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Exact solutions of stochastic generalized long–short wave resonance equations with cubic–quintic nonlinearity via the new sub-ODE method

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Abstract. We study a stochastic generalized long–short wave resonance system with cubic–quintic nonlinearities, perturbation terms, and multiplicative white noise in the Itô sense. Using a traveling-wave reduction combined with the newly proposed sub-ODE method, we construct explicit families of solitary-wave and elliptic-function solutions, including elevation, depression, and singular branches. The analysis reveals how stochastic effects shrink the existence domains of these solutions, while a Hamiltonian linearization provides Vakhitov–Kolokolov- and Grillakis–Shatah–Strauss-type stability criteria. The results enrich nonlinear wave theory and offer insights relevant to solitary-wave applications in fiber optics and related technologies.

Keywords: embedded solitons, long–short wave, new sub-ODE method, cubic–quintic nonlinearity, multiplicative white noise.

1 Introduction

Long–short wave (LS) resonance is one of the most interesting phenomena in nonlinear dynamics characterized by complicated interactions of three waves with second-order nonlinearity. This kind of resonance gives a new viewpoint to understand the coupling among different wave modes in systems. The seminal work by Benney laid the building blocks for formulating nonlinear partial differential equations that describe long- and short-wave dynamics with precision [5]. Building on this foundation, by deriving LS-type equations, authors like Djordjevic and Redekopp have advanced our understanding of this resonance [8]. These equations make clear the essential mechanisms underlying resonance interactions: how nonlinear interaction with long waves compensates for the dispersion inherent in short waves. This dynamic equilibrium is essential for the development of long waves, which in turn depend on the self-interactions of short waves, and indicates the critical importance of nonlinear processes in determining the behavior of wave propagation [14]. The presence of stochastic effects, especially multiplicative white noise, brings extra layers of complexity into the dynamics of solitons. Multiplicative white noise, defined as random fluctuations scaling with signal amplitude, acts directly on the intensity of solitons [2, 9, 10]. The perturbations found can cause the destabilization of solitons, alter their properties and velocity or trajectory parameters that are essential in maintaining signal integrity and minimizing error rates in optical communication systems [4, 15, 16]. Recent studies have demonstrated the presence and stability of solitons within cubic–quintic nonlinear systems. For instance, research has shown that the introduction of fractional derivatives can stabilize soliton textures across a wide range of parameters, confirming the effectiveness of variational methods combined with stability criteria in predicting soliton behavior [17]. Additionally, the analysis of modulation instability in cubic–quintic systems highlights the critical role of these nonlinear interactions in determining the stability of solitary waves, particularly in multi-dimensional contexts [6].

This work addresses these problems by deriving accurate solutions for stochastic generalized LS resonant equations with cubic–quintic strong nonlinearities having perturbation terms and driven by multiplicative white noise, as interpreted in the Itô sense. The present study adopts the new sub-ODE method, which offers a pioneering analytical framework to investigate the interactions between nonlinear wave resonance, soliton behavior, and stochastic influences. The results afford fresh perspectives on the theoretical comprehension of such systems, leading to important implications for fundamental research in nonlinear wave theory and stochastic dynamical systems.

Governing model. The stochastic generalized long–short wave resonant equations (SLSWR), which incorporate cubic–quintic nonlinearities, perturbation terms, and multiplicative white noise in the Itô sense, is presented for the first time as

$$\begin{aligned} &\mathrm{i}S_t + \alpha S_{xx} - \left(\gamma|S|^2 + \delta|S|^4\right)S + \sigma S\frac{\mathrm{d}\mathcal{W}(t)}{\mathrm{d}t} \\ &\quad = LS + \mathrm{i}\left[\lambda_1 S_x + \mu_1\left(|S|^2 S\right)_x + \theta_1\left(|S|^2\right)_x S + \nu_1|S|^2 S_x\right], \end{aligned} \tag{1}$$

$$L_t + \beta\left(|S|^2\right)_x = 0, \tag{2}$$

where $S(x,t)$ represents the complex-valued wave function, while $L(x,t)$ is the real-valued wave function corresponding to the amplitude of the long wave, where $\mathrm{i}=\sqrt{-1}$. The parameters λ_1, μ_1, θ_1, and ν_1 are constants representing the coefficients of nonlinear terms, while $\alpha,\gamma,\delta,\sigma,\beta\in\mathbb{R}$, where σ represents the coefficient of noise strength, and $\mathcal{W}(t)$ denotes the standard Wiener process. It is useful to provide a definition of $\mathcal{W}(t)$ [1, 3]. Wiener process (or Brownian motion) is a stochastic process $\{\mathcal{W}(t)\}_{t\geqslant 0}$ with the properties given below:

- $\mathcal{W}(0)=0$: the process initiates from the origin.
- Given $0\leqslant s<t$, the change $\mathcal{W}(t)-\mathcal{W}(s)$ follows a normal distribution with a mean of 0 and a variance of $t-s$, that is, $\mathcal{W}(t)-\mathcal{W}(s)\sim N(0,\, t-s)$.
- The process has independent increments: for any $0\leqslant t_1<t_2<\cdots<t_n$, the increments $\mathcal{W}(t_2)-\mathcal{W}(t_1)$, $\mathcal{W}(t_3)-\mathcal{W}(t_2)$, …, $\mathcal{W}(t_n)-\mathcal{W}(t_{n-1})$ are mutually independent.
- $\mathcal{W}(t)$ has continuous trajectories.

The multiplicative noise in Eq. (1) characterizes a process where the phase of the excitation is disturbed. If $\sigma=\lambda_1=\mu_1=\theta_1=\nu_1=0$, Eqs. (1) and (2) were investigated in [20] through the application of different methods. Also, Eqs. (1) and (2) have been studied in [19] when $\delta=\gamma=\lambda_1=\mu_1=\theta_1=\nu_1=0$.

The objective of this paper is to develop exact solutions for Eqs. (1) and (2) for SLSWR equations incorporating cubic–quintic nonlinearities and having perturbation terms affected by multiplicative white noise in the Itô sense. By employing a new sub-ODE method, we seek to enhance the understanding of these complex interactions and their implications for wave dynamics in various applications within photonics and beyond. The organization of this work is as follows: In Section 2, we present the mathematical modelling of Eqs. (1) and (2). In Section 3, the solution to Eqs. (1) and (2) using new sub-ODE method is provided. In Section 4, we completed this study with the conclusions.

2 Mathematical analysis

To obtain the exact solution of Eqs. (1) and (2), we apply the following transformation:

$$S(x,t)=\vartheta(\zeta)\mathrm{e}^{\mathrm{i}[-\kappa x+\chi t+\sigma\mathcal{W}(t)-\sigma^2 t]},\qquad L(x,t)=\varpi(\zeta), \tag{3}$$

where $\chi,\kappa\in\mathbb{R}$ represent the wave number and frequency of the soliton, respectively. The functions $\vartheta(\zeta)$ and $\varpi(\zeta)$ are real-valued and describe the shape of the soliton pulses. The variable ζ is defined as

$$\zeta=x-\varrho t, \tag{4}$$

where ϱ represents the soliton velocity. Inserting Eqs. (3)–(4) into (1) yields

$$\begin{aligned}&-\mathrm{i}\varrho\vartheta'-\big(\chi-\sigma^2\big)\vartheta+\alpha\big(\vartheta''-2\mathrm{i}\kappa\vartheta'-\kappa^2\vartheta\big)-\gamma\vartheta^3-\delta\vartheta^5-\vartheta\varpi\\&\quad=\mathrm{i}\big[\lambda_1(\vartheta'-\mathrm{i}\kappa\vartheta)+\mu_1\big(3\vartheta^2\vartheta'-\mathrm{i}\kappa\vartheta^3\big)+2\theta_1\vartheta^2\vartheta'+\nu_1\vartheta^2(\vartheta'-\mathrm{i}\kappa\vartheta)\big].\end{aligned} \tag{5}$$

Inserting Eqs. (3) and (4) into (2), we get $-\varrho\varpi' + 2\beta\vartheta\vartheta' = 0$. One time integrating this equation, concludes

$$\varpi(\zeta) = \frac{\beta}{\varrho}\vartheta^2. \tag{6}$$

From Eq. (5) the imaginary part is given by

$$-(\varrho + 2\alpha\kappa + \lambda_1)\vartheta' - (3\mu_1 + 2\theta_1 + \nu_1)\vartheta^2\vartheta' = 0. \tag{7}$$

