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Synchronization of quaternion-valued BAM neural networks with time-varying delays: A matrix measure approach*

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Abstract. This paper studies the global asymptotic synchronization (GAS) and global exponential synchronization (GES) problems of a class of quaternion-valued bidirectional associative memory neural networks (QVBAMNNs) with time-varying delays. Based on the matrix measure approach and Halanay inequality, sufficient conditions for GAS of unbounded time-varying delay system and GES of bounded time-varying delay system were established. Different from existing methods, this approach does not require the construction of Lyapunov functions and obtains the criteria for system synchronization expressed in terms of norms and measures. Finally, the validity of the proposed theoretical results and the feasibility of the control strategy are verified through numerical simulations.

Keywords: GAS, GES, QVBAMNNs, matrix measure approach, time-varying delays.

1 Introduction

Bidirectional associative memory (BAM), as a crucial associative memory model, exhibits significant value in pattern recognition, image processing, and data compression [9, 16, 22–24]. Compared with traditional unidirectional networks, BAM achieves pattern

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storage and retrieval through a two-layer feedback neural network structure. The core feature of this algorithm lies in establishing a bidirectional mapping relationship between the input pattern and the output pattern. Even in the presence of noise interference or incomplete input data, the system can still accurately restore the original storage pattern. Moreover, in practical applications, BAM neural networks are often affected by time delays, which may arise from signal transmission delays, communication delays between neurons, or hardware limitations [18, 19, 26, 27]. This time-delay characteristic may not only trigger oscillatory behaviors in the nervous system but also exert a significant influence on its information processing capabilities. Consequently, the dynamical analysis of time-delay BAM neural networks has received extensive attention.

Quaternions, as hypercomplex numbers, possess a unique noncommutative algebraic structure. By introducing quaternion algebra into neural networks, which can more effectively process multichannel data such as three-dimensional spatial rotations and color images. This overcomes the limitations of traditional real-valued neural networks (RVNNs) in representing high-dimensional features and modeling intrinsic geometric relationships. Studies have shown that QVNN has significant advantages in computer vision, three-dimensional pose estimation, and signal processing [4, 5, 17]. By integrating quaternion algebra into the BAM network, the QVBAMNNs model not only retains the bidirectional associative memory capability but also significantly enhances its capacity for representing high-dimensional data, making it well-suited for multimodal information processing. However, due to the complexity of the double-layer structure of the BAM model, the study of the dynamic characteristics of QVBAMNNs faces significant challenges. Current research primarily focuses on stability analysis [14,20,29,30], while research on dynamic behavior synchronization control is relatively scarce and needs further improvement.

Synchronization, as a fundamental characteristic of complex dynamical systems, plays a vital role in neural network studies, reflecting both cooperative neuron interactions and system stability. The synchronization process involves complex nonlinear dynamics that increase computational demands and pose theoretical challenges. When accounting for practical factors like time-varying delays and stochastic disturbances, synchronization behaviors become even more intricate. Consequently, investigating synchronization phenomenon in time-delay neural networks is of great significance. Recent studies have explored various synchronization types including exponential synchronization [2, 25], projected synchronization [12], and finite-time synchronization [3, 15] through intermittent, event-triggered, and adaptive control methods.

In recent years, the matrix measure approach has gained widespread popularity in the analysis of neural network dynamics due to its unique advantages [7, 8, 10]. The matrix measure approach has the following advantages: (i) eliminating complex Lyapunov function construction and intricate integration calculations; (ii) providing more accurate, less conservative results through sign-sensitivity compared to matrix norms; (iii) offering flexible numerical ranges for processing mixed weight matrices. This method has been successfully applied to various synchronization problems, including quasisynchronization in delayed coupled networks [7], projection quasisynchronization in complex-valued networks with proportional delays [10], and synchronization in fractional-order delayed networks [8]. To the best of our knowledge, the synchronization problem of QVBAMNNs with time-varying delays has not been explored through the matrix measure approach, thus this work is considered both necessary and valuable.

