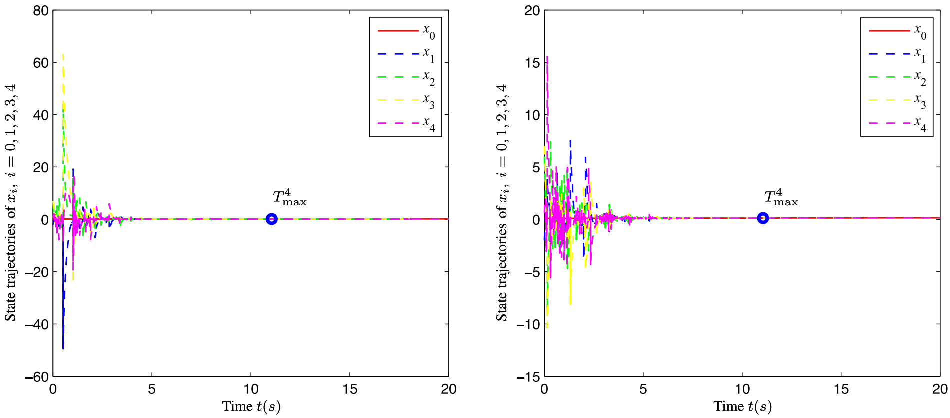


Figure 2. Fixed-time leader–follower consensus with $\tau = 0.1$ s under controller (8).

The nonlinear dynamics of the leader and followers is expressed as (4), where the control protocol $u_i(t)$ for ith follower is given by (8). The inherent dynamics are set as $f(x_i(t), t) = 0.02 \sin(x_i(t))$, $i = 0, 1, 2, 3, 4$. It can be easily verified that $f(x_i(t), t)$ satisfies Assumption 2, where the constant $l = 0.02$. The leader's and followers' initial conditions are randomly selected as $x_0(0) = 0.1$, $x_1(0) = -2$, $x_2(0) = 6$, $x_3(0) = 7$, $x_4(0) = 0.6$, respectively. For satisfying sufficient conditions (9) and (10), the maximum and minimum eigenvalues of L_B are calculated, then $k_1 = 0.1$, $k_2 = 1.08$, and $k_3 = 1.05$ can be selected to guarantee fixed-time leader–follower consensus. Furthermore, for calculating the upper bound of ST, $\xi = 0.11$ and $\eta = 1.12$ are chosen. The relevant parameters are chosen through multiple tests conducted within a specified range. Until now, by direct computation, all conditions in Theorem 1 are satisfied. Consequently, the fixed-time leader–follower consensus is reached. Ulteriorly, from Theorem 1 it can be calculated that the upper bound of ST is estimated as $T^4_{\max} = 11.0548$ s. In the simulations, let $\tau = 0.1$ s, 0.5 s, and 1 s, respectively. The profiles of all agents are exhibited in Figs. 2–3. It can be found that the 4 followers eventually not only achieve consensus tracking in a fixed time but also converge to the same value, which is the value of the leader. Those simulation results demonstrate that the fixed-time leader–follower consensus of this MAS is reached. Thereby, we verify the effectiveness of the established theoretical results.

Moreover, according to Corollaries 1 and 2, using the same parameters ξ, η, k_2, and k_3 as in the above example and setting $k_1 = 0.1$, the upper bounds of the ST are estimated as $T^5_{\max} = 11.0975$ s and $T^6_{\max} = 11.0722$ s, respectively, based on primitive computation.



(a) (b)

Figure 3. Fixed-time leader–follower consensus under controller (8): (a) $\tau = 0.5$ s; (b) $\tau = 1$ s.

Table 1. Comparisons for STs.

Ref.	$k_1 = 0.1$	$k_1 = 0.2$	$k_1 = 0.5$	$k_1 = 1.5$
[3]	11.0722	5.9943	5.9182	6.0613
[22]	11.0975	6.2086	6.2086	6.2086
Theorem 1	11.0548	4.0355	4.0161	4.0521

They are all larger than $T^4_{\max} = 11.0548$ s, which implies that the estimation of the upper bound of ST in our research is more precise in a sense than those in [3, 22]. This result is in accord with Remark 7. Furthermore, when the parameter k_1 takes different values, $\xi = 0.1$, $\eta = 2.9$, k_2, and k_3 satisfy the conditions of Theorem 1, the estimation values of ST are shown in Table 1. It is apparent that the improved lemma is more effective and superior than the existing lemmas in [3, 22]. Furthermore, it clearly shows that the ST is relevant to the coefficient of the time-delayed term. Consequently, the influences of the delay on the fixed-time consensus tracking process can be further examined.

Based on the above analysis, the new fixed-time stability lemma can not only be used to reach the fixed-time leader–follower consensus of MASs but also provide a smaller upper bound of ST than those existing lemmas. Hence, some former fixed-time stability lemmas, such as [2, 3, 22], are improved, and some consensus analysis of MASs by using those former lemmas, such as [19, 20, 28], can also be extended. It is believed that the new lemma can provide some guidance for the fixed-time control in the complex network systems.

Especially worth pointing out is that an interesting phenomenon is found in the simulations. When we take $\tau = 0.1$ s, 0.5 s, 1 s, with the same ξ, η, k_1, k_2, and k_3, the trajectories of the leader and followers reach consensus within $T^* = 2.12$ s, 7.33 s, and 9.41 s under the control protocol (8), respectively. Those practical STs in all cases are smaller than $T^4_{\max}$, and they are different, which indicates that time delay may affect the consensus of delayed MASs. This is a problem that will be a research direction in the future.

5 Conclusions

This article has studied the fixed-time consensus problem of delayed nonlinear first-order leader–follower MASs under undirected interaction topology. A novel fixed-time stability lemma is presented, where the delay term is considered. Then a new class of nonlinear and discontinuous control protocol is proposed. Based on it, the followers achieve the agreement with the leader in a fixed time. Simultaneously, the estimated upper bound of ST is directly calculated with the determined system parameters. One numerical simulation is presented, and its results indicate that the followers with time delay can keep up with the leader within a fixed time. In addition, when the delay parameter τ takes different values, the actual STs are much smaller than the theoretical upper bound. In the future, efforts will be made to implement the presented lemma into the time-delayed nonlinear MASs with general directed and switching topology, even second-order and higher-order nonlinear time-delayed MASs. Furthermore, the influence of time delay on systems will also be a future research direction.

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