On applying linearly independece on (7), we get

$$\varrho = -2\alpha\kappa - \lambda_1, \qquad 3\mu_1 + 2\theta_1 + \nu_1 = 0.$$

As derived from Eq. (5), the real part can be written as

$$\begin{aligned}\alpha\vartheta'' &- \bigl(\chi - \sigma^2 + \kappa\lambda_1 + \alpha\kappa^2\bigr)\vartheta - (\gamma + \kappa\mu_1 + \kappa\nu_1)\vartheta^3 \\ &- \delta\vartheta^5 - \vartheta\varpi = 0.\end{aligned} \tag{8}$$

Substituting Eq. (6) into (8), we have

$$\begin{aligned}\alpha\varrho\vartheta'' &- \varrho\bigl(\chi - \sigma^2 + \kappa\lambda_1 + \alpha\kappa^2\bigr)\vartheta - \bigl[(\mu_1 + \nu_1)\varrho\kappa + \varrho\gamma + \beta\bigr]\vartheta^3 \\ &- \varrho\delta\vartheta^5 = 0.\end{aligned} \tag{9}$$

Considering the homogeneous balance between ϑ'' and ϑ^5 in Eq. (9), we obtain $N = 1/2$. Now, we use the wave transformation $\vartheta(\zeta) = U^{1/2}(\zeta)$, where $U(\zeta)$ is a new function of ζ. Now Eq. (9) reduces to the following ODE:

$$\begin{aligned}2\alpha\varrho UU'' &- \alpha\varrho U'^2 - 4\varrho\bigl(\chi - \sigma^2 + \kappa\lambda_1 + \alpha\kappa^2\bigr)U^2 \\ &- 4\bigl[(\mu_1 + \nu_1)\varrho\kappa + \varrho\gamma + \beta\bigr]U^3 - 4\delta\varrho U^4 = 0.\end{aligned} \tag{10}$$

Lets us now solve Eq. (10) using the new sub-ODE method.

3 New sub-ODE method

In this section, we give the main steps of this method [16].

Step 1. Based on this method, we have the formal solution of Eq. (10) as follows:

$$U(\xi) = \mu F^m(\xi), \quad \mu > 0, \tag{11}$$

where m is a parameter, and $F(\xi)$ satisfies the nonlinear ODE

$$\begin{aligned}F'^2(\xi) = \mathrm{AF}^{2-2p}(\xi) + \mathrm{BF}^{2-p}(\xi) + \mathrm{CF}^2(\xi) \\ + \mathrm{DF}^{2+p}(\xi) + \mathrm{EF}^{2+2p}(\xi), \quad p > 0,\end{aligned} \tag{12}$$

where A, B, C, D, and E are constants.

Step 2. We determine m in (11) using the homogeneous balance principle.

Step 3. We substitute (11) along with (12) into Eq. (10), collect all the coefficients of $F^{mj}(\xi)[F'(\xi)]^s$, $j = 0, 1, 2, \dots$, $s = 0, 1$, and set them to zero. We get a set of algebraic equations, which can be solved to find A, B, C, D, E, and μ.

Step 4. We solve the system of algebraic equations obtained in Step 3, using the Maple, to find A, B, C, D, E, μ ,and v.

Step 5. It is well known that Eq. (12) has the following solutions.

Type 1. If $A = B = D = 0$, then one gets:

(i) When $C > 0$ and $E < 0$, we have the bright soliton solution

$$F(\xi) = \left[\varepsilon\sqrt{-\frac{C}{E}}\operatorname{sech}(p\sqrt{C}\xi)\right]^{1/p}, \quad \varepsilon = \pm 1.$$

(ii) When $C < 0$ and $E > 0$, we have the periodic wave solution

$$F(\xi) = \left[\varepsilon\sqrt{-\frac{C}{E}}\sec(p\sqrt{-C}\xi)\right]^{1/p}, \quad \varepsilon = \pm 1.$$

(iii) When $C = 0$ and $E > 0$, we have the rational solution

$$F(\xi) = \left[\frac{\varepsilon}{p\sqrt{E}\xi}\right]^{1/p}, \quad \varepsilon = \pm 1.$$

Type 2. If $B = D = 0$, $A = C^2/(4B)$, then one gets:

(i) When $C < 0$ and $E > 0$, we have the dark soliton solution

$$F(\xi) = \left[\varepsilon\sqrt{-\frac{C}{2E}}\tanh\left(p\sqrt{-\frac{C}{2}}\xi\right)\right]^{1/p}.$$

(ii) When $C > 0$ and $E > 0$, we have the periodic wave solution

$$F(\xi) = \left[\varepsilon\sqrt{\frac{C}{2E}}\tan\left(p\sqrt{\frac{C}{2}}\xi\right)\right]^{1/p}.$$

Type 3. If $B = D = 0$, then one gets:

(i) When $C(2m^2 - 1) > 0$, $E < 0$, $0 < m < 1$, and $A = C^2m^2(m^2 - 1)/(E(2m^2 - 1)^2)$, we have the Jacobian elliptic function (JEF) solution

$$F(\xi) = \left[\varepsilon\sqrt{-\frac{Cm^2}{E(2m^2-1)}}\operatorname{cn}\left(p\sqrt{\frac{C}{2m^2-1}}\xi\right)\right]^{1/p}.$$

(ii) When $C > 0$, $E < 0$, $0 < m < 1$, and $A = C^2(1 - m^2)/(E(2 - m^2)^2)$, we have the JEF solution

$$F(\xi) = \left[\varepsilon\sqrt{-\frac{C}{E(2-m^2)}}\operatorname{dn}\left(p\sqrt{\frac{C}{2-m^2}}\xi\right)\right]^{1/p}.$$

(iii) When $C < 0$, $E > 0$, $0 < m < 1$, and $A = C^2m^2/(E(m^2+1)^2)$, we have the JEF solution

$$F(\xi) = \left[\varepsilon\sqrt{-\frac{Cm^2}{E(m^2+1)}}\,\mathrm{sn}\left(p\sqrt{-\frac{C}{m^2+1}}\xi\right)\right]^{1/p}.$$

Type 4. If $A = B = E = 0$, then one gets:

(i) When $C > 0$ and $D < 0$, we have the bright soliton solution

$$F(\xi) = \left[-\frac{C}{D}\operatorname{sech}^2\left(\frac{p}{2}\sqrt{C}\xi\right)\right]^{1/p}.$$

(ii) When $C < 0$ and $D > 0$, we have the periodic solution

$$F(\xi) = \left[-\frac{C}{D}\sec^2\left(\frac{p}{2}\sqrt{-C}\xi\right)\right]^{1/p}.$$

(iii) When $C = 0$ and $D > 0$, we have the rational solution

$$F(\xi) = \left[\frac{4}{D(p\xi)^2}\right]^{1/p}.$$

Type 5. If $C = E = 0$ and $D > 0$, we have the Weierstrass elliptic function WEF solution as

$$F(\xi) = \left[\wp\left(\frac{p}{2}\sqrt{D}\xi, g_2, g_3\right)\right]^{1/p},$$

where $g_2 = -4B/D$ and $g_3 = -4A/D$.

Type 6. If $B = D = 0$, then one gets the WEF solutions as:

(i) When $g_2 = (4C^2 - 12AE)/3$ and $g_3 = 4C(-2C^2 + 9AE)/27$, we have

$$F(\xi) = \left[\frac{1}{E}\wp(p\xi, g_2, g_3) - \frac{C}{3E}\right]^{1/(2p)} \quad \text{or} \quad F(\xi) = \left[\frac{3A}{3\wp(p\xi, g_2, g_3) - C}\right]^{1/(2p)}.$$

(ii) When $g_2 = C^2/12 + AE$ and $g_3 = C(36AE - C^2)/216$, we have

$$F(\xi) = \left[\frac{6\sqrt{A}\wp(p\xi, g_2, g_3) + C\sqrt{A}}{3\wp'(p\xi, g_2, g_3)}\right]^{1/p} \quad \text{or} \quad F(\xi) = \left[\frac{3\sqrt{E^{-1}}\wp'(p\xi, g_2, g_3)}{6\wp(p\xi, g_2, g_3) + C}\right]^{1/p},$$

where $\wp'(p\xi, g_2, g_3) = \mathrm{d}\wp(p\xi, g_2, g_3)/\mathrm{d}x$, and g_2, g_3 are called invariants of the WEF.