Based on the aforementioned considerations, this paper investigates GAS and GES for a class of QVBAMNNs with time-varying delays. The main contributions of this article are as follows:

(i) Using quaternion decomposition, the system is transformed into an equivalent real-valued form, which enables the classical matrix measure and Halanay inequality to be reapplied within the real number domain. Sufficient conditions are established for GAS with unbounded delays and GES with bounded delays.

(ii) A global synchronization method without the need to construct a Lyapunov function was adopted. By analyzing the error system and designing feedback controllers, concise synchronization criteria are derived, avoiding complex quaternion-domain constructions and simplifying the analysis.

(iii) Based on the matrix measure approach, the controller gain matrix does not need to be symmetric, nor does it need to be positive definite or negative definite. The final obtained synchronization conditions are presented in the form of norms and metrics, and are independent of the delay, which is more verifiable and concise than those from traditional methods.

The remainder of the paper is organized as follows. Section 2 presents the necessary lemmas, assumptions, and model descriptions. Section 3 establishes sufficient conditions for global synchronization of QVBAMNNs using the matrix measure and Halanay inequality. Section 4 provides two numerical examples supporting the theoretical results. Section 5 concludes the paper.

Notations. This paper uses $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$, $\mathbb{Q}$, $\mathbb{Q}^n$, $\mathbb{Q}^{n\times n}$ to represent the real numbers, n-dimensional real vectors, $n\times n$ real matrices, quaternions, n-dimensional quaternion vectors, and $n\times n$ quaternion matrices, respectively. Furthermore, X^{T} represents the transpose of matrix X.

2 Preliminaries

Quaternions are a type of hypercomplex numbers that generalize complex numbers. A quaternion $\aleph\in\mathbb{Q}$ can be expressed as $\aleph=\aleph^R+\aleph^I\mathrm{i}+\aleph^J\mathrm{j}+\aleph^K\mathrm{k}$, where $\aleph^R,\aleph^I,\aleph^J,\aleph^K\in\mathbb{R}$, the imaginary units i, j, and k comply with Hamilton's multiplication rules, which are defined as follows: $\mathrm{i}^2=\mathrm{j}^2=\mathrm{k}^2=-1$, $\mathrm{jk}=-\mathrm{kj}=\mathrm{i}$, $\mathrm{ki}=-\mathrm{ik}=\mathrm{j}$, $\mathrm{ij}=-\mathrm{ji}=\mathrm{k}$. The modulus of the quaternion $\aleph$: $|\aleph|=\sqrt{\aleph\aleph^*}=\sqrt{(\aleph^R)^2+(\aleph^I)^2+(\aleph^J)^2+(\aleph^K)^2}$. For the quaternions $\aleph=\aleph^R+\aleph^I\mathrm{i}+\aleph^J\mathrm{j}+\aleph^K\mathrm{k}$ and $\Im=\Im^R+\Im^I\mathrm{i}+\Im^J\mathrm{j}+\Im^K\mathrm{k}$, their addition rule is as follows:

$$\aleph+\Im=\left(\aleph^R+\Im^R\right)+\left(\aleph^I+\Im^I\right)\mathrm{i}+\left(\aleph^J+\Im^J\right)\mathrm{j}+\left(\aleph^K+\Im^K\right)\mathrm{k}.$$

The multiplication between them is defined by Hamilton's multiplication rules as

$$\begin{aligned}\aleph\Im={}&\left(\aleph^R\Im^R-\aleph^I\Im^I-\aleph^J\Im^J-\aleph^K\Im^K\right)+\left(\aleph^R\Im^I+\aleph^I\Im^R+\aleph^J\Im^K-\aleph^K\Im^J\right)\mathrm{i}\\&+\left(\aleph^R\Im^J-\aleph^I\Im^K+\aleph^J\Im^R+\aleph^K\Im^I\right)\mathrm{j}+\left(\aleph^R\Im^K+\aleph^I\Im^J-\aleph^J\Im^I+\aleph^K\Im^R\right)\mathrm{k}.\end{aligned}$$

In this paper, we consider a class of QVBAMNNs described by the following delay differential equations:

$$\begin{aligned}\dot{x}_i(t) &= -c_i x_i(t) + \sum_{j=1}^{n} n_{ij} f_j\big(y_j(t)\big) + \sum_{j=1}^{n} m_{ij} f_j\big(y_j\big(t-\delta_j(t)\big)\big)\\ &\quad + h_i, \quad i = 1,2,\dots,m,\\ \dot{y}_j(t) &= -d_j y_j(t) + \sum_{i=1}^{m} q_{ji} g_i\big(x_i(t)\big) + \sum_{i=1}^{m} p_{ji} g_i\big(x_i\big(t-\tau_i(t)\big)\big)\\ &\quad + z_j, \quad j = 1,2,\dots,n.\end{aligned} \tag{1}$$

System (1) can be rewritten in the form of the following vector matrix:

$$\begin{aligned}\dot{x}(t) &= -Cx(t) + Nf\big(y(t)\big) + Mf\big(y\big(t-\delta(t)\big)\big) + H,\\ \dot{y}(t) &= -Dy(t) + Qg\big(x(t)\big) + Pg\big(x\big(t-\tau(t)\big)\big) + Z.\end{aligned} \tag{2}$$

Model (1) is made up of two neural fields F_x and F_y, where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^{\mathrm{T}}$ and $y(t) = (y_1(t), y_2(t), \dots, y_m(t))^{\mathrm{T}}$ are the state vectors in F_x and the jth neuron in F_y, respectively. $C = \mathrm{diag}\{c_1, c_2, \dots, c_n\}$, $D = diag\{d_1, d_2, \dots, d_m\}$ are self-connection positive definite diagonal weight matrices, $N = (n_{ij})_{n\times m} \in \mathbb{Q}^{n\times m}$, $M = (m_{ij})_{n\times m} \in \mathbb{Q}^{n\times m}$, $Q = (q_{ji})_{m\times n} \in \mathbb{Q}^{m\times n}$, $P = (p_{ji})_{m\times n} \in \mathbb{Q}^{m\times n}$ are the interconnection weight matrices. $f(y(t)) = (f_1(y_1(t)), f_2(y_2(t)), \dots, f_m(y_m(t)))^{\mathrm{T}}$ and $g(x(t)) = (g_1(x_1(t)), g_2(x_2(t)), \dots, g_n(x_n(t)))^{\mathrm{T}}$ are activation functions. $\delta(t)$ and $\tau(t)$ are the corresponding time-varying delays. H and Z are the external input vector.

The initial value condition for system (1) is

$$\begin{aligned}x_i(s_1) &= \phi_i(s_1), \quad s_1 \in \big[t_0 - \tau(t),\, t_0\big],\ i = 1,2,\dots,m,\\ \phi_i(s_1) &= \phi_i^R(s_1) + \phi_i^I(s_1)\mathrm{i} + \phi_i^J(s_1)\mathrm{j} + \phi_i^K(s_1)\mathrm{k},\\ y_j(s_2) &= \varphi_j(s_2), \quad s_2 \in \big[t_0 - \delta(t),\, t_0\big],\ j = 1,2,\dots,n,\\ \varphi_j(s_2) &= \varphi_j^R(s_2) + \varphi_j^I(s_2)\mathrm{i} + \varphi_j^J(s_2)\mathrm{j} + \varphi_j^K(s_2)\mathrm{k}.\end{aligned}$$

The state vector in (1) consists of quaternions denoted respectively

$$\begin{aligned}x_i(t) &= x_i^R(t) + x_i^I(t)\mathrm{i} + x_i^J(t)\mathrm{j} + x_i^K(t)\mathrm{k},\\ y_j(t) &= y_j^R(t) + y_j^I(t)\mathrm{i} + y_j^J(t)\mathrm{j} + y_j^K(t)\mathrm{k}.\end{aligned}$$

Similarly, the activation functions $f(\cdot)$ and $g(\cdot)$ can be expressed as respectively

$$\begin{aligned}f_j(t) &= f_j^R(t) + f_j^I(t)i + f_j^J(t)j + f_j^K(t)k,\\ g_i(t) &= g_i^R(t) + g_i^I(t)i + g_i^J(t)j + g_i^K(t)k,\end{aligned}$$

where $f^\theta(\cdot) = (f_1^\theta(\cdot), f_2^\theta(\cdot), \dots, f_n^\theta(\cdot))^{\mathrm{T}}$, $g^\theta(\cdot) