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Adaptive synchronization of quaternion-valued inertial neural networks with uncertain parameters and a flexible time delay: A novel non-separation and non-reduction approach*

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Abstract. In this paper, a direct method is presented for analyzing the adaptive synchronization of quaternion-valued inertial neural networks (QVINNs) with uncertain parameters and time delay. By this method, the forms of time delays and the Lyapunov function in this paper are flexible, and in stability analysis, the requirement for negative definiteness of Lyapunov functions will be weakened. In addition, a more generalized adaptive synchronization controller has been proposed for QVINNs, which no longer requires multiple calculations of the control gain. Since most of the previous adaptive control methods did not impose requirements on the convergence speed of the QVINNs, a class of improved adaptive controllers is designed to address the slow synchronization speed of QVINNs. Finally, numerical simulations are conducted to validate the synchronization conditions.

Keywords: quaternion, adaptive synchronization, non-separation approach, delayed inertial neural networks.

1 Introduction

The Irish mathematician Hamilton first introduced the concept of quaternions in 1843. It emerged from his exploration of a new mathematical structure intended to expand the

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concept of complex numbers. A key difference between this innovative structure and the realms of real and complex numbers lies in the inclusion of a real component alongside three imaginary components. Furthermore, the commutative property of multiplication, which holds for real and complex numbers, does not universally hold for quaternions. As a consequence, quaternions are less commonly utilized compared to real and complex numbers. In recent decades, numerous scholars have diligently worked to address the difficulties arising from the non-commutativity of quaternion multiplication, resulting in significant advancements. For instance, in the field of color image processing, Chen et al. introduced a quaternion matrix for color image analysis [4]. Similarly, Shen et al. explored the application of singular value decomposition of quaternion matrices to address vector sensor signal processing issues in geology [23]. Lastly, in the field of new energy sources, Knolmajer et al. researched a quaternion-based approach for irradiance calculation in estimating energy output from solar power plants [8].

With escalating interest in machine learning and artificial intelligence, neural networks have surfaced as a noteworthy focal point for researchers, proffering a promising avenue for tackling intricate problems [7]. It became gradually apparent that real-valued neural networks (RVNNs) are ill-equipped to cope with high-dimensional challenges. For example, Parcollet et al. illustrated that real-valued convolutional neural networks, trained on grayscale images, failed to capture color information [18]. Consequently, scholars shifted their focus to complex-valued neural networks (CVNNs), and numerous outstanding achievements have been made in this field [3, 6]. Currently, CVNNs are recognized as an efficient method for processing two-dimensional input vectors primarily applied in signal processing, this is because complex numbers can simultaneously represent both the phase and amplitude of signals. Despite this, the application of complex numbers is constrained to two-dimensional space, posing a challenge in meeting the demands of three-dimensional representation. Over the past two decades, quaternion-valued neural networks (QVNNs) have emerged as a potentially advantageous alternative due to their demonstrated capacity to effectively handle multidimensional data. Significant insights into the dynamic behavior of QVNNs have been extensively derived in some literature [19–21]. To circumvent the non-commutativity of quaternion multiplication, a common and effective approach is to decompose a QVNN equivalently into four RVNNs or two CVNNs. Then one can proceed to analyse the problem smoothly using the multiplication rules for real and complex numbers. However, this separation method results in a larger system with an increased number of RVNNs or CVNNs, significantly amplifying the researcher's workload. Consequently, a novel trend has emerged in processing QVNNs using a direct quaternion method. Compared to the separation method, the non-separation approach excels in preserving the unique properties of quaternions, thereby enhancing computational efficiency. Some outstanding non-separation approaches have been obtained in [15, 24, 25, 28].

In 1986, Babcock and his colleagues pioneered the integration of inductors into neural networks, using second-order differential equations to illustrate the dynamic model. This pioneering work gave rise to a novel network model known as inertial neural networks (INNs), with diverse applications across various domains. This technology is primarily applied to biological models, including the study of subthreshold behavior of axons in squid giant neurons [17] and capillary resonance models [2], but it also covers other areas such as image encryption techniques [11, 22] and secure communications [10]. Furthermore, the field of networking has seen significant advancements and discoveries as evidenced by a plethora of recent studies [9, 26, 27, 31]. The introduction of inertial terms endows quaternion-valued neural networks with more complex asymptotic behaviors such as chaos and bifurcation. Addressing these issues holds practical significance, as it may play a unique role in fields like aircraft attitude control and image encryption/decryption. The success of these results hinges on employing a step-down technique in dealing with inertial neural networks. This method involves transforming a second-order network model into two first-order models by substituting variables. While this method effectively handles the complexity associated with second-order variables through variable substitution, it imposes a substantial workload. How to address inertial terms without resorting to variable substitution is one of the key challenges of this paper.

Numerous scholars have generated a substantial body of high-quality outcomes rooted in the non-reduction approach [13, 14, 16]. However, these achievements often overlook the influence of an infinite time-varying delay on the system. While Singh's research group had integrated the impact of an infinite time-varying delay, its methodologies are restricted by the necessity for the time-varying delay to be derivable and for the derivative of the time-varying delay not to exceed 1 [24]. In regard to controller design, in [27], authors incorporated a delay term, which introduces an element of uncertainty into the control system. In [24, 26, 28], the designed controllers did not include a delay, but multiple integrators are used to calculate the control gain, which may present challenges in practical implementation. Additionally, deriving the results requires the prior design of intricate Lyapunov functions, along with addressing the non-commutativity of quaternion multiplication and handling more general time-delay terms, posing considerable challenges to both theoretical derivation and practical application. Given these challenges, this study enhances and refines the current approaches, while introducing a novel control technique. The main contributions of this paper are given in the following statements.

(i) A new non-separation and non-reduction approach is utilized for the synchronization of QVINNs, this avoids the difficulty of restoring the controller to the original system in the reduction method, and the conservative assumptions about the activation function in the separation method.
(ii) Due to the unique inertial term present in inertial neural networks, the method mentioned in literature [29] is no longer applicable to QVINNs. In this paper, we construct an auxiliary function that does not include the integral term from Literature [24], and its advantage is that it does not require additional processing to handle delay.
(iii) Different from existing results [24, 29], the model studied in this paper is more general. Some of parameters in the model are uncertain, and the time delay considered can be of various forms. The time delay can manifest as constant delay, bounded time-varying delay, unbounded time-varying delay with bounded and differentiable derivatives, or unbounded time-varying delay with nondifferentiable derivatives.

(iv) In existing results [1, 24, 28], their adaptive controller involved multiple control gains for the convenience of theoretical derivation. However, in the control strategy in this paper, only one control gain is needed, which makes our controller design simpler.

(v) In previous studies on the adaptive synchronization of inertial neural networks, there were no requirements for the convergence speed of the error system. In this paper, the γ-type function is defined, and a class of improved adaptive control method is proposed, which enables the error system to achieve faster convergence.

The remainder of this paper is structured with the following contents: in Section 2, a class of QVNN models is presented, and some preparatory work is outlined; in Section 3, sufficient conditions for synchronization and adaptive synchronization of QVINNs are explored; in Section 4, the feasibility of the presented theorem is verified through three numerical examples; and in Section 5, the main contributions and future work of this paper are summarized.

Notations. $\mathbb{I} = \{1, 2, 3, \dots, n\}$, the set of all real numbers is denoted as $\mathbb{R}$, and the set of all quaternion numbers is denoted as $\mathbb{Q}$. For any $u \in \mathbb{Q}$, $u = u^r + \mathcal{I}u^i + \mathcal{J}u^j + \mathcal{K}u^k$, $\mathrm{Re}(u) = u^r$, $\overline{u} = u^r - \mathcal{I}u^i - \mathcal{J}u^j - \mathcal{K}u^k$ is the conjugate of u, in which $u^r, u^i, u^j, u^k \in \mathbb{R}$, and $\mathcal{I}^2 = \mathcal{J}^2 = \mathcal{K}^2 = -1$, $\mathcal{I}\mathcal{J} = -\mathcal{J}\mathcal{I}$, $\mathcal{J}\mathcal{K} = -\mathcal{K}\mathcal{J}$, $\mathcal{K}\mathcal{I} = -\mathcal{I}\mathcal{K}$, where $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$ are imaginary units. $\|\cdot\|$ means the 2-norm. $\mathbb{Q}^n$ is expressed as a vector space with n-dimensional quaternions. The set of continuous and bounded functions from $(t_0, +\infty]$ to $\mathbb{R}$ is defined as $C((t_0, +\infty], \mathbb{R})$.

2 Preliminaries

Consider a class of QVINNs with delay as follows:

$$\ddot{u}_i(t) = -\alpha_i u_i(t) - \beta_i \dot{u}_i(t) + \sum_{j=1}^{n} a_{ij} f_j\big(u_j(t)\big) + \sum_{j=1}^{n} b_{ij} g_j\big(u_j\big(t - \delta(t)\big)\big), \tag{1}$$

where $i \in \mathbb{I}$, $0 < \hat{\alpha}_i \leqslant \alpha_i \leqslant \hat{\hat{\alpha}}_i$, and $0 < \hat{\beta}_i \leqslant \beta_i \leqslant \hat{\hat{\beta}}_i$. $u_i(t) \in \mathbb{Q}$ represents the state of the ith neurons at a time t. The second-order derivative $\ddot{u}_i(t)$ is called the inertial term. $a_{ij}, b_{ij} \in \mathbb{Q}$ are quaternion-valued connection weights. $\delta(t)$ denotes the time-varying delay. $f_i(\cdot)$ and $g_i(\cdot)$ represent the quaternion-valued activation functions. The initial values related to system (1) are given by $u_i(s) = \kappa_i(s)$, $\dot{u}_i(s) = \vartheta_i(s)$, $s \in (-\infty, 0]$, where $\kappa_i(s), \vartheta_i(s) \in \mathcal{Q}((-\infty, 0], \mathbb{Q})$.