Type 7. If $A = B = 0$, then one gets:

(i) When $C > 0$, $D < 2C$, and $E = D^2/(4C) - C$, we have the bright soliton solution

$$F(\xi) = \left[\frac{1}{\cosh(p\sqrt{C}\xi) - \frac{D}{2C}}\right]^{1/p}.$$

(ii) When $C > 0$, $E > 0$, and $D = -2\sqrt{CE}$, we have the dark soliton solution

$$F(\xi) = \left[\frac{1}{2}\sqrt{\frac{C}{E}}\left[1 + \varepsilon \tanh\left(\frac{p}{2}\sqrt{C}\xi\right)\right]\right]^{1/p}.$$

(iii) When $C = 0$ and $E < 0$, we have the rational solution

$$F(\xi) = \left[\frac{4D}{(pD\xi)^2 - 4E}\right]^{1/n}.$$

Type 8. If $A = B = 0$, then one gets:
(i) When $C > 0$ and $D^2 - 4CE > 0$, we have the bright soliton solution

$$F(\xi) = \left[\frac{2C}{\varepsilon\sqrt{D^2 - 4CE}\cosh(p\sqrt{C}\xi) - D}\right]^{1/p}.$$

(ii) When $C > 0$ and $D^2 - 4CE < 0$, we have the singular soliton solution

$$F(\xi) = \left[\frac{2C}{\varepsilon\sqrt{-(D^2 - 4CE)}\sinh(p\sqrt{C}\xi) - D}\right]^{1/p}.$$

(iii) When $C > 0$ and $D^2 - 4CE = 0$, we have the dark and singular soliton solutions

$$F(\xi) = \left[-\frac{C}{D}\left\{1 + \varepsilon \tanh\left(\frac{p}{2}\sqrt{C}\xi\right)\right\}\right]^{1/p},$$
$$F(\xi) = \left[-\frac{C}{D}\{1 + \varepsilon \coth\left(\frac{p}{2}\sqrt{C}\xi\right)\}\right]^{1/p}.$$

(iv) When $C > 0$, we have the soliton solutions

$$F(\xi) = \left[-\frac{CD\operatorname{sech}^2(\frac{p}{2}\sqrt{C}\xi)}{D^2 - CE[1 + \varepsilon\tanh(\frac{p}{2}\sqrt{C}\xi)]^2}\right]^{1/p},$$
$$F(\xi) = \left[\frac{CD\operatorname{csch}^2(\frac{p}{2}\sqrt{C}\xi)}{D^2 - CE[1 + \varepsilon\coth(\frac{p}{2}\sqrt{C}\xi)]^2}\right]^{1/p}.$$

(v) When $C > 0$ and $E > 0$, we have the soliton solutions

$$F(\xi) = \left[-\frac{C\operatorname{sech}^2(\frac{p}{2}\sqrt{C}\xi)}{D + 2\varepsilon\sqrt{CE}\tanh(\frac{p}{2}\sqrt{C}\xi)}\right]^{1/p},$$
$$F(\xi) = \left[\frac{C\operatorname{csch}^2(\frac{p}{2}\sqrt{C}\xi)}{D + 2\varepsilon\sqrt{CE}\coth(\frac{p}{2}\sqrt{C}\xi)}\right]^{1/p}.$$

(vi) When $C < 0$ and $D^2 - 4CE > 0$, we have the periodic wave solution

$$F(\xi) = \left[\frac{2C\sec(p\sqrt{-C}\xi)}{\varepsilon\sqrt{D^2-4CE} - D\sec(p\sqrt{-C}\xi)}\right]^{1/p},$$

$$F(\xi) = \left[\frac{2C\csc(p\sqrt{-C}\xi)}{\varepsilon\sqrt{D^2-4CE} - D\csc(p\sqrt{-C}\xi)}\right]^{1/p}.$$

(vii) When $C < 0$ and $E > 0$, we have the periodic wave solutions

$$F(\xi) = \left[-\frac{C\sec^2(\frac{p}{2}\sqrt{-C}\xi)}{D + 2\varepsilon\sqrt{-CE}\tan(\frac{p}{2}\sqrt{-C}\xi)}\right]^{1/p},$$

$$F(\xi) = \left[-\frac{C\csc^2(\frac{p}{2}\sqrt{-C}\xi)}{D + 2\varepsilon\sqrt{-CE}\cot(\frac{p}{2}\sqrt{-C}\xi)}\right]^{1/p}.$$

Type 9. If $A = 0$, $B = 8C^2/(27D)$, and $E = D^2/(4C)$, then one gets:

(i) When $C < 0$, we have the soliton solutions

$$F(\xi) = \left[-\frac{8C\tanh^2(\frac{p}{2}\sqrt{-\frac{C}{3}}\xi)}{3D[3+\tanh^2(\frac{p}{2}\sqrt{-\frac{C}{3}}\xi)]}\right]^{1/p}, \quad F(\xi) = \left[-\frac{8C\coth^2(\frac{p}{2}\sqrt{-\frac{C}{3}}\xi)}{3D[3+\coth^2(\frac{p}{2}\sqrt{-\frac{C}{3}}\xi)]}\right]^{1/p}.$$

(ii) When $C < 0$, we have the periodic wave solutions

$$F(\xi) = \left[\frac{8C\tan^2(\frac{p}{2}\sqrt{\frac{C}{3}}\xi)}{3D[3-\tan^2(\frac{p}{2}\sqrt{\frac{C}{3}}\xi)]}\right]^{1/p}, \quad F(\xi) = \left[\frac{8C\cot^2(\frac{p}{2}\sqrt{\frac{C}{3}}\xi)}{3D[3-\cot^2(\frac{p}{2}\sqrt{\frac{C}{3}}\xi)]}\right]^{1/p}.$$

Type 10. If $A = 0$, then one gets:

(i) When $E > 0$, $0 < m < 1$, $B = D^3(m^2-1)/(32m^2E^2)$, and $C = D^4(5m^4-1)/(16m^2E)$, we have the JEF solutions

$$F(\xi) = \left[-\frac{D}{4E}\left\{1 + \varepsilon\,\mathrm{sn}\left(\frac{pD}{4m}\sqrt{\frac{1}{E}}\xi\right)\right\}\right]^{1/p},$$

$$F(\xi) = \left[-\frac{D}{4E}\left\{1 + \frac{\varepsilon}{m\,\mathrm{sn}(\frac{pD}{4m}\sqrt{\frac{1}{E}}\xi)}\right\}\right]^{1/p}.$$

(ii) When $E > 0$, $0 < m < 1$, $B = D^3(1-m^2)/(32E^2)$, and $C = D^2(5-m^2)/(16E)$, we have the JEF solutions

$$F(\xi) = \left[-\frac{D}{4E}\left\{1 + \varepsilon m\,\mathrm{sn}\left(\frac{pD}{4}\sqrt{\frac{1}{E}}\xi\right)\right\}\right]^{1/p},$$

$$F(\xi) = \left[-\frac{D}{4E}\left\{1 + \frac{\varepsilon}{\mathrm{sn}(\frac{pD}{4}\sqrt{\frac{1}{E}}\xi)}\right\}\right]^{1/p}.$$

(iii) When $E < 0$, $0 < m < 1$, $B = D^3/(32m^2E^2)$, and $C = D^2(4m^2+1)/(16m^2E)$, we have the JEF solution

$$F(\xi) = \left[-\frac{D}{4E}\left\{1 + \varepsilon\,\mathrm{cn}\left(\frac{pD}{4m}\sqrt{-\frac{1}{E}}\xi\right)\right\}\right]^{1/p}.$$

(iv) When $E < 0$, $0 < m < 1$, $B = m^2D^3/(32E^2(m^2-1))$, and $C = D^2(5m^2-4)/(16E(m^2-1))$, we have the JEF solution

$$F(\xi) = \left[-\frac{D}{4E}\left\{1 + \frac{\varepsilon}{\sqrt{1-m^2}}\,\mathrm{dn}\left(\frac{pD}{4}\sqrt{-\frac{1}{(1-m^2)E}}\xi\right)\right\}\right]^{1/p}.$$

(v) When $E > 0$, $0 < m < 1$, $B = D^3/(32E^2(1-m^2))$, and $C = D^2(4m^2-5)/(16E(m^2-1))$, we have the JEF solution

$$F(\xi) = \left[-\frac{D}{4E}\left\{1 + \frac{\varepsilon}{\mathrm{cn}(\frac{pD}{4}\sqrt{\frac{1}{(1-m^2)E}}\xi)}\right\}\right]^{1/p}.$$

(vi) When $E < 0$, $0 < m < 1$, $B = m^2D^3/(32E^2)$, and $C = D^2(m^2+4)/(16E)$, we have the JEF solutions

$$F(\xi) = \left[-\frac{D}{4E}\left\{1 + \varepsilon\,\mathrm{dn}\left(\frac{pD}{4}\sqrt{-\frac{1}{E}}\xi\right)\right\}\right]^{1/p},$$

$$F(\xi) = \left[-\frac{D}{4E}\left\{1 + \frac{\varepsilon\sqrt{1-m^2}}{\mathrm{dn}(\frac{pD}{4}\sqrt{-\frac{1}{E}}\xi)}\right\}\right]^{1/p}.$$

Type 11. If $A = E = 0$, then one gets:

(i) When $D < 0$, $(2m^2-1)C > 0$, $0 < m < 1$, and $B = C^2m^2(m^2-1)/(D(2m^2-1)^2)$, we have the JEF solution

$$F(\xi) = \left[-\frac{m^2C}{D(2m^2-1)}\,\mathrm{cn}^2\left(\frac{p}{2}\sqrt{\frac{C}{2m^2-1}}\xi\right)\right]^{1/p}.$$

(ii) When $D > 0$, $C < 0$, $0 < m < 1$, and $B = C^2m^2/(D(m^2+1)^2)$, we have the JEF solution

$$F(\xi) = \left[-\frac{m^2C}{D(m^2+1)}\,\mathrm{sn}^2\left(\frac{p}{2}\sqrt{-\frac{C}{m^2+1}}\xi\right)\right]^{1/p}.$$

(iii) When $D < 0$, $C > 0$, $0 < m < 1$, and $B = C^2(1-m^2)/(D(2-m^2)^2)$, we have

$$F(\xi) = \left[-\frac{C}{D(2-m^2)}\,\mathrm{dn}^2\left(\frac{p}{2}\sqrt{\frac{C}{2-m^2}}\xi\right)\right]^{1/p}.$$

Step 6. We substitute the values of A, B, C, D, E, μ, and v as well as the above solutions $F(\xi)$ into (11), and we have the exact solutions of Eqs. (1) and (2).

4 Solitary wave solutions

Consider the homogeneous balance between UU'' and U^4 in Eq. (10). We get

$$m + m + 2p = 4m \quad\Longrightarrow\quad m = p.$$

Now, Eq. (10) has the formal solution

$$U(\xi) = \mu F^p(\xi). \tag{13}$$

Inserting (13) together with (12) into Eq. (10) and isolating the coefficients of $F^{s_1 p}(\xi) \times [F'(\xi)]^{s_2}$, $s_1 = 0, 1, \dots, 4$, $s_2 = 0, 1$, the following system of equations is obtained:

$$\begin{aligned}
F^{4p}(\xi)&\colon\ 4\delta\mu^2 - 3E\alpha p^2 = 0,\\
F^{3p}(\xi)&\colon\ \bigl[4(\mu_1 + \nu_1)k\mu + 4\gamma\mu - 2\alpha D p^2\bigr]v + 4\beta\mu = 0,\\
F^{2p}(\xi)&\colon\ \bigl(Cp^2 - 4k^2\bigr)\alpha + 4\sigma^2 - 4k\lambda_1 - 4\omega = 0,\\
F^{0}(\xi)&\colon\ A\alpha p^2 = 0.
\end{aligned} \tag{14}$$

With the aid of Maple or Mathematica, solving system (14) leads to the following results:

$$A = 0, \qquad C = \frac{4(\alpha\kappa^2 + \kappa\lambda_1 - \sigma^2 + \chi)}{\alpha p^2},$$
$$D = \frac{2\mu(\varrho\kappa\mu_1 + \varrho\kappa\nu_1 + \gamma\varrho + \beta)}{\alpha p^2\varrho}, \qquad E = \frac{4\delta\mu^2}{3\alpha p^2}.$$

Case 1. When $A = B = D = 0$, we get the following results:

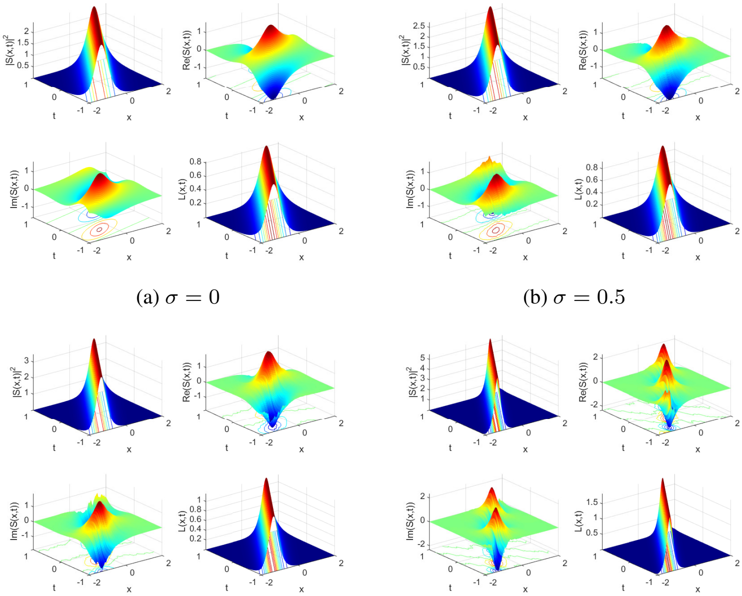
$$\begin{aligned}
&C = \frac{4(\alpha\kappa^2 + \kappa\lambda_1 - \sigma^2 + \chi)}{\alpha p^2}, \qquad E = \frac{4\delta\mu^2}{3\alpha p^2},\\
&\beta = -(\kappa\mu_1 + \kappa\nu_1 + \gamma)\varrho.
\end{aligned} \tag{15}$$

From (13) and (15) and the new sub-ODE method [18] the elevation solitary wave solutions are obtained as

$$\begin{aligned}
S(x,t) &= \left[\epsilon\sqrt{\frac{-3\Xi}{\delta}}\,\mathrm{sech}\left(2\sqrt{\frac{\Xi}{\alpha}}\zeta\right)\right]^{1/2} \mathrm{e}^{\mathrm{i}[-\kappa x + \chi t + \sigma\mathcal{W}(t) - \sigma^2 t]},\\
L(x,t) &= \frac{\epsilon\beta}{\varrho}\sqrt{\frac{-3\Xi}{\delta}}\,\mathrm{sech}\left(2\sqrt{\frac{\Xi}{\alpha}}\zeta\right),
\end{aligned} \tag{16}$$

provided that $\delta\Xi < 0$, $\alpha\Xi > 0$, $\epsilon = \pm 1$, and $\Xi = \alpha\kappa^2 + \kappa\lambda_1 - \sigma^2 + \chi$.

Figure 1 shows the effect of the noise parameter on the elevation solitary wave (16) with respect to $\epsilon = \lambda_1 = \chi = \beta = 1$, $\alpha = -1$, $\kappa = 2$, and $\delta = 0.5$.



(a) $\sigma = 0$ (b) $\sigma = 0.5$

(c) $\sigma = 1$ (d) $\sigma = 2$

Figure 1. Bright soliton solution of Eq.(16) with different values of noises.

Case 2. When $A = B = 0$ and $E = D^2/(4C) - C$, then we have

$$C = \frac{4\varXi}{\alpha p^2}, \qquad D = -\frac{24\varXi(\kappa\mu_1\varrho + \kappa\nu_1\varrho + \gamma\varrho + \beta)}{\alpha p^2\sqrt{9(\kappa\mu_1\varrho + \kappa\nu_1\varrho + \gamma\varrho + \beta)^2 - 48\varrho^2\varXi\delta}},$$
$$\mu = -\frac{12\varrho\varXi}{\sqrt{9(\kappa\mu_1\varrho + \kappa\nu_1\varrho + \gamma\varrho + \beta)^2 - 48\varrho^2\varXi\delta}}, \tag{17}$$

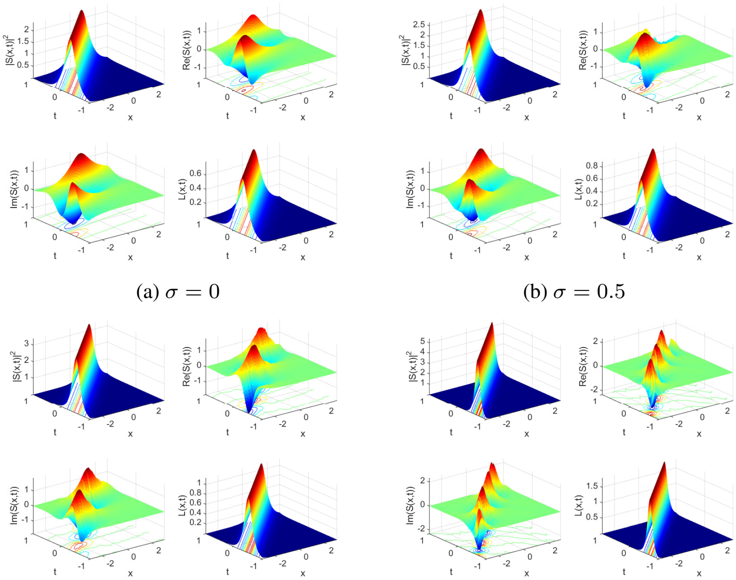
provided that $9(\kappa\mu_1\varrho + \kappa\nu_1\varrho + \gamma\varrho + \beta)^2 - 48\delta\varrho^2\varXi > 0$, and $\varXi = \alpha\kappa^2 + \kappa\lambda_1 - \sigma^2 + \chi$.

From (13) and (17) and the new sub-ODE method [18] the elevation solitary wave solutions are obtained as

$$S(x,t) = \left[-\frac{12\varrho\varXi}{\sqrt{Q_1}\cosh(2\sqrt{\frac{\varXi}{\alpha}}\zeta) + 3(\mu_1+\nu_1)\kappa\varrho + 3\gamma\varrho + 3\beta}r\right]^{1/2} \mathrm{e}^{\mathrm{i}[-\kappa x + \chi t + \sigma\mathcal{W}(t) - \sigma^2 t]},$$

$$L(x,t) = -\frac{12\beta\varXi}{\sqrt{Q_1}\cosh(2\sqrt{\frac{\varXi}{\alpha}}\zeta) + 3(\mu_1+\nu_1)\kappa\varrho + 3\gamma\varrho + 3\beta},$$

provided that $Q_1 = 9(\kappa\mu_1\varrho + \kappa\nu_1\varrho + \gamma\varrho + \beta)^2 - 48\delta\varrho^2\varXi > 0$, $\alpha\varXi > 0$, and $\varXi = \alpha\kappa^2 + \kappa\lambda_1 - \sigma^2 + \chi$.



(a) $\sigma = 0$ (b) $\sigma = 0.5$

(c) $\sigma = 1$ (d) $\sigma = 2$

Figure 2. Bright soliton solution of Eq.(18) with different values of noises.

Case 3. When $A = B = 0$ and $C > 0$, we get the soliton solutions

$$S(x,t) = \left[-\frac{24\varrho Q_2 \varXi \operatorname{sech}^2(\sqrt{\frac{\varXi}{\alpha}}\zeta)}{12Q_2^2 - 16\varrho^2\delta\varXi\{1 + \epsilon\tanh(\sqrt{\frac{\varXi}{\alpha}}\zeta)\}^2}\right]^{1/2} e^{i[-\kappa x + \chi t + \sigma\mathcal{W}(t) - \sigma^2 t]},$$

$$L(x,t) = -\frac{24\beta Q_2 \varXi \operatorname{sech}^2(\sqrt{\frac{\varXi}{\alpha}}\zeta)}{12Q_2^2 - 16\varrho^2\delta\varXi\{1 + \epsilon\tanh(\sqrt{\frac{\varXi}{\alpha}}\zeta)\}^2},$$

provided that $Q_2 = \kappa\mu_1\varrho + \kappa\nu_1\varrho + \gamma\varrho + \beta$, $\alpha\varXi > 0$, $\epsilon = \pm 1$, and $\varXi = \alpha\kappa^2 + \kappa\lambda_1 - \sigma^2 + \chi$.

Figure 2 shows the effect of the noise parameter on the elevation solitary wave (18) with respect to $\epsilon = \kappa = \beta = \gamma = \mu_1 = 1$, $\alpha = -0.5$, $\lambda_1 = \chi = 0.5$, $\nu_1 = 0.1$, and $\delta = 0.9$.

Case 4. When $A = B = 0$ and $C > 0$, we get the following results:

(i) If $D^2 - 4CE > 0$, then the elevation solitary wave solutions are obtained as

$$S(x,t) = \left[\frac{12\varrho\varXi}{\epsilon\sqrt{Q_1}\{2\cosh^2(\sqrt{\frac{\varXi}{\alpha}}\zeta) - 1\} - 3Q_2}\right]^{1/2} e^{i[-\kappa x + \chi t + \sigma\mathcal{W}(t) - \sigma^2 t]},$$

$$L(x,t) = \frac{12\beta\varXi}{\epsilon\sqrt{Q_1}\{2\cosh^2(\sqrt{\frac{\varXi}{\alpha}}\zeta) - 1\} - 3Q_2},$$

provided that $\alpha\varXi > 0$, $Q_1 > 0$, $\epsilon = \pm 1$, and $\varXi = \alpha\kappa^2 + \kappa\lambda_1 - \sigma^2 + \chi$.

(ii) If $D^2 - 4CE < 0$, then the singular soliton solutions are obtained as

$$S(x,t) = \left[\frac{12\varrho\Xi}{\epsilon\sqrt{-Q_1}\sinh(2\sqrt{\frac{\Xi}{\alpha}}\zeta) - 3Q_2}\right]^{1/2} \mathrm{e}^{\mathrm{i}[-\kappa x+\chi t+\sigma\mathcal{W}(t)-\sigma^2 t]},$$

$$L(x,t) = \frac{12\beta\Xi}{\epsilon\sqrt{-Q_1}\sinh(2\sqrt{\frac{\Xi}{\alpha}}\zeta) - 3Q_2},$$

provided that $\alpha\Xi > 0$, $Q_1 < 0$, $\epsilon = \pm 1$, and $\alpha\kappa^2 + \kappa\lambda_1 - \sigma^2 + \chi$.

(iii) If $E > 0$, then we have the soliton solutions

$$S(x,t) = \left[-\frac{12\varrho\Xi\,\mathrm{sech}^2(\sqrt{\frac{\Xi}{\alpha}}\zeta)}{6Q_2 + 8\epsilon\varrho\sqrt{3\delta\Xi}\tanh(\sqrt{\frac{\Xi}{\alpha}}\zeta)}\right]^{1/2} \mathrm{e}^{\mathrm{i}[-\kappa x+\chi t+\sigma\mathcal{W}(t)-\sigma^2 t]},$$

$$L(x,t) = -\frac{12\beta\Xi\,\mathrm{sech}^2(\sqrt{\frac{\Xi}{\alpha}}\zeta)}{6Q_2 + 8\epsilon\varrho\sqrt{3\delta\Xi}\tanh(\sqrt{\frac{\Xi}{\alpha}}\zeta)},$$

provided that $\delta\Xi > 0$, $\alpha\Xi > 0$, $\epsilon = \pm 1$, and $\Xi = \alpha\kappa^2 + \kappa\lambda_1 - \sigma^2 + \chi$.

(iv) If $D^2 - 4CE = 0$, then we have the following results:

$$C = \frac{4(\alpha\kappa^2 + \kappa\lambda_1 - \sigma^2 + \chi)}{\alpha p^2}, \qquad D = \frac{2\mu(\kappa\mu_1\varrho + \kappa\nu_1\varrho + \gamma\varrho + \beta)}{\alpha p^2\varrho},$$

$$E = \frac{(\kappa\mu_1\varrho + \kappa\nu_1\varrho + \gamma\varrho + \beta)^2\mu^2}{4\varrho^2\alpha p^2(\alpha\kappa^2 + \kappa\lambda_1 - \sigma^2 + \chi)}$$

along with the constraint condition

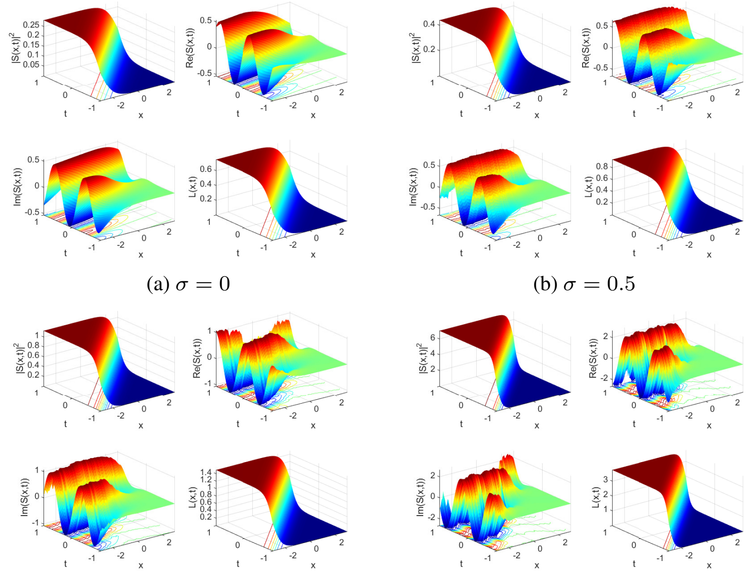
$$\delta = \frac{3(\kappa\mu_1\varrho + \kappa\nu_1\varrho + \gamma\varrho + \beta)^2}{16\varrho^2(\alpha\kappa^2 + \kappa\lambda_1 - \sigma^2 + \chi)}. \tag{18}$$

From (13) and (18) and the new sub-ODE method [18] the depression solitary wave solutions are obtained as

$$\begin{aligned} S(x,t) &= \left[-\frac{2\varrho\Xi}{\kappa\mu_1\varrho + \kappa\nu_1\varrho + \gamma\varrho + \beta}\left\{1 + \epsilon\tanh\left(\sqrt{\frac{\Xi}{\alpha}}\zeta\right)\right\}\right]^{1/2} \\ &\quad\times \mathrm{e}^{\mathrm{i}[-\kappa x+\chi t+\sigma\mathcal{W}(t)-\sigma^2 t]}, \\ L(x,t) &= -\frac{2\beta\Xi}{\kappa\mu_1\varrho + \kappa\nu_1\varrho + \gamma\varrho + \beta}\left\{1 + \epsilon\tanh\left(\sqrt{\frac{\Xi}{\alpha}}\zeta\right)\right\}, \end{aligned} \tag{19}$$

provided that $\alpha\Xi > 0$, $\epsilon = \pm 1$, and $\Xi = \alpha\kappa^2 + \kappa\lambda_1 - \sigma^2 + \chi$.

Figure 3 shows the effect of the noise parameter on the depression solitary wave (19) with respect to $\epsilon = \lambda_1 = \chi = \beta = \gamma = \mu_1 = \nu_1 = 1$, $\alpha = -1$, and $\kappa = 2$.



(a) $\sigma = 0$ (b) $\sigma = 0.5$

(c) $\sigma = 1$ (d) $\sigma = 2$

Figure 3. Dark soliton solution of Eq.(19) with different values of noises.

Case 5. When $A = 0$, $B = 8C^2/(27D)$, and $E = D^2/(4C)$, we have the following results:

$$C = \frac{4(\alpha\kappa^2 + \kappa\lambda_1 - \sigma^2 + \chi)}{\alpha p^2}, \qquad D = \frac{2\mu(\kappa\mu_1\varrho + \kappa\nu_1\varrho + \gamma\varrho + \beta)}{\alpha p^2\varrho},$$

$$E = \frac{(\kappa\mu_1\varrho + \kappa\nu_1\varrho + \gamma\varrho + \beta)^2\mu^2}{4\varrho^2\alpha p^2(\alpha\kappa^2 + \kappa\lambda_1 - \sigma^2 + \chi)}$$

along with the constraint condition

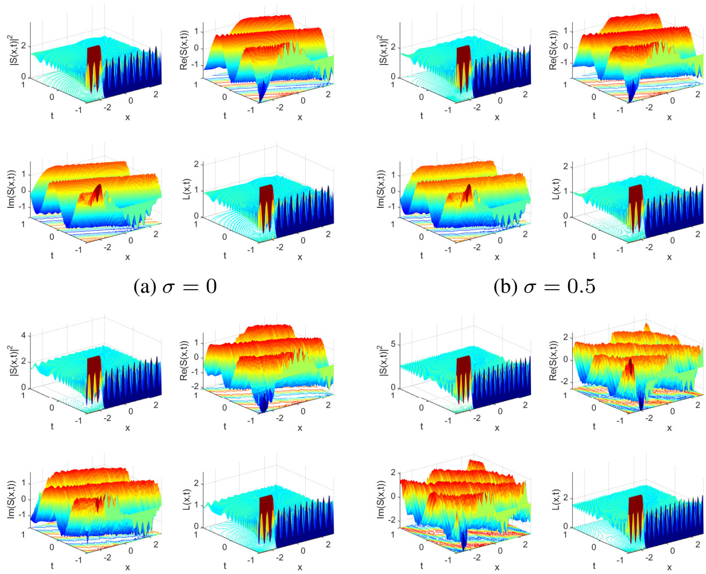
$$\delta = \frac{3(\kappa\mu_1\varrho + \kappa\nu_1\varrho + \gamma\varrho + \beta)^2}{16\varrho^2(\alpha\kappa^2 + \kappa\lambda_1 - \sigma^2 + \chi)}. \tag{20}$$

From (13) and (20) and new sub-ODE method [18] we have the soliton solutions as

$$S(x,t) = \left[\frac{-16\varrho\varXi\tanh^2(\sqrt{-\frac{\varXi}{3\alpha}}\zeta)}{3(\kappa\mu_1\varrho + \kappa\nu_1\varrho + \gamma\varrho + \beta)\{3 + \tanh^2(\sqrt{-\frac{\varXi}{3\alpha}}\zeta)\}}\right]^{1/2} \times \mathrm{e}^{\mathrm{i}[-\kappa x + \chi t + \sigma\mathcal{W}(t) - \sigma^2 t]},$$

$$L(x,t) = -\frac{16\beta\varXi\tanh^2(\sqrt{-\frac{\varXi}{3\alpha}}\zeta)}{3(\kappa\mu_1\varrho + \kappa\nu_1\varrho + \gamma\varrho + \beta)\{3 + \tanh^2(\sqrt{-\frac{\varXi}{3\alpha}}\zeta)\}},$$

provided that $\alpha\varXi < 0$ and $\varXi = \alpha\kappa^2 + \kappa\lambda_1 - \sigma^2 + \chi$.



(a) $\sigma = 0$ (b) $\sigma = 0.5$

(c) $\sigma = 1$ (d) $\sigma = 2$

Figure 4. JEF solution of Eq.(21) with different values of noises.

Case 6. When $A = 0$ and $0 < m < 1$, we have the following JEF solutions:

(i) If $E > 0$, $B = D^3(m^2 - 1)/(32m^2E^2)$, and $C = D^2(5m^2 - 1)/(16m^2E)$, then we have the JEF solutions as

$$\begin{aligned} S(x,t) &= \left[-\frac{8m^2\varrho\varXi\{1 + \epsilon\,\mathrm{sn}(2\sqrt{\frac{\varXi}{\alpha(5m^2-1)}}\zeta)\}}{(5m^2-1)(\kappa\mu_1\varrho + \kappa\nu_1\varrho + \gamma\varrho + \beta)}\right]^{1/2} \\ &\quad \times \mathrm{e}^{\mathrm{i}[-\kappa x + \chi t + \sigma\mathcal{W}(t) - \sigma^2 t]}, \\ L(x,t) &= -\frac{8m^2\beta\varXi\{1 + \epsilon\,\mathrm{sn}(2\sqrt{\frac{\varXi}{\alpha(5m^2-1)}}\zeta)\}}{(5m^2-1)(\kappa\mu_1\varrho + \kappa\nu_1\varrho + \gamma\varrho + \beta)}, \end{aligned} \tag{21}$$

provided that $\alpha(5m^2 - 1)\varXi > 0$, $\epsilon = \pm 1$, and $\varXi = \alpha\kappa^2 + \kappa\lambda_1 - \sigma^2 + \chi$.

Figure 4 shows the effect of noise parameter to the JEF solutions (21) with respect to $\epsilon = \gamma = \mu_1 = \nu_1 = 1$, $\alpha = -2$, $\lambda_1 = \beta = 3$, $\chi = -3$, $\kappa = 2$, and $m = 0.5$.

(ii) If $E > 0$, $B = D^3(1 - m^2)/(32E^2)$, and $C = D^2(5 - m^2)/(16E)$, consequently, we have

$$\begin{aligned} S(x,t) = \Bigg[&\frac{8\varrho\varXi}{(\kappa\mu_1\varrho + \kappa\nu_1\varrho + \gamma\varrho + \beta)(m^2 - 5)} \\ &\times \left\{1 + \epsilon m\,\mathrm{sn}\left(2\sqrt{\frac{-\varXi}{\alpha(m^2-5)}}\zeta\right)\right\}\Bigg]^{1/2} \mathrm{e}^{\mathrm{i}[-\kappa x + \chi t + \sigma\mathcal{W}(t) - \sigma^2 t]}, \end{aligned}$$

$$L(x,t)=\frac{8\beta\Xi}{(\kappa\mu_1\varrho+\kappa\nu_1\varrho+\gamma\varrho+\beta)(m^2-5)}\left\{1+\epsilon m\,\mathrm{sn}\left(2\sqrt{\frac{-\Xi}{\alpha(m^2-5)}}\zeta\right)\right\},$$

provided that $\alpha(m^2-5)\Xi<0$, $\epsilon=\pm1$, and $\Xi=\alpha\kappa^2+\kappa\lambda_1-\sigma^2+\chi$.

(iii) If $E<0$, $B=D^3/(32m^2E^2)$, and $C=D^2(4m^2+1)/(16m^2E)$, then we have

$$\begin{aligned}S(x,t)&=\left[\frac{-8\varrho\Xi}{(\kappa\mu_1\varrho+\kappa\nu_1\varrho+\gamma\varrho+\beta)(4m^2+1)}\right.\\&\quad\left.\times\left\{1+\epsilon\,\mathrm{cn}\left(\frac{2}{m}\sqrt{\frac{-\Xi}{\alpha(4m^2+1)}}\zeta\right)\right\}\right]^{1/2}\mathrm{e}^{\mathrm{i}[-\kappa x+\chi t+\sigma\mathcal{W}(t)-\sigma^2t]},\qquad(22)\\L(x,t)&=\frac{-8\beta\Xi}{(\kappa\mu_1\varrho+\kappa\nu_1\varrho+\gamma\varrho+\beta)(4m^2+1)}\left\{1+\epsilon\,\mathrm{cn}\left(\frac{2}{m}\sqrt{\frac{-\Xi}{\alpha(4m^2+1)}}\zeta\right)\right\}.\end{aligned}$$

Setting $m\to1^-$ in (22) leaves us the elevation solitary wave solution

$$\begin{aligned}S(x,t)&=\left[-\frac{8\varrho\Xi}{5(\kappa\mu_1\varrho+\kappa\nu_1\varrho+\gamma\varrho+\beta)}\right.\\&\quad\left.\times\left\{1+\epsilon\,\mathrm{sech}\left(2\sqrt{\frac{-\Xi}{5\alpha}}\zeta\right)\right\}\right]^{1/2}\mathrm{e}^{\mathrm{i}[-\kappa x+\chi t+\sigma\mathcal{W}(t)-\sigma^2t]},\qquad(23)\\L(x,t)&=-\frac{8\beta\Xi}{5(\kappa\mu_1\varrho+\kappa\nu_1\varrho+\gamma\varrho+\beta)}\left\{1+\epsilon\,\mathrm{sech}\left(2\sqrt{\frac{-\Xi}{5\alpha}}\zeta\right)\right\},\end{aligned}$$

provided that $\alpha\Xi<0$, $\epsilon=\pm1$, and $\Xi=\alpha\kappa^2+\kappa\lambda_1-\sigma^2+\chi$.

(iv) If $E<0$, $B=m^2D^3/(32E^2(m^2-1))$, and $C=D^2(5m^2-4)/(16E(m^2-1))$, consequently, we have

$$\begin{aligned}S(x,t)&=\left[-\frac{8\varrho\Xi\sqrt{m^2-1}}{(\kappa\mu_1\varrho+\kappa\nu_1\varrho+\gamma\varrho+\beta)(5m^2-4)}\right.\\&\quad\left.\times\left\{\sqrt{m^2-1}+\epsilon\,\mathrm{dn}\left(2\sqrt{\frac{\Xi}{\alpha(5m^2-4)}}\zeta\right)\right\}\right]^{1/2}\mathrm{e}^{\mathrm{i}[-\kappa x+\chi t+\sigma\mathcal{W}(t)-\sigma^2t]},\\L(x,t)&=-\frac{8\beta\Xi\sqrt{m^2-1}}{(\kappa\mu_1\varrho+\kappa\nu_1\varrho+\gamma\varrho+\beta)(5m^2-4)}\\&\quad\times\left\{\sqrt{m^2-1}+\epsilon\,\mathrm{dn}\left(2\sqrt{\frac{\Xi}{\alpha(5m^2-4)}}\zeta\right)\right\},\end{aligned}$$

provided that $\alpha\Xi(5m^2-4)>0$, $\epsilon=\pm1$, and $\Xi=\alpha\kappa^2+\kappa\lambda_1-\sigma^2+\chi$.

(v) If $E > 0$, $B = D^3/(32E^2(1 - m^2))$, and $C = D^2(4m^2 - 5)/(16E(m^2 - 1))$, consequently, we have the JEF solution as

$$S(x,t) = \left[-\frac{8\varrho(m^2-1)\varXi}{(\kappa\mu_1\varrho + \kappa\nu_1\varrho + \gamma\varrho + \beta)(4m^2-5)} \times \left\{1 + \frac{\epsilon}{\mathrm{cn}(2\sqrt{\frac{-\varXi}{\alpha(4m^2-5)}}\zeta)}\right\}\right]^{1/2} \mathrm{e}^{\mathrm{i}[-\kappa x + \chi t + \sigma \mathcal{W}(t) - \sigma^2 t]},$$

$$L(x,t) = -\frac{8\beta(m^2-1)\varXi}{(\kappa\mu_1\varrho + \kappa\nu_1\varrho + \gamma\varrho + \beta)(4m^2-5)}\left\{1 + \frac{\epsilon}{\mathrm{cn}(2\sqrt{\frac{-\varXi}{\alpha(4m^2-5)}}\zeta)}\right\},$$

provided that $\alpha(4m^2 - 5)\varXi < 0$, $\epsilon = \pm 1$, and $\varXi = \alpha\kappa^2 + \kappa\lambda_1 - \sigma^2 + \chi$.

(vi) If $E < 0$, $B = m^2D^3/(32E^2)$, and $C = D^2(m^2 + 4)/(16E)$, consequently, we have

$$S(x,t) = \left[-\frac{8\varrho\varXi}{(\kappa\mu_1\varrho + \kappa\nu_1\varrho + \gamma\varrho + \beta)(m^2+4)} \times \left\{1 + \epsilon\,\mathrm{dn}\left(2\sqrt{\frac{-\varXi}{\alpha(m^2+4)}}\zeta\right)\right\}\right]^{1/2} \mathrm{e}^{\mathrm{i}[-\kappa x + \chi t + \sigma \mathcal{W}(t) - \sigma^2 t]}, \tag{24}$$

$$L(x,t) = -\frac{8\beta\varXi}{(\kappa\mu_1\varrho + \kappa\nu_1\varrho + \gamma\varrho + \beta)(m^2+4)}\left\{1 + \epsilon\,\mathrm{dn}\left(2\sqrt{\frac{-\varXi}{\alpha(m^2+4)}}\zeta\right)\right\},$$

provided that $\alpha\varXi < 0$, $\epsilon = \pm 1$, and $\varXi = \alpha\kappa^2 + \kappa\lambda_1 - \sigma^2 + \chi$.

Setting $m \to 1^-$ in (24), we recover the same elevation solitary waves as (23).

In addition to the mathematical derivations, the physical properties of the obtained soliton solutions merit further exploration. Elevation and depression solitary waves, for instance, correspond to localized intensity enhancements or depletions in the short-wave component, which can directly affect energy transport in optical fibers and resonant media. Singular solutions, on the other hand, may represent extreme events or wave collapse scenarios. The role of multiplicative noise is particularly significant: it effectively shifts the frequency parameter $\varLambda$, thereby shrinking the parameter windows where solitons exist. This implies that stronger noise intensity can destabilize or suppress solitary waves, altering their propagation velocity, amplitude, and robustness. These effects suggest that noise not only modifies the stability landscape but also determines which types of soliton structures are physically realizable. In shallow-water hydrodynamics, fast and unresolved processes such as gusty wind stress and small-scale pressure variability act on the free surface and are commonly represented as temporally white stochastic forcings when their correlation times are short compared to the resolved dynamics [12]. This approach is mathematically standard for the shallow-water equations with random wind perturbations [7] and consistent with fluctuating-hydrodynamics theory, which introduces white-in-time random stresses at the continuum level [13]. Our white-noise term is therefore a physically justified idealization of high-frequency, subgrid processes acting on shallow-water flows [11].

5 Stability analysis

In this section, we analyze the stability properties of the solitary-wave solutions derived previously. We begin with the deterministic setting and derive the Hamiltonian spectral problem obtained by linearizing the governing equations. The spectral structure is then connected to orbital stability through the classical Vakhitov–Kolokolov (VK) and Grillakis–Shatah–Strauss (GSS) criteria, which depend on the slope of the conserved short-wave power. After establishing the slope condition, we apply it to the explicit solitary-wave families and distinguish between neutrally stable and spectrally stable branches. We then consider modulational instability around constant-amplitude backgrounds and finally describe the role of multiplicative white noise, showing how it shifts the existence and stability domains. This organization provides a coherent picture of stability for both deterministic and stochastic regimes.

We first consider the deterministic spectral problem. Setting $\sigma = 0$, we perturb the short-wave solution as

$$S(x,t) = \mathrm{e}^{\mathrm{i}(-\kappa x+\chi t)}\bigl(\vartheta(\zeta) + \varepsilon[u(\zeta)\mathrm{e}^{st} + \mathrm{i}v(\zeta)\mathrm{e}^{st}]\bigr),$$
$$L(x,t) = \varpi(\zeta) + \varepsilon w(\zeta)\mathrm{e}^{st}$$

with $s \in \mathbb{C}$ and $0 < \varepsilon \ll 1$. Linearizing the long-wave equation

$$L_t + \beta\bigl(|S|^2\bigr)_x = 0$$

in the moving frame gives

$$-\varrho w' + 2\beta(\vartheta u)' = 0 \quad \Longrightarrow \quad w = \frac{2\beta}{\varrho}\vartheta u$$

for localized perturbations. Substituting this expression into the linearized short-wave equation yields the Hamiltonian eigenvalue problem

$$s\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$

where the self-adjoint Schrödinger-type operators are

$$\begin{aligned} L_- &= -\alpha\partial_{\zeta\zeta} + \Lambda + G_1\vartheta^2 + \delta\vartheta^4, \\ L_+ &= -\alpha\partial_{\zeta\zeta} + \Lambda + 3G_1\vartheta^2 + 5\delta\vartheta^4 - \frac{2\beta}{\varrho}\vartheta^2 \end{aligned} \tag{25}$$

with $G_1 := \gamma + \kappa\mu_1 + \kappa\nu_1$. The LS-coupling term appears only in L_+, as dictated by the relation $w = (2\beta/\varrho)\vartheta u$. These operators satisfy $L_-\vartheta = 0$, $L_+\vartheta' = 0$, reflecting gauge and translation symmetries, which are essential for the VK/GSS framework.

The VK/GSS slope condition is based on the conserved short-wave power

$$M(\Lambda) := \int_{-\infty}^{\infty} \vartheta^2(\zeta;\Lambda)\,\mathrm{d}\zeta.$$

Under standard assumptions,

$$\frac{\mathrm{d}M}{\mathrm{d}\Lambda} < 0 \implies \text{orbital (spectral) stability}, \qquad \frac{\mathrm{d}M}{\mathrm{d}\Lambda} > 0 \implies \text{instability}.$$

Applying this criterion to explicit solitary-wave families, we begin with case 1 elevation solitons. For this branch,

$$\vartheta^2(\zeta) = \varepsilon\sqrt{\frac{-3(\alpha\kappa^2 + \kappa\lambda_1 - \sigma^2 + \chi)}{\delta}}\,\operatorname{sech}^2\left(2\sqrt{\frac{\alpha\kappa^2 + \kappa\lambda_1 - \sigma^2 + \chi}{\alpha}}\zeta\right),$$

which leads to

$$M(\Lambda) = \varepsilon\sqrt{\frac{-3}{\delta}}\frac{\pi}{2}\sqrt{\alpha}.$$

This expression is independent of Λ, implying $\mathrm{d}M/\mathrm{d}\Lambda = 0$. Hence the case 1 family is marginally (neutrally) stable in the VK sense. For the remaining elevation and depression branches, the expressions for ϑ^2 involve additive constants or rational sech / $\tanh$ combinations, giving explicit Λ-dependence. In these cases, within the existence domains, one obtains $\mathrm{d}M/\mathrm{d}\Lambda < 0$, indicating VK/GSS stability.

We next study modulational instability (MI) around constant-amplitude backgrounds. Linearizing about $\vartheta \equiv \vartheta_0$ gives the side-band dispersion relation

$$s^2(\xi) = \left(\alpha\xi^2 + G_1 + 2\delta\vartheta_0^2\right)\xi^2 - \frac{\beta}{\varrho}\vartheta_0^2.$$

The LS coupling introduces the additional contribution $-(\beta/\varrho)\vartheta_0^2$, which can act stabilizing or destabilizing depending on the sign, in agreement with the VK picture. Finally, we incorporate the effect of multiplicative white noise. The Itô gauge $\mathrm{e}^{\mathrm{i}[\sigma W(t) - \sigma^2 t]}$ transfers all σ-dependence into the shifted parameter

$$\Lambda(\sigma) = \chi - \sigma^2 + \kappa\lambda_1 + \alpha\kappa^2.$$

Thus, the profile equation and the operators (25) retain their form but are evaluated at $\Lambda(\sigma)$. As a result:

- The existence windows shrink as σ increases.
- The Case 1 elevation branch remains neutrally stable wherever it exists since M is constant.
- Mean-square stability reduces to the deterministic VK/GSS classification evaluated at $\Lambda(\sigma)$.

The linearization in the standing-wave frame yields a Hamiltonian system with self-adjoint blocks $L_\pm$. The correct placement of the LS-coupling term in L_+ ensures preservation of the symmetry kernels and allows the VK/GSS framework to classify stability of all explicit solitary-wave families derived earlier.

6 Conclusions

In this study, we derived new exact solutions of the stochastic LSWR equations with cubic–quintic nonlinearities, perturbation terms, and multiplicative white noise in the Itô sense. Using the new sub-ODE method, we obtained a broad family of solitary-wave and elliptic-function solutions, including elevation, depression, and singular branches, and analyzed their stability through the Vakhitov–Kolokolov/GSS framework. These results significantly enrich the theory of nonlinear wave resonance in stochastic environments.

Beyond their theoretical interest, the findings hold potential for practical applications. In particular, the explicit solitary-wave solutions provide useful insights for designing and controlling soliton-based technologies in optical fiber communications, nonlinear photonics, and signal transmission systems, where stochastic perturbations and higher-order nonlinearities often play a crucial role. Understanding how multiplicative noise affects the existence and stability of solitary waves can guide the development of more robust optical devices and communication protocols.

Future research may extend this work in several directions. First, exploring higher-dimensional generalizations of the stochastic LSWR model would provide a richer understanding of wave interactions in multidimensional media. Second, incorporating other types of stochastic forcing or fractional-order derivatives may reveal new dynamical behaviors. Finally, numerical simulations and experimental validations would complement the analytical results, helping to bridge the gap between theory and practical implementation.

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Conflicts of interest. The authors declare no conflicts of interest.

Appendix. Symbols and parameters

Symbol	Definition / Meaning
α	Dispersion coefficient of the short-wave component
β	Coupling coefficient between long and short waves
ϵ	A parameter in sub-ODE method with values ± 1
γ	Cubic nonlinear coefficient in the short-wave equation
δ	Quintic nonlinear coefficient in the short-wave equation
λ_1	Perturbation parameter associated with the short wave
μ_1	Perturbation parameter associated with higher-order effects
ν_1	Perturbation parameter in the short-wave interaction
θ_1	Additional perturbation constant
ϱ	Constant related to the long-wave velocity

Symbol	Definition / Meaning
κ	Wave number in the traveling-wave ansatz
χ	Frequency parameter in the phase of the short wave
σ	Noise intensity parameter (strength of multiplicative white noise)
$W(t)$	Standard Wiener process representing Itô white noise
$S(x,t)$	Short-wave complex envelope
$L(x,t)$	Long-wave component
$\vartheta(\zeta)$	Short-wave profile in the traveling-wave reduction
$\varpi(\zeta)$	Long-wave profile in the traveling-wave reduction

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