With system (1) considered as the drive system, the corresponding response system is described as follows:

$$\begin{aligned}\ddot{v}_i(t) = {}& -\alpha_i v_i(t) - \beta_i \dot{v}_i(t) + \sum_{j=1}^{n} a_{ij} f_j\big(v_j(t)\big) + \sum_{j=1}^{n} b_{ij} g_j\big(v_j\big(t - \delta(t)\big)\big) \\ & + U_i(t),\end{aligned} \tag{2}$$

where $U_i(t)$ is the controller that needs to be designed, and the remaining parameters retain their meanings as in system (1). The initial values related to system (2) are given by $v_i(s) = K_i(s)$, $\dot{v}_i(s) = \Theta_i(s)$, $s \in (-\infty, 0]$, where $K_i(s), \Theta_i(s) \in \mathcal{Q}((-\infty, 0], \mathbb{Q})$.

Remark 1. In contrast to papers [27] and [1], the controller in this paper does not incorporate time delay terms or sign functions. In contrast to papers [24] and [28], the controller in this paper utilises an adaptive rule. This design renders the controller implementation simpler and more straightforward, avoiding potential discontinuities and system jitter issues that sign functions may induce.

$e_i(t) = v_i(t) - u_i(t)$ is defined as the synchronization error for the drive-response systems (1) and (2). A description of the corresponding error function is provided as follows:

$$\begin{aligned} \ddot{e}_i(t) = {} & -\alpha_i e_i(t) - \beta_i \dot{e}_i(t) + \sum_{j=1}^{n} a_{ij} F_j\bigl(e_j(t)\bigr) + \sum_{j=1}^{n} b_{ij} G_j\bigl(e_j\bigl(t - \delta(t)\bigr)\bigr) \\ & + U_i(t), \end{aligned} \tag{3}$$

where $F_j(e_j(t)) = f_j(v_j(t)) - f_j(u_j(t))$, $G_j(e_j(t)) = g_j(v_j(t)) - g_j(u_j(t))$. Now, we give some assumptions for activation functions and delays as follows.

Assumption 1. For any $j \in \mathbb{I}$ and $q, p \in \mathbb{Q}$, there exist two constants $l_j^f, l_j^g \geqslant 0$ such that the activation functions in systems (1) and (2) satisfy the following inequalities:

$$\bigl\|f_j(q) - f_j(p)\bigr\| \leqslant l_j^f \|q - p\|, \qquad \bigl\|g_j(q) - g_j(p)\bigr\| \leqslant l_j^g \|q - p\|.$$

Assumption 2. Time-varying delay $\delta(t)$ satisfies $\delta(t) > 0$ and

$$\lim_{t \to +\infty} \bigl(t - \delta(t)\bigr) = +\infty.$$

Remark 2. The delay conditions considered here have various manifestations, which can be bounded delays [30], proportional delays [1, 32], unbounded delays that are not differentiable and differentiable unbounded delays with bounded derivatives [24].

Definition 1. The drive-response systems (1) and (2) are said to be globally asymptotically synchronized if $\lim_{t\to\infty} \|e(t)\| = 0$, where $e(t) = (e_1(t), e_2(t), \dots, e_n(t))$.

Lemma 1. (*See* [29].) *Let $p(t) \geqslant 0$ be a continuous function satisfying*

$$\dot{p}(t) \leqslant -\alpha p(t) + \beta p\bigl(t - \delta(t)\bigr),$$

where $\alpha > \beta \geqslant 0$. Then $p(t)$ converges asymptotically to zero if $\lim_{t\to+\infty}(t - \delta(t)) = +\infty$.

Remark 3. When analysing certain stability issues, researchers often employ Lyapunov method. This involves constructing a positive definite Lyapunov function $V(e_i(t))$ and proving the negative definiteness of its derivative to demonstrate the stability of the equilibrium point. However, the approach adopted in this paper differs from this line of reasoning. We construct an auxiliary function $p(t)$ that is larger than the norm of the error $e_i(t)$. By showing that $p(t)$ asymptotically converges to zero, it is demonstrated that the norm of the error $e_i(t)$ also asymptotically converges to zero.

Lemma 2 [Barbalat lemma]. *Let a function $f(t)$ be uniformly continuous, and let $\lim_{t\to+\infty}\int_{t_0}^{t} f(s)\,\mathrm{d}s$ exist and be finite. Then $\lim_{t\to+\infty} f(t)=0$.*

Lemma 3. (*See* [25, *Lemma* 2].) *If $x, y \in \mathbb{Q}$ and $\varepsilon \in \mathbb{R}^{+}$, we obtain*

$$xy+\bar{y}\bar{x} \leqslant \varepsilon x\bar{x}+\frac{1}{\varepsilon} y\bar{y}.$$

Lemma 4. *If $x, y \in \mathbb{Q}$ and $\varepsilon \in \mathbb{R}^{+}$, the following inequality holds*:

$$|\bar{x}y+\bar{y}x| \leqslant \varepsilon\bar{x}x+\frac{1}{\varepsilon}\bar{y}y.$$

Proof. In fact, there exists a conclusion that $|\bar{x}y+\bar{y}x| = 2|\operatorname{Re}(\bar{x}y)|$, then two scenarios will be discussed.

Case 1. When $\operatorname{Re}(\bar{x}y) \geqslant 0$, $\bar{x}y+\bar{y}x \geqslant 0$ and $|\bar{x}y+\bar{y}x| = \bar{x}y+\bar{y}x$. Based on Lemma 3, the following conclusion holds: $|\bar{x}y+\bar{y}x| = \bar{x}y+\bar{y}x \leqslant \varepsilon x\bar{x}+y\bar{y}/\varepsilon$.

Case 2. When $\operatorname{Re}(\bar{x}y) < 0$, $\bar{x}y+\bar{y}x < 0$ and $|\bar{x}y+\bar{y}x| = -\bar{x}y-\bar{y}x$. It is evident that $\overline{(\sqrt{\varepsilon}x+y/\sqrt{\varepsilon})}(\sqrt{\varepsilon}x+y/\sqrt{\varepsilon}) \geqslant 0$ is true. Then $-\bar{x}y-\bar{y}x \leqslant \varepsilon\bar{x}x+\bar{y}y/\varepsilon$ must also be true. Therefore, $|\bar{x}y+\bar{y}x| = -\bar{x}y-\bar{y}x \leqslant \varepsilon x\bar{x}+y\bar{y}/\varepsilon$. Then the proof is complete. $\square$

Remark 4. It is clear that Lemma 3 is a special case of Lemma 4. In fact, $x\bar{y}+y\bar{x} < |x\bar{y}+y\bar{x}|$ is always true, and when $|\bar{x}y+\bar{y}x| \leqslant \varepsilon\bar{x}x+\bar{y}y/\varepsilon$ holds, Lemma 3 clearly holds.

3 Main results

This section presents a novel synchronization method for QVINNs with infinite time-varying delays, which employs a controller that is more readily implementable than existing approaches.

Theorem 1. *Suppose that Assumptions* 1 *and* 2 *hold. Systems* (1) *and* (2) *are globally asymptotically synchronized under the feedback controller $U_i(t) = -k_i(e_i(t)+\dot{e}_i(t))$ if the following condition holds*:

$$A+B<0,$$

where

$$A_i^1 = -2+\frac{1}{\varepsilon}\left(|-\alpha_i+\beta_i|+\sum_{j=1}^{n} a_{ji}\overline{a_{ji}}\left(l_i^f\right)^2\right),$$

$$A_i^2 = -2(\beta_i+k_i-1)+\varepsilon\left(|-\alpha_i+\beta_i|+2n\right),$$

$$A_i = \max_{i=1,\dots,n}\left\{A_i^1, A_i^2\right\}, \qquad B_i = \varepsilon\sum_{j=1}^{n}\overline{b_{ji}}b_{ji}\left(l_i^g\right)^2,$$

$$A = \max_{i=1,\dots,n}\{A_1, A_2, \dots, A_n\}, \qquad B = \max_{i=1,\dots,n}\{B_1, B_2, \dots, B_n\},$$

$$k = \max_{i=1,\dots,n}\{k_i\}, \quad \varepsilon \in \mathbb{R}^{+}.$$

Proof. Let

$$p(t) = \sum_{i=1}^{n} \overline{e_i(t)} e_i(t) + \overline{\big(\dot{e}_i(t) + e_i(t)\big)} \big(\dot{e}_i(t) + e_i(t)\big).$$

The function $p(t)$ takes the derivative with respect to time t, which can then be obtained by substituting Eq. (3):

$$\begin{aligned}
\dot{p}(t) &= \sum_{i=1}^{n} \big\{ \overline{\big(\ddot{e}_i(t) + \dot{e}_i(t)\big)} \big(\dot{e}_i(t) + e_i(t)\big) + \overline{\big(\dot{e}_i(t) + e(t)\big)} \big(\ddot{e}_i(t) + \dot{e}_i(t)\big) \\
&\quad + \overline{\dot{e}_i(t)} e_i(t) + \overline{e_i(t)} \dot{e}_i(t) \big\} \\
&= \sum_{i=1}^{n} \Bigg\{ \Bigg(\overline{-\alpha_i e_i(t) - (\beta_i - 1)\dot{e}_i(t) + \sum_{j=1}^{n} a_{ij} F_j\big(e_j(t)\big)} \\
&\quad \overline{+ \sum_{j=1}^{n} b_{ij} G_j\big(e_j\big(t - \delta(t)\big)\big) - k_i\big(e_i(t) + \dot{e}_i(t)\big)} \Bigg) \big(e_i(t) + \dot{e}_i(t)\big) \\
&\quad + \overline{\big(\dot{e}_i(t) + e_i(t)\big)} \Bigg(-\alpha_i e_i(t) - (\beta_i - 1)\dot{e}_i(t) + \sum_{j=1}^{n} a_{ij} F_j\big(e_j(t)\big) \\
&\quad + \sum_{j=1}^{n} b_{ij} G_i\big(t - \delta(t)\big) - k_i\big(e_i(t) + \dot{e}_i(t)\big) \Bigg) + \overline{\dot{e}_i(t)} e_i(t) + \overline{e_i(t)} \dot{e}_i(t) \Bigg\} \\
&= \sum_{i=1}^{n} \Bigg\{ -2\overline{e_i(t)} e_i(t) + \big\{ \overline{e_i(t)} \big(e_i(t) + \dot{e}_i(t)\big) + \overline{\big(e_i(t)} + \overline{\dot{e}_i(t)\big)} e_i(t) \big\} \\
&\quad + \big(l - \alpha_i + \beta_i - 1\big) \big\{ \overline{e_i(t)} \big(e_i(t) + \dot{e}_i(t)\big) + \overline{\big(e_i(t) + \dot{e}_i(t)\big)} e_i(t) \big\} \\
&\quad - 2(\beta_i + k_i - 1) \overline{\big(e_i(t) + \dot{e}_i(t)\big)} \big(e_i(t) + \dot{e}_i(t)\big) \\
&\quad + \Bigg\{ \overline{\sum_{j=1}^{n} a_{ij} F_j(e_j)} (e_i(t) + \dot{e}_i(t)) + \overline{(e_i(t) + \dot{e}_i(t))} \sum_{j=1}^{n} a_{ij} F_j(e_j) \Bigg\} \\
&\quad + \Bigg\{ \overline{\sum_{j=1}^{n} b_{ij} G_j(e_i - \delta(t))} \big(e_i(t) + \dot{e}_i(t)\big) + \overline{(e_i(t) + \dot{e}_i(t))} \sum_{j=1}^{n} b_{ij} G_j\big(t - \delta(t)\big) \Bigg\} \Bigg\} \\
&= \sum_{i=1}^{n} \Bigg\{ -2\overline{e_i(t)} e_i(t) + (-\alpha_i + \beta_i) \big\{ \overline{e_i(t)} \big(e_i(t) + \dot{e}_i(t)\big) \overline{\big(e_i(t) + \dot{e}_i(t)\big)} e_i(t) \big\} \\
&\quad + \Bigg\{ \overline{\sum_{j=1}^{n} a_{ij} F_j\big(e_j(t)\big)} \big(e_i(t) + \dot{e}_i(t)\big) + \overline{\big(e_i(t) + \dot{e}_i(t)\big)} \sum_{j=1}^{n} a_{ij} F_j\big(e_j(t)\big) \Bigg\} \\
&\quad + \Bigg\{ \overline{\big(e_i(t) + \dot{e}_i(t)\big)} \sum_{j=1}^{n} b_{ij} G_i\big(e_j\big(t - \delta(t)\big)\big) + \overline{\sum_{j=1}^{n} b_{ij} G_i\big(e_j\big(t - \delta(t)\big)\big)} \big(e_i(t) + \dot{e}_i(t)\big) \Bigg\} \\
&\quad - 2(\beta_i + k_i - 1) \overline{\big(e_i(t) + \dot{e}_i(t)\big)} \big(e_i(t) + \dot{e}_i(t)\big) \Bigg\}.
\end{aligned}$$

According to Lemmas 3, 4 and Assumption 1, there exist constants ε, $l_j^f, l_j^g > 0$ such that

$$
\begin{aligned}
&(-\alpha_i+\beta_i)\bigl\{\overline{e_i(t)}\bigl(e_i(t)+\dot e_i(t)\bigr)+\overline{\bigl(e_i(t)+\dot e_i(t)\bigr)}e_i(t)\bigr\}\\
&\quad\leqslant \bigl|(-\alpha_i+\beta_i)\bigl\{\overline{e_i(t)}\bigl(e_i(t)+\dot e_i(t)\bigr)+\overline{\bigl(e_i(t)+\dot e_i(t)\bigr)}e_i(t)\bigr\}\bigr|\\
&\quad\leqslant \bigl|(-\alpha_i+\beta_i)\bigr|\biggl\{\varepsilon\overline{\bigl(e_i(t)+\dot e_i(t)\bigr)}\bigl(e_i(t)+\dot e_i(t)\bigr)+\frac{1}{\varepsilon}\dot e_i(t)e_i(t)\biggr\}.
\end{aligned}
$$

Similar proofs lead to the following two conclusions:

$$
\begin{aligned}
&\Biggl\{\overline{\sum_{j=1}^n a_{ij}F_j\bigl(e_j(t)\bigr)}\bigl(e_i(t)+\dot e_i(t)\bigr)+\overline{\bigl(e_i(t)+\dot e_i(t)\bigr)}\sum_{j=1}^n a_{ij}F_j\bigl(e_j(t)\bigr)\Biggr\}\\
&\quad\leqslant \varepsilon\overline{\bigl(e_i(t)+\dot e_i(t)\bigr)}\bigl(e_i(t)+\dot e_i(t)\bigr)+\frac{1}{\varepsilon}\sum_{j=1}^n\overline{a_{ij}F_j\bigl(e_j(t)\bigr)}a_{ij}F_j\bigl(e_j(t)\bigr)\\
&\quad\leqslant \varepsilon\overline{\bigl(e_i(t)+\dot e_i(t)\bigr)}\bigl(e_i(t)+\dot e_i(t)\bigr)+\frac{1}{\varepsilon}\sum_{j=1}^n\overline{a_{ij}}a_{ij}\bigl(l_j^f\bigr)^2\overline{e_j(t)}e_j(t)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Biggl\{\overline{\sum_{j=1}^n b_{ij}G_j\bigl(e_j\bigl(t-\delta(t)\bigr)\bigr)}\bigl(e_i(t)+\dot e_i(t)\bigr)+\overline{\bigl(e_i(t)+\dot e_i(t)\bigr)}\sum_{j=1}^n b_{ij}G_j\bigl(e_j\bigl(t-\delta(t)\bigr)\bigr)\Biggr\}\\
&\quad\leqslant \varepsilon\overline{\bigl(e_i(t)+\dot e_i(t)\bigr)}\bigl(e_i(t)+\dot e_i(t)\bigr)+\frac{1}{\varepsilon}\sum_{j=1}^n\overline{b_{ij}}b_{ij}\bigl(l_j^g\bigr)^2\overline{e_j\bigl(t-\delta(t)\bigr)}e_j\bigl(t-\delta(t)\bigr).
\end{aligned}
$$

This demonstrates that the following equation is true:

$$
\begin{aligned}
\dot p(t)\leqslant{}&\Biggl\{-2+\frac{1}{\varepsilon}\Biggl(|-\alpha_i+\beta_i|+\sum_{j=1}^n a_{ji}\overline{a_{ji}}\bigl(l_i^f\bigr)^2\Biggr)\Biggr\}\overline{e_i(t)}e_i(t)\\
&+\bigl\{-2(\beta_i+k_i-1)+\varepsilon\bigl(|-\alpha_i+\beta_i|+2n\bigr)\bigr\}\overline{\bigl(e_i(t)+\dot e_i(t)\bigr)}\bigl(e_i(t)+\dot e_i(t)\bigr)\\
&+\frac{1}{\varepsilon}\sum_{j=1}^n\overline{b_{ji}}b_{ji}\bigl(l_i^g\bigr)^2\overline{e_i\bigl(t-\delta(t)\bigr)}e_i\bigl(t-\delta(t)\bigr)\\
<{}&A_i^1\overline{e_i(t)}e_i(t)+A_i^2\overline{\bigl(e_i(t)+\dot e_i(t)\bigr)}\bigl(e_i(t)+\dot e_i(t)\bigr)+B_ip\bigl(t-\delta(t)\bigr)\\
<{}&A_ip(t)+B_ip\bigl(t-\delta(t)\bigr)<Ap(t)+Bp\bigl(t-\delta(t)\bigr).
\end{aligned}
\tag{4}
$$

In light of the definitions of A and B, as well as the findings of Lemma 2, it can be inferred that $\lim_{t\to\infty}p(t)=0$; consequently,

$$
\lim_{t\to\infty}\sum_{i=1}^n\overline{e_i(t)}e_i(t)\leqslant\lim_{t\to\infty}p(t)=0.
$$

Then, according to Definition 1, the proof is complete. □

Remark 5. In Theorem 1, an innovative method is proposed that addresses the impact of the delay term on the system without the use of a Lyapunov function. This approach relaxes the delay constraints, differing from the approaches proposed in [24] and [28], and is less prevalent in QVINNs synchronization studies.

Remark 6. If the delay is restricted to be either a constant delay or a bounded delay, it can be asserted that $V(t)$ converges exponentially to zero. Specifically, there exist constants a and b such that $p(t) = a\mathrm{e}^{-b(t-t_0)}$. This assertion stems from the requirement that when Eq. (4) holds, the inequality $\dot{p}(t) \leqslant Ap(t) + B\sup_{t-\delta\leqslant s\leqslant t}(p(t))$ must also hold, where $B > 0$, $A + B < 0$. It is evident based on the classical Halanay inequality [12].

Remark 7. Addressing the non-commutativity of quaternion multiplication and handling delay terms are central challenges in this paper. In contrast to [17, 18], our delay assumptions are considerably more flexible. Moreover, Lemma 4, which deals with quaternion non-commutativity, together with the use of the max function to manage parameter uncertainty, represents one of the novel contributions of this work.

Corollary 1. *Provided that the delay δ(t) is either bounded or constant and Assumption* 1 *is satisfied, the global exponential synchronization of systems* (1) *and* (2) *can be achieved by the controller* $U_i(t) = -k_i(e_i(t) + \dot{e}_i(t))$.

In order to reduce the waste of control gain, an adaptive controller has been designed as follows for achieving drive-response synchronization:

$$U_i(t) = -k(t)\big(e_i(t) + \dot{e}_i(t)\big), \tag{5}$$

where

$$\dot{k}(t) = \varsigma\sum_{i=1}^{n}\big\{\overline{e_i(t)}e_i(t) + \overline{\big(\dot{e}_i(t) + e_i(t)\big)}\big(\dot{e}_i(t) + e_i(t)\big)\big\}, \quad \varsigma, k(0) > 0.$$

Theorem 2. *If Assumptions* 1 *and* 2 *hold, systems* (1) *and* (2) *will achieve drive-response synchronization with an adaptive controller* (5).

Proof. Let $p(t) = \sum_{i=1}^{n}\overline{(\dot{e}_i(t) + e_i(t))}(\dot{e}_i(t) + e_i(t)) + \overline{e_i(t)}e_i(t)$. The function $p(t)$ takes the derivative with respect to time t, which can be obtained by substituting Eq. (3):

$$\begin{aligned}\dot{p}(t) = \sum_{i=1}^{n}\Bigg\{&-2\overline{e_i(t)}e_i(t) + (-\alpha_i + \beta_i)\big\{\overline{e_i(t)}\big(e_i(t) + \dot{e}_i(t)\big)\overline{\big(e_i(t) + \dot{e}_i(t)\big)}e_i(t)\big\}\\ &- 2\big(\beta_i + k(t) - 1\big)\overline{\big(e_i(t) + \dot{e}_i(t)\big)}\big(e_i(t) + \dot{e}_i(t)\big)\\ &+ \Bigg\{\overline{\sum_{j=1}^{n}a_{ij}F_j\big(e_j(t)\big)}\big(e_i(t) + \dot{e}_i(t)\big) + \overline{\big(e_i(t) + \dot{e}_i(t)\big)}\sum_{j=1}^{n}a_{ij}F_j\big(e_j(t)\big)\Bigg\}\\ &+ \Bigg\{\overline{\big(e_i(t) + \dot{e}_i(t)\big)}\sum_{j=1}^{n}b_{ij}G_i\big(e_j\big(t - \delta(t)\big)\big)\\ &+ \overline{\sum_{j=1}^{n}b_{ij}G_i\big(e_j\big(t - \delta(t)\big)\big)}\big(e_i(t) + \dot{e}_i(t)\big)\Bigg\}\Bigg\}.\end{aligned}$$

Based on Lemma 3, the following inequality is established:

$$\begin{aligned}
\dot{p}(t) \leqslant{}& \left\{-2 + \frac{1}{\varepsilon}\left(|-\alpha_i + \beta_i| + \sum_{j=1}^{n} a_{ji}\overline{a_{ji}}\big(l_i^f\big)^2\right)\right\}\overline{e_i(t)}e_i(t) \\
&+ \big\{-2(\beta_i + k(t) - 1) + \varepsilon\big(|-\alpha_i + \beta_i| + 2n\big)\big\}\overline{\big(e_i(t) + \dot{e}_i(t)\big)}\big(e_i(t) + \dot{e}_i(t)\big) \\
&+ \frac{1}{\varepsilon}\sum_{j=1}^{n} \overline{b_{ji}}b_{ji}\big(l_i^g\big)^2\overline{e_i\big(t - \delta(t)\big)}e_i\big(t - \delta(t)\big) \\
\leqslant{}& \max_{i=1,\dots,n}\left\{-2 + \frac{1}{\varepsilon}\left(|-\alpha_i + \beta_i| + \sum_{j=1}^{n} a_{ji}\overline{a_{ji}}\big(l_i^f\big)^2\right),\right. \\
&\left. -2\big(\beta_i + k(t) - 1\big) + \varepsilon\big(|-\alpha_i + \beta_i| + 2n\big)\right\}p(t) + Qp\big(t - \delta(t)\big), \quad (6)
\end{aligned}$$

where $Q = \varepsilon \max_{i=1,\dots,n} \sum_{j=1}^{n}\{\overline{b_{ji}}b_{ji}\big(l_i^g\big)^2\}$.

Suppose the following equation holds:

$$\begin{aligned}
&\max_{i=1,\dots,n}\left\{-2 + \frac{1}{\varepsilon}\left(|-\alpha_i + \beta_i| + \sum_{j=1}^{n} a_{ji}\overline{a_{ji}}\big(l_i^f\big)^2\right),\right. \\
&\qquad\left. -2\big(\beta_i + k(t) - 1\big) + \varepsilon\big(|\beta_i - \alpha_i| + 2n\big)\right\} \\
&= -2 + \frac{1}{\varepsilon}\left(|-\alpha_i + \beta_i| + \sum_{j=1}^{n} a_{ji}\overline{a_{ji}}\big(l_i^f\big)^2\right).
\end{aligned}$$

Inequality (6) is equivalent to the following inequality:

$$\begin{aligned}
\dot{p}(t) \leqslant{}& \max_{i=1,\dots,n}\left\{-2 + \frac{1}{\varepsilon}\left(|-\alpha_i + \beta_i| + \sum_{j=1}^{n} a_{ji}\overline{a_{ji}}\big(l_i^f\big)^2\right)\right\}p(t) \\
&+ Qp\big(t - \delta(t)\big).
\end{aligned}$$

When ε is sufficiently large, the following equation must hold:

$$-2 + \frac{1}{\varepsilon}\left(|-\alpha_i + \beta_i| + \sum_{j=1}^{n} a_{ji}\overline{a_{ji}}\big(l_i^f\big)^2 + \sum_{j=1}^{n} \overline{b_{ji}}b_{ji}\big(l_i^g\big)^2\right) < 0.$$

It follows that

$$-2 + \frac{1}{\varepsilon}\left(|-\alpha_i + \beta_i| + \sum_{j=1}^{n} a_{ji}\overline{a_{ji}}\big(l_i^f\big)^2\right) + Q < 0,$$

and by Lemma 1, we conclude that $\lim_{t\to+\infty} p(t) = 0$ and $\lim_{t\to+\infty} \sum_{i=1}^{n} \overline{e_i(t)}e_i(t) \leqslant \lim_{t\to+\infty} p(t) = 0$. According to Definition 1, systems (1) and (2) are synchronized under the controller (5).

Another suppose is that

$$\begin{aligned}&\max_{i=1,\dots,n}\left\{-2+\frac{1}{\varepsilon}\left(|-\alpha_i+\beta_i|+\sum_{j=1}^{n}a_{ji}\overline{a_{ji}}\big(l_i^f\big)^2\right),\right.\\&\qquad\left.-2\big(\beta_i+k(t)-1\big)+\varepsilon\big(|\beta_i-\alpha_i|+2n\big)\right\}\\&=-2\big(\beta_i+k(t)-1\big)+\varepsilon\big(|\beta_i-\alpha_i|+2n\big).\end{aligned}$$

Inequality (6) is equivalent to the following inequality:

$$\dot p(t)\leqslant\max_{i=1,\dots,n}-2\big(\beta_i+k(t)-1\big)+\varepsilon\big(|\alpha_i+\beta_i|+2n\big)p(t)+Qp\big(t-\delta(t)\big).$$

Suppose k^* is a constant and satisfies

$$k^*>\frac{1}{2}\max_{i=1,\dots,n}\big\{-2(\beta_i-1)+\varepsilon\big(|\beta_i-\alpha_i|+2n\big)+Q\big\}.$$

If there exists t^* such that $k(t^*)=k^*$, by the monotonicity of $k(t)$, it follows that $k(t)>k^*$ when $t>t^*$. In this instance, the following inequality is satisfied:

$$\begin{aligned}&-k(t)+\frac{1}{2}\max_{i=1,\dots,n}\big\{-2(\beta_i-1)+\varepsilon\big(|\beta_i-\alpha_i|+2n\big)+Q\big\}\\&\quad<-k^*+\frac{1}{2}\max_{i=1,\dots,n}\big\{-2(\beta_i-1)+\varepsilon\big(|\beta_i-\alpha_i|+2n\big)+Q\big\}<0.\end{aligned}$$

As in the proof of Theorem 1, it can be shown with great ease that $\lim_{t\to+\infty}p(t)=0$, $\lim_{t\to+\infty}\sum_{i=1}^{n}\overline{e_i(t)}e_i(t)=0$, and that systems (1) and (2) achieve drive-response synchronization under the adaptive controller (5).

Conversely, for any $0<k(t)<k^*$, let $M(t)=p(t)+(k(t)-k^*)^2/\varsigma$. Then the derivative of $M(t)$ with respect to time t is

$$\begin{aligned}\dot M(t)\leqslant{}&\max_{i=1,\dots,n}\big\{-2\big(\beta_i+k(t)-1\big)+\varepsilon\big(|\beta_i-\alpha_i|+2n\big)\big\}p(t)+Qp\big(t-\delta(t)\big)\\&+2\big(k(t)-k^*\big)\big[\overline{e_i(t)}e_i(t)+\overline{\big(\dot e(t)+e_i(t)\big)}\big(\dot e(t)+e_i(t)\big)\big].\end{aligned}$$

Thus,

$$\begin{aligned}\dot M(t)&\leqslant\max_{i=1,\dots,n}\big\{-2\big(\beta_i+k(t)-1\big)+\varepsilon\big(|\beta_i-\alpha_i|+2n\big)\big\}p(t)\\&\quad+Qp\big(t-\delta(t)\big)+2\big(k(t)-k^*\big)p(t)\\&=\max_{i=1,\dots,n}\big\{-2(\beta_i+k^*-1)+\varepsilon\big(|\beta_i-\alpha_i|+2n\big)\big\}p(t)\\&\quad+Qp\big(t-\delta(t)\big).\end{aligned}$$

By the monotonicity of $k(t)$, it can be demonstrated that

$$\big(k\big(t-\delta(t)\big)-k^*\big)^2>\big(k(t)-k^*\big)^2$$

and

$$M\big(t-\delta(t)\big) = p\big(t-\delta(t)\big) + \sum_{i=1}^{n} \frac{1}{\varsigma}\big(k\big(t-\delta(t)\big) - k^*\big)^2$$
$$> p\big(t-\delta(t)\big) + \sum_{i=1}^{n} \frac{1}{\varsigma}\big(k(t) - k^*\big)^2. \tag{7}$$

The following conclusion is reached by relating Eq. (7):

$$\begin{aligned}
\dot{M}(t) &< \max_{i=1,\dots,n}\big\{-2(\beta_i + k^* - 1) + \varepsilon\big(|\beta_i - \alpha_i| + 2n\big)\big\}p(t) \\
&\quad + QM\big(t-\delta(t)\big) - Q\sum_{i=1}^{n}\frac{1}{\varsigma}\big(k(t)-k^*\big)^2 \\
&= \max_{i=1,\dots,n}\big\{-2(\beta_i + k^* - 1) + \varepsilon\big(|\beta_i - \alpha_i| + 2n\big)\big\}p(t) \\
&\quad + QM\big(t-\delta(t)\big) - QM(t) + Qp(t) \\
&\leqslant \max_{i=1,\dots,n}\big\{-2\big(\beta_i + k^* - 1) + \varepsilon\big(|\beta_i - \alpha_i| + 2n\big) + Q\big\}p(t) \\
&\quad - \frac{1}{2}QM(t) + \frac{1}{2}QM\big(t-\delta(t)\big) \\
&\leqslant -\frac{1}{2}QM(t) + \frac{1}{2}QM\big(t-\delta(t)\big).
\end{aligned} \tag{8}$$

Define $\mathcal{M} = \sup_{t\in[0,T]} M(t)$. Since $k(t)$, $e_i(t)\overline{e_i}(t)$, and $\dot{e}_i(t)\overline{\dot{e}_i(t)}$ are continuous, $\mathcal{M}$ is bounded on the region $[0,T]$. It is evident that $M(t) \leqslant \mathcal{M}$ on the interval $[0,T]$. Furthermore, it is demonstrated that this conclusion remains valid on the interval $(T,+\infty)$.

Suppose that $t_n^* > T$ satisfies $M(t_n^*) = \mathcal{M}$. In the event that no such series exists, the trajectory of $M(t)$ can be discussed in two cases.

Case 1. $M(t) < \mathcal{M}$, $t \in (T,+\infty)$. It can be concluded that $M(t) \leqslant \mathcal{M}$ is a valid proposition.

Case 2. $M(t) > \mathcal{M}$, $t \in (T,+\infty)$. It can be demonstrated that at this juncture T, $M(T)$ must be equal to $\mathcal{M}$. Otherwise, at the moment T, the left side $M(t) < \mathcal{M}$, at the moment T, the right side $M(t) > \mathcal{M}$, which is in contradiction with the continuity of $M(t)$. Consider the derivative at time T. According to Eq. (8), the function is nonincreasing at time T, which contradicts the assumed trajectory in case 2.

It is now shown that the conclusion $M(t) \leqslant \mathcal{M}$ still holds for the existence of the sequence t_n^*. For the defined time series $t_n{}^*$, the inequality

$$\dot{M}(t_n^*) < -\frac{1}{2}QM(t_n^*) + \frac{1}{2}QM\big(t_n^* - \delta(t_n^*)\big)$$

is established. When analysing the smallest t_1^* in the sequence t_n^*, by the definition of t_n^*, we obtain the following equation:

$$\dot{M}(t_1^*) < -\frac{1}{2}QM(t_1^*) + \frac{1}{2}QM\big(t_1^* - \delta(t_1^*)\big) = -\frac{1}{2}Q\mathcal{M} + \frac{1}{2}QM\big(t_1^* - \delta(t_1^*)\big).$$

If $t_1^* - \delta(t_1^*) \leqslant T$, then $M(t_1^* - \delta(t_1^*)) < \mathcal{M}$, $\dot{M}(t_1^*) < 0$. Otherwise, $t_1^* - \delta(t_1^*) > T$. The conclusion that $\dot{M}(t_1^*) < 0$ clearly holds when $M(t_1^* - \delta(t_1^*)) \leqslant \mathcal{M}$. Conversely, if $M(t_1^* - \delta(t_1^*) > \mathcal{M}$, one can obtain $\dot{M}(T) < 0$ and $M(t_1^* - \delta(t_1^*) > \mathcal{M}$. Then, in accordance with the mediator theorem for continuous functions, there exists a t_0 on the interval $(T,\, t_1^* - \delta(t_1^*))$ that satisfies the condition, which is inconsistent with the definition of the minimum of t_1. Thus on the interval $t \in (T, t_1^*]$, it is concluded that $M(t) \leqslant \mathcal{M}$. Similarly, it can be shown that on the following intervals $(t_1^*, t_2^*]$, $(t_2^*, t_3^*]$, $\dots$, and $(t_n^*, +\infty)$, $M(t) \leqslant \mathcal{M}$ always holds.

The above analytical procedure shows that $M(t)$ is a bounded function, which in turn shows that $p(t)$ must also be a bounded function. It is already known that $0 < k(t) < k^*$, so by Eq. (6) it can be concluded that $\dot{p}(t)$ is an upper bounded function. Therefore, $p(t)$ is said to be uniformly continuous on the interval $[0, +\infty)$.

The construction of $p(t)$ allows for the derivation of the following result:

$$p(t) = \sum_{i=1}^{n} \overline{\big(\dot{e}_i(t) + e_i(t)\big)}\big(\dot{e}_i(t) + e_i(t)\big) + \overline{e_i(t)}e_i(t) = \frac{1}{\varsigma}\dot{k}(t). \tag{9}$$

Integrating Eq. (9) on $[0, +\infty)$, we can get

$$\lim_{t\to+\infty} \int_0^t p(t)\,\mathrm{d}t = \frac{1}{\varsigma} \lim_{t\to+\infty} \big(k(t) - k(0)\big) < +\infty.$$

This also yields $\lim_{t\to+\infty} p(t) = 0$ by virtue of Lemma 2, which serves to demonstrate that $\lim_{t\to+\infty} \sum_{i=1}^{n} \overline{e_i(t)}e_i(t) = 0$. $\square$

Although globally asymptotic synchronization is achieved, the desired synchronization speed may not be attained by this adaptive method alone. Therefore, the concept of a γ-type function is defined and introduced for the purpose of analyzing the error system, inspired by the methods in [29] and [5].

Definition 2. A function $\mathcal{F}(t) \in C([t_0, +\infty), R+)$ is called a γ-type function if it fulfills the subsequent conditions:

(i) $\mathcal{F}(t)$, $\mathcal{F}'(t)$ are differentiable, and $\mathcal{F}'(t)/\mathcal{F}(t)$, $\mathcal{F}''(t)/\mathcal{F}(t)$ have bounds on $[t_0, +\infty)$,

(ii) $\lim_{t\to\infty} \mathcal{F}(t) = +\infty$.

Theorem 3. *Suppose Assumptions* 1 *and* 2 *hold. Furthermore, there exists an γ-type function $\gamma(t)$ that satisfies the condition* $0 < \overline{\lim}_{t\to\infty}\dot{\gamma}(t)/(\gamma(t - \delta(t))) < +\infty$. *Then it can be concluded that systems* (1) *and* (2) *will achieve drive-response synchronization under the controller*

$$U_i(t) = -k(t)\bigg((1 + \frac{\dot{\gamma}(t)}{\gamma(t)})e_i(t) + \dot{e}_i(t)\bigg)$$

with the following adaptive law:

$$\dot{k}(t) = \varsigma \sum_{i=1}^{n} \big(\overline{E_i}(t)E_i(t) + \overline{\big(\dot{E}_i(t) + E_i(t)\big)}\big(E_i(t) + \dot{E}_i(t)\big)\big),$$

where $\varsigma > 0$ *and* $E_i(t) = \gamma(t)e_i(t)$.

Proof. From the definition of $E_i(t)$, the time derivative of $E_i(t)$ is $\dot{E}_i(t) = \dot{\gamma}(t)\, e_i(t) + \gamma(t)\, \dot{e}_i(t)$. Differentiating $\dot{E}_i(t)$ once more with respect to time, we obtain

$$\begin{aligned}
\ddot{E}_i(t) &= \ddot{\gamma}(t)e_i(t) + \dot{\gamma}(t)\dot{e}_i(t) + \dot{\gamma}(t)\dot{e}_i(t) + \gamma(t)\ddot{e}_i(t) \\
&= \frac{\ddot{\gamma}(t)}{\gamma(t)}E_i(t) + 2\frac{\dot{\gamma}(t)}{\gamma(t)}\left(\dot{E}_i(t) - \frac{\dot{\gamma}(t)}{\gamma(t)}E_i(t)\right) - \left(\alpha_i - \beta_i\left(1 + \frac{\dot{\gamma}(t)}{\gamma(t)}\right)\right)E_i(t) \\
&\quad - \beta_i\big(E_i(t) + \dot{E}_i(t)\big) + \gamma(t)\sum_{j=1}^{n} a_{ij}F_j\big(e_j(t)\big) \\
&\quad + \gamma(t)\sum_{j=1}^{n} b_{ij}G_j\big(e_j\big(t - \delta(t)\big)\big) - k(t)\big(E_i(t) + \dot{E}_i(t)\big) \\
&= \left(-\alpha_i + \beta_i\left(1 + \frac{\dot{\gamma}(t)}{\gamma(t)}\right) + \frac{\ddot{\gamma}(t)}{\gamma(t)} - \left(\frac{\dot{\gamma}(t)}{\gamma(t)}\right)^2 - 2\frac{\dot{\gamma}(t)}{\gamma(t)}\right)E_i(t) \\
&\quad - \left(\beta_i - 2\frac{\dot{\gamma}(t)}{\gamma(t)}\right)\big(E_i(t) + \dot{E}_i(t)\big) + \gamma(t)\sum_{j=1}^{n} a_{ij}F_j\big(e_j(t)\big) \\
&\quad + \gamma(t)\sum_{j=1}^{n} b_{ij}G_j\big(e_j\big(t - \delta(t)\big)\big) - k(t)\big(E_i(t) + \dot{E}_i(t)\big).
\end{aligned}$$

Define the function

$$\hat{p}(t) = \sum_{i=1}^{n} \overline{\big(\dot{E}_i(t) + E_i(t)\big)}\big(\dot{E}_i(t) + E_i(t)\big) + \overline{E_i(t)}E_i(t).$$

Then the derivative of $V(t)$ with respect to time t can be expressed as

$$\begin{aligned}
\dot{\hat{p}}(t) &= \sum_{i=1}^{n} \big[\overline{\dot{E}_i(t)}E_i(t) + \overline{E_i(t)}\dot{E}_i(t) + \overline{\big(\ddot{E}_i(t) + \dot{E}_i(t)\big)}\big(\dot{E}_i(t) + E_i(t)\big) \\
&\quad + \overline{\big(\dot{E}_i(t) + E_i(t)\big)}\big(\ddot{E}_i(t) + \dot{E}_i(t)\big)\big] \\
&= -2\overline{E}_i(t)E_i(t) + \big(\overline{E_i(t)}\big(E_i(t) + \dot{E}_i(t)\big) + \overline{\big(E_i(t) + \dot{E}_i(t)\big)}E_i(t)\big) \\
&\quad + \left(-\alpha_i + \beta_i\left(1 + \frac{\dot{\gamma}(t)}{\gamma(t)}\right)\frac{\ddot{\gamma}(t)}{\gamma(t)} - \left(\frac{\dot{\gamma}(t)}{\gamma(t)}\right)^2 - 2\frac{\dot{\gamma}(t)}{\gamma(t)} - 1\right)\big(\overline{E_i(t)}\big(E_i(t) + \dot{E}_i(t)\big) \\
&\quad + \overline{\big(E_i(t) + \dot{E}_i(t)\big)}E_i(t)\big) - 2\left(\beta_i - 2\frac{\dot{\gamma}(t)}{\gamma(t)} - \frac{1}{2}\right)\overline{\big(E_i(t) + \dot{E}_i(t)\big)}\big(E_i(t) + \dot{E}_i(t)\big)
\end{aligned}$$

$$
\begin{aligned}
&+ \left(\overline{\gamma(t)\sum_{j=1}^{n} F_j\big(e_j(t)\big)}\big(E_i(t)+\dot{E}_i(t)\big) + \overline{\big(E_i(t)+\dot{E}_i(t)\big)}\gamma(t)\sum_{j=1}^{n} F_j\big(e_j(t)\big)\right) \\
&+ \left(\overline{\gamma(t)\sum_{j=1}^{n} b_{ij} G_j\big(e_j\big(t-\delta(t)\big)\big)}\big(E_i(t)+\dot{E}_i(t)\big)\right. \\
&\left.+ \overline{\big(E_i(t)+\dot{E}_i(t)\big)}\gamma(t)\sum_{j=1}^{n} b_{ij} G_j\big(e_j\big(t-\delta(t)\big)\big)\right) \\
&- 2k(t)\overline{\big(E_i(t)+\dot{E}_i(t)\big)}\big(E_i(t)+\dot{E}_i(t)\big) \\
\leqslant{}& -2 + \frac{1}{\varepsilon}\left(\left|-\alpha_i + \beta_i\left(1+\frac{\dot{\gamma}(t)}{\gamma(t)}\right) + \frac{\ddot{\gamma}(t)}{\gamma(t)} - \left(\frac{\dot{\gamma}(t)}{\gamma(t)}\right)^2 - 2\frac{\dot{\gamma}(t)}{\gamma(t)}\right|\right. \\
&\left.+ \sum_{j=1}^{n} \overline{a_{ji}} a_{ji}\big(l_i^f\big)^2\right)\overline{E}_i(t)E_i(t) + \left(-2\beta_i - 4\frac{\dot{\gamma}(t)}{\gamma(t)} + 1 - 2k(t)\right. \\
&\left.+ \varepsilon\left(2n + \left|-\alpha_i + \beta_i\left(1+\frac{\dot{\gamma}(t)}{\gamma(t)}\right) + \frac{\ddot{\gamma}(t)}{\gamma(t)} - \left(\frac{\dot{\gamma}(t)}{\gamma(t)}\right)^2 - \frac{\dot{\gamma}(t)}{\gamma(t)}\right|\right)\right) \\
&\times \overline{\big(\dot{E}_i(t)+E_i(t)\big)}\big(\dot{E}_i(t)+E_i(t)\big) \\
&+ \frac{1}{\varepsilon}\left(\frac{\gamma(t)}{\gamma(t-\delta(t))}\right)^2 \sum_{j=1}^{n} \overline{b_{ji}} b_{ji}\big(l_i^g\big)^2 \overline{E_i\big(t-\delta(t)\big)} E_i\big(t-\delta(t)\big) \\
\leqslant{}& C(t)\hat{V}(t) + Q\hat{V}\big(t-\delta(t)\big),
\end{aligned}
$$

where

$$
\begin{aligned}
C(t) &= \max_{i=1,\dots,n}\big\{A(t), B(t)\big\} \quad \text{and} \quad Q = \max_{i=1,\dots,n} \frac{1}{\varepsilon}\frac{\gamma(t)}{\gamma\big(t-\delta(t)\big)}\big(l_i^g\big)^2 \sum_{j=1}^{n} \overline{b}_{ji} b_{ji}, \\
A(t) &= -2 + \frac{1}{\varepsilon}\left(\left|-\alpha_i + \beta_i\left(1+\frac{\dot{\gamma}(t)}{\gamma(t)}\right) + \frac{\ddot{\gamma}(t)}{\gamma(t)} - \left(\frac{\dot{\gamma}(t)}{\gamma(t)}\right)^2 - 2\frac{\dot{\gamma}(t)}{\gamma(t)}\right|\right. \\
&\quad \left.+ \sum_{j=1}^{n} \overline{a_{ji}} a_{ji}\big(l_i^f\big)^2\right), \\
B(t) &= -2\beta_i - 4\frac{\dot{\gamma}(t)}{\gamma(t)} + 1 - 2k(t) + \varepsilon\left(2n + \left|-\alpha_i + \beta_i\left(1+\frac{\dot{\gamma}(t)}{\gamma(t)}\right) + \frac{\ddot{\gamma}(t)}{\gamma(t)}\right.\right. \\
&\quad \left.\left. - \left(\frac{\dot{\gamma}(t)}{\gamma(t)}\right)^2 - 2\frac{\dot{\gamma}(t)}{\gamma(t)}\right|\right).
\end{aligned}
$$

In accordance with the definition of the γ-type function and the assumptions of Theorem 3, there exist several positive numbers ω_0, ω_1, ω_2, and ω_3 that satisfy the following inequality:

$$
\omega_0 < \frac{\dot{\gamma}(t)}{\gamma(t)} \leqslant \omega_1, \qquad \frac{\ddot{\gamma}(t)}{\gamma(t)} \leqslant \omega_2, \qquad \frac{\dot{\gamma}(t)}{\gamma\big(t-\delta(t)\big)} \leqslant \omega_3.
$$

$\overline{k}$ is a constant that satisfies the following inequality:

$$A(t)+\frac{1}{\varepsilon}\left(\frac{\gamma(t)}{\gamma\bigl(t-\delta(t)\bigr)}\right)^2 \leqslant A(t)+\frac{1}{\varepsilon}(\omega_3)^2Q<0,$$

and

$$\begin{aligned}2\overline{k} &> -2\beta_i-4\omega_0+1+\varepsilon\left(2n+\left|-\alpha_i+\beta_i\left(1+\frac{\dot\gamma(t)}{\gamma(t)}\right)+\frac{\ddot\gamma(t)}{\gamma(t)}\right.\right.\\ &\quad\left.\left.-\left(\frac{\dot\gamma(t)}{\gamma(t)}\right)^2-2\frac{\dot\gamma(t)}{\gamma(t)}\right|\right)+\frac{1}{\varepsilon}(\omega_3)^2Q\\ &= w(t).\end{aligned}$$

$|-\alpha_i+\beta_i(1+\dot\gamma(t)/\gamma(t))+\ddot\gamma(t)/\gamma(t)-(\dot\gamma(t)/\gamma(t))^2-2\dot\gamma(t)/\gamma(t)|$ is bounded because $\dot\gamma(t)/\gamma(t)$ and $\ddot\gamma(t)/\gamma(t)$ are bound. There must exist a finite $\overline{k}$ satisfying the inequality above. Similar to proof of Theorem 2, this proof will be completed in two stages.

Firstly, if there exists a value of $t>\overline{T}$ for which $k(t)=\overline{k}$, it follows that $k(t)\geqslant\overline{k}$ when $t>\overline{T}$,

$$\begin{aligned}&C(t)+\left(\frac{\dot\gamma(t)}{\gamma(t-\delta(t))}\right)^2Q\\ &\quad\leqslant\max_{i=1,\dots,n}\left(A(t)+\frac{1}{\varepsilon}(\omega_3)^2Q,\ w(t)-k(t)+\frac{1}{\varepsilon}(\omega_3)^2Q\right)\\ &\quad\leqslant\max_{i=1,\dots,n}\left(A(t)+\frac{1}{\varepsilon}(\omega_3)^2Q,\ w(t)-\overline{k}+\frac{1}{\varepsilon}(\omega_3)^2Q\right).\end{aligned}$$

By applying the results of Lemma 1, it can be demonstrated that $\lim_{t\to\infty}\hat p(t)=0$, which indicates the existence of a positive number M_1 such that $\hat p(t)<M_1$ when $t>\overline{T}$. Consequently, $\sum_{i=1}^n\overline{E_i(t)}E_i(t)\leqslant M_1$ and $\sum_{i=1}^n\overline{e_i(t)}e_i(t)\leqslant M_1/\gamma^2(t)$.

Secondly, we will discuss the case that $k(t)<\overline{k}$. Let $\hat{\hat p}(t)=\hat p(t)+(k(t)-\overline{k})^2/\varsigma$. The same proof process is employed as in Theorem 2, and we can get that $\lim_{t\to+\infty}\hat{\hat p}(t)=0$, which indicates the existence of a positive number M_2 such that $\hat{\hat p}(t)<M_2$ when $t>0$. Consequently, $\sum_{i=1}^n\overline{E_i(t)}E_i(t)\leqslant M_2$ and $\sum_{i=1}^n\overline{e_i(t)}e_i(t)\leqslant M_2/\gamma^2(t)$. Then let $M=\max\{M_1,M_2\}$. Obviously, $\sum_{i=1}^n\overline{e_i(t)}e_i(t)\leqslant M/\gamma^2(t)$, and thus the proof is completed. □

Remark 8. As demonstrated in papers [24] and [28], adaptive control was found to achieve synchronization. However, it should be noted that no requirements were specified with regard to the synchronization speed. In contradistinction to papers [24] and [28], the present paper proposes a faster adaptive synchronization control strategy.

Corollary 2. *Suppose that the delay is given by $\delta(t)=\sqrt{t}+|\sin t|$ and the γ-type function by $\mathcal{F}(t)=(t+1)^3$. Then the drive-response synchronization of systems* (1) *and* (2) *can be achieved under the following adaptive controller $U_i(t)$. Moreover, the resulting synchronization rate is superior to that obtained in Theorem* 2.

Here

$$U_i(t) = -k(t)\left(1 + \frac{3}{t+1}\right)\bigl(e_i(t) + \dot{e}_i(t)\bigr), \qquad E_i(t) = (t+1)^3 e_i(t),$$

and the adaptive gain $k(t)$ evolves according to

$$\dot{k}(t) = \varsigma \sum_{i=1}^{n} \overline{E_i}(t)E_i(t) + \overline{\bigl(\dot{E}_i(t) + E_i(t)\bigr)}\bigl(\dot{E}_i(t) + E_i(t)\bigr), \quad \varsigma = 1.$$

Proof. The whole proof is divided into two parts.

Firstly, prove that the objective function is an γ-type function.

$$\mathcal{F}(t) = (t+1)^3, \qquad \dot{\mathcal{F}}(t) = 3(t+1)^2, \qquad \ddot{\mathcal{F}}(t) = 6(t+1),$$
$$\frac{\dot{\mathcal{F}}(t)}{\mathcal{F}(t)} = \frac{3}{t+1}, \qquad \frac{\ddot{\mathcal{F}}(t)}{\mathcal{F}(t)} = \frac{6}{(t+1)^2},$$

and

$$0 < \frac{\dot{\mathcal{F}}(t)}{\mathcal{F}(t)} \leqslant 3, \qquad 0 < \frac{\ddot{\mathcal{F}}(t)}{\mathcal{F}(t)} \leqslant 6.$$

Secondly, verify whether the γ-type function satisfies the conditions stated in Theorem 3,

$$\varlimsup_{t\to\infty} \frac{\mathcal{F}(t)}{\mathcal{F}\bigl(t-\delta(t)\bigr)} = \frac{(t+1)^3}{(t+1-\delta(t))^3}, \qquad \varlimsup_{t\to\infty} \frac{\mathcal{F}(t)}{\mathcal{F}\bigl(t-\delta(t)\bigr)} \leqslant 1.$$

Therefore, according to the definition of the γ-type function and Theorem 3, we conclude that systems (1) and (2) are synchronized. Moreover, this form of synchronization is more efficient than that guaranteed by Theorem 2, as will be illustrated in the simulation experiments. $\square$

Remark 9. Corollary 2 highlights that the selection of a γ-type function is dependent on the intrinsic delays embedded within the model. In scenarios involving bounded delays, the option of γ-type function of exponential that converge more rapidly is viable. Conversely, in cases of unbounded delays, utilizing a γ-type function of exponential form may lead to an unstable error system.

4 Numerical simulations

Example 1. Consider two-dimensional QVINNs (1) and (2) as the drive system and the response system, respectively. Their parameters are set as follows: $0.9 < \alpha_1 < 1.1$, $1.9 < \alpha_2 < 2.1$, $0.4 < \beta_1 < 0.6$, $0.6 < \beta_2 < 0.8$,

$$f_1(x) = \tanh(x), \qquad f_2(x) = 0.5\tanh(x),$$
$$g_1(x) = 0.4\tanh(x), \qquad g_2(x) = 0.3\tanh(x), \qquad \delta(t) = \sqrt{t} + |\sin t|,$$

$$a_{2\times 2} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad b_{2\times 2} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

where

$$\begin{aligned} a_{11} &= -0.25 + 0.1\mathcal{I} - 0.5\mathcal{J} + 0.15\mathcal{K}, & b_{11} &= 0.1 - 0.2\mathcal{I} + 0.45\mathcal{J} + 0.2\mathcal{K}, \\ a_{12} &= -0.15 + 0.5\mathcal{I} - 0.2\mathcal{J} - 0.17\mathcal{K}, & b_{12} &= -0.2 + 0.2\mathcal{I} - 0.25\mathcal{J} - 0.15\mathcal{K}, \\ a_{21} &= -0.45 + 0.3\mathcal{I} - 0.3\mathcal{J} + 0.35\mathcal{K}, & b_{21} &= 0.05 - 0.1\mathcal{I} + 0.22\mathcal{J} + 0.1\mathcal{K}, \\ a_{22} &= -0.50 + 0.2\mathcal{I} - 1.0\mathcal{J} + 0.30\mathcal{K}, & b_{22} &= -0.6 + 0.6\mathcal{I} + 0.75\mathcal{J} - 0.45\mathcal{K}. \end{aligned}$$

The initial state of the system can be expressed as

$$\begin{aligned} \kappa_1 &= 1 + 0.1\mathcal{I} - 0.3\mathcal{J} + 0.5\mathcal{K}, & \vartheta_1 &= 0.5 + 0.45\mathcal{I} + 0.3\mathcal{J} + 0.3\mathcal{K}, \\ \kappa_2 &= 1.5 + 0.6\mathcal{I} + 0.2\mathcal{J} + 1\mathcal{K}, & \vartheta_2 &= 0.7 + 0.65\mathcal{I} + 0.5\mathcal{J} + 0.4\mathcal{K}, \\ K_1 &= -0.35 - 0.25\mathcal{I} + 0.3\mathcal{J} - 0.1\mathcal{K}, & \Theta_1 &= -0.1 - 0.33\mathcal{I} - 0.2\mathcal{J} - 0.2\mathcal{K}, \\ K_2 &= -0.85 - 0.75\mathcal{I} - 0.2\mathcal{J} - 0.6\mathcal{K}, & \Theta_2 &= -0.3 - 0.53\mathcal{I} - 0.4\mathcal{J} - 0.3\mathcal{K}. \end{aligned}$$

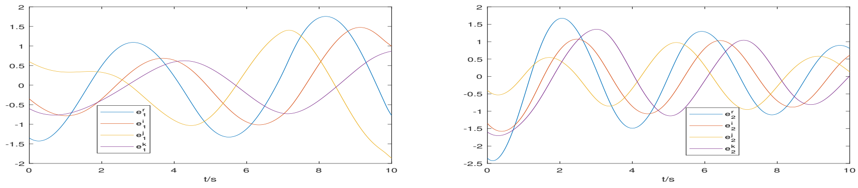
In the absence of a controller, the drive and response systems exhibit different dynamic behaviors in Fig.1. By Theorem 1, the condition k_i can be readily identified, and let us assume that $U_i(t) = -25(e_i(t) + \dot{e}_i(t))$ and $\varepsilon = 1$.

In accordance with the aforementioned assumptions and conditions, with the controller as above, the dynamic behavior of systems (1) and (2) is illustrated in Figs. 2 and 3. The behavior of e_1 and e_2 can be described as in Fig. 4.

In Example 1, by observing Figs. 2, 3, and 4, it can be observed that for any given initial state, the drive systems u_1 and u_2, as well as the response systems v_1 and v_2, always attain the same state with the intervention of the controller. This implies that the error system ultimately converges to 0 and that the drive-response systems ultimately achieve synchronization.

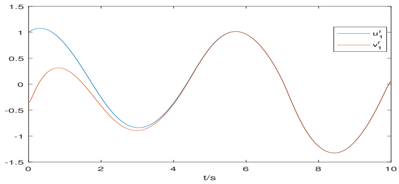
Remark 10. In the numerical simulations of [28], a separable activation function was employed, which differs from the nonseparable function $\tanh(x)$ used in Example 1 of this paper. Our approach is more universally applicable.

Remark 11. The case being considered involves nondifferentiable delays. In fact, when the nondifferentiable part is removed, the system is transformed into a scenario that is differentiable but with an unbounded derivative. At this point, our analysis remains valid.

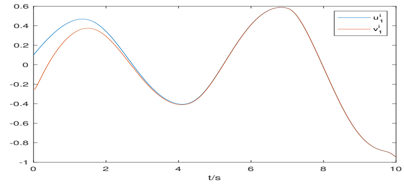


(a) Behavior of e_1^r, e_1^i, e_1^j and e_1^k (b) Behavior of e_2^r, e_2^i, e_2^j and e_2^k

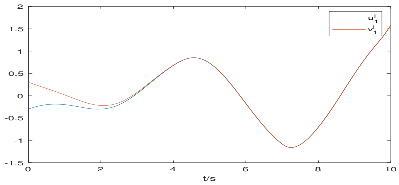
Figure 1. Dynamical behavior of e_1 and e_2 without controller.



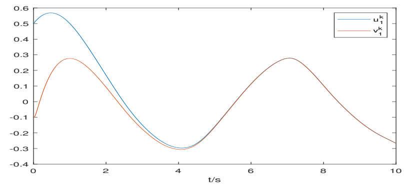
(a) u_1^r and v_1^r



(b) u_1^i and v_1^i

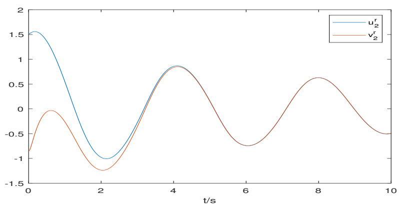


(c) u_1^j and v_1^j

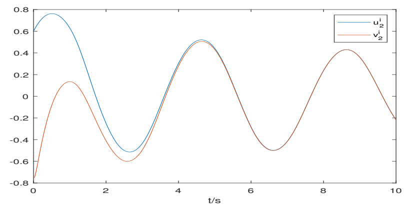


(d) u_1^k and v_1^k

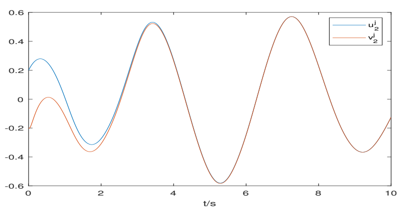
Figure 2. Dynamical behavior of the different components of u_1 and v_1.



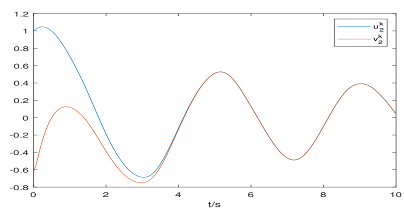
(a) u_2^r and v_2^r



(b) u_2^i and v_2^i

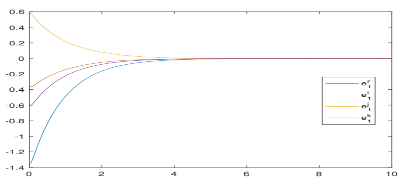


(c) u_2^j and v_2^j

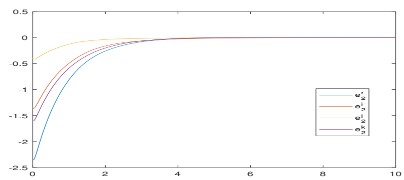


(d) u_2^k and v_2^k

Figure 3. Dynamical behavior of the different components of u_2 and v_2.



(a) Dynamic behavior of e_1

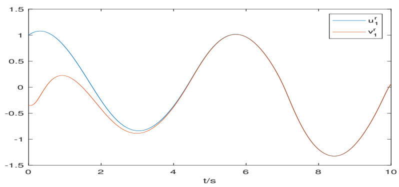


(b) Dynamic behavior of e_2

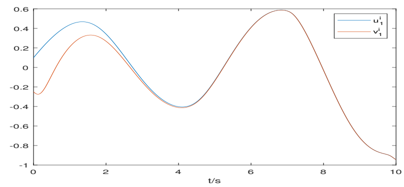
Figure 4. Dynamic behavior of e_1 and e_2.

Example 2. In this example, we continue to analyze the two-dimensional numerical model (from Example 1) in order to study adaptive synchronization between systems (1) and (2). The adaptive feedback controller is chosen as $U_i(t) = -k(t)(e_i(t) + \dot{e}_i(t))$, and its adaptive law for $\dot{k}(t)$ is given by (5) with the parameter $\varsigma = 1$.

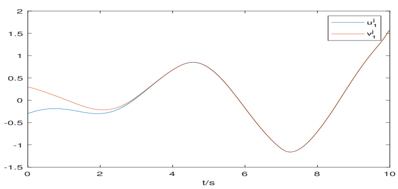
The dynamic performance of systems (1) and (2) under the influence of response controller is depicted in Figs. 5 and 6. The observations of Figs. 5–7 indicate that the drive-response system eventually achieves synchronized as a result of the applied controller. It is noteworthy that the trajectories of the same variables differed between Examples 1 and 2. This occurred because the gain conditions outlined in Theorem 1 were not strict, resulting in a portion of the gain being effectively unused. See Remark 12 for details.



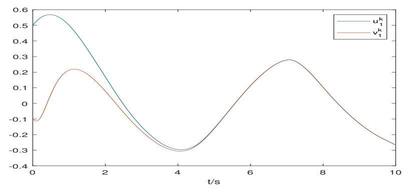
(a) Dynamic behavior of u_1^r and v_1^r



(b) Dynamic behavior of u_1^i and v_1^i

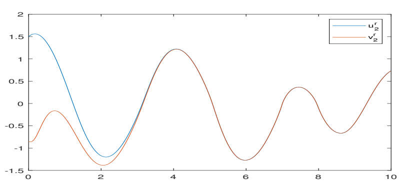


(c) Dynamic behavior of u_1^j and v_1^j

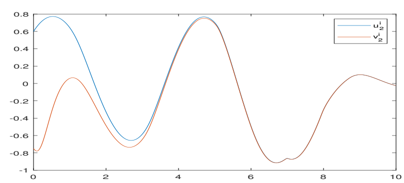


(d) Dynamic behavior of u_1^k and v_1^k

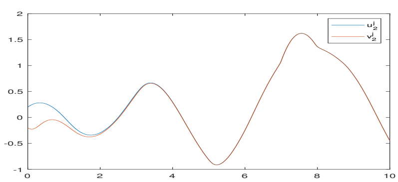
Figure 5. Dynamical behavior of the different components of u_1 and v_1



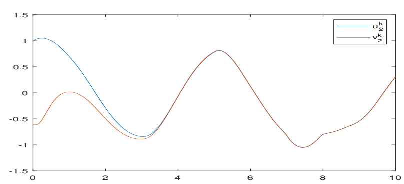
(a) Dynamic behavior of u_2^r and v_2^r



(b) Dynamic behavior of u_2^i and v_2^i

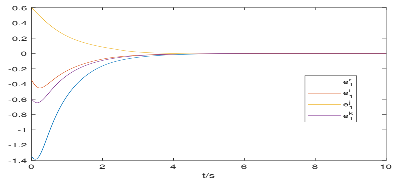


(c) Dynamic behavior of u_2^j and v_2^j

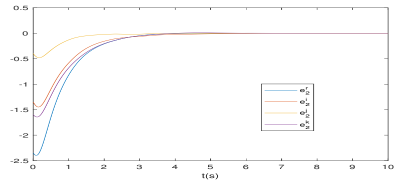


(d) Dynamic behavior of u_2^k and v_2^k

Figure 6. Dynamical behavior of the different components of u_2 and v_2.



(a) Dynamic behavior of e_1

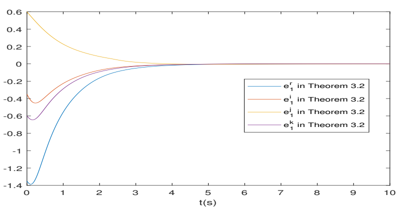


(b) Dynamic behavior of e_2

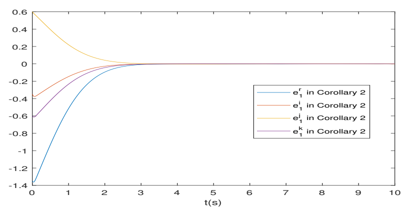
Figure 7. Dynamic behavior of e_1 and e_2.

Example 3. The objective of this numerical experiment is twofold: firstly, to illustrate the superiority of adaptive controllers in reducing gain wastage; secondly, to highlight the accelerating effect of Theorem 3 on the synchronization rate of the system.

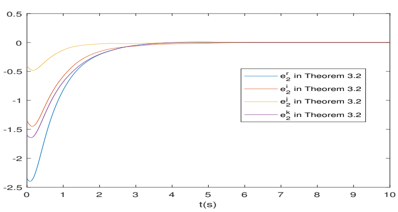
In order to facilitate comparison, the results obtained under the conditions of Corollary 2 are contrasted with the data set provided by Example 2. Due to space limitations, only some of the comparison results are presented in Figs. 8, 9, and 10.



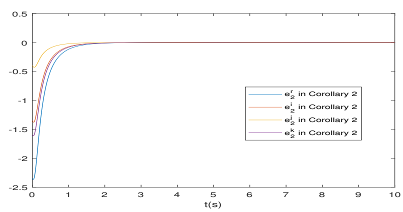
(a) e_1 in Theorem 2



(b) e_1 in Corollary 2

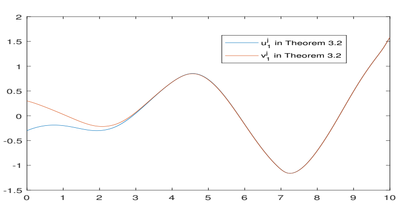


(c) e_2 in Theorem 2

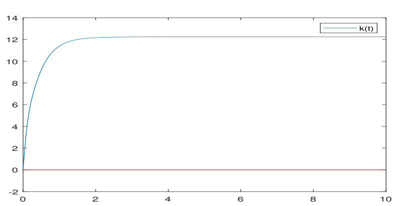


(d) e_2 in Corollary 2

Figure 8. e_1 and e_2 Under Different Controllers (Theorem 2 and Corollary 2).

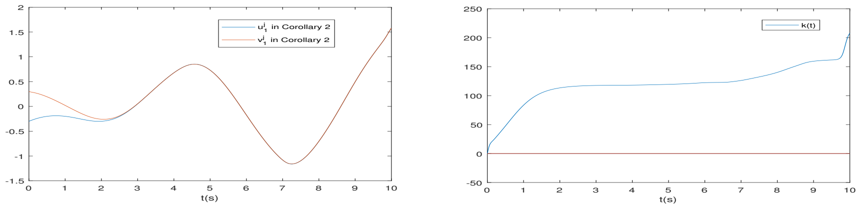


(a) u_1^j and v_1^j



(b) $k(t)$

Figure 9. u_1^j, v_1^j, and $k(t)$ under different controllers (Theorem 2).



(a) u_1^j and v_1^j (b) $k(t)$

Figure 10. u_1^j, v_1^j and $k(t)$ under different controllers (Corollary 2).

The purpose of Example 3 is to demonstrate that the synchronization rate is accelerated by the γ-type function. In Example 3, the time required for the error function to reach 0 is approximately half of that in Example 2.

Remark 12. In fact, it is clear from Figs. 9 and 10 that the minimum gain that would allow the system to achieve synchronization without requiring convergence speed is 12, but the control gain designed in Example 1 is 25, which results in wasted gain.

5 Conclusion

This paper investigates the synchronization of a special class of QVINNs. An innovative non-separation approach is employed, and a novel class of positive-definite functions is introduced to replace the Lyapunov functions in the analytical process. This novel approach, uncommon in QVINN research, avoids the numerous constraints typically imposed on the delay term. Utilizing this strategy, a unique controller is designed that excludes the sign function and delay terms, only one integral function is used in the calculation of the control gain. Furthermore, to enhance synchronization efficiency, γ-type function is introduced into the error system. While this technique accelerates synchronization, identifying a suitable γ-type function poses challenges, and addressing the limitations of this function will be a key focus of future research.

Author contributions. All Authors (X.Y., T.P., J.X., Z.T., Z.L., and L. W.) have contributed as follows: methodology, T.P.; formal analysis, X.Y.; software, Z.T.; validation, X.Y.; writing – original draft preparation, X.Y.; writing – review & editing, J.X., Z.L., and L.W.. All authors have read and approved the published version of the manuscript.

Conflicts of interest. The authors declare that there is no known competing financial interests to influence the work in this paper.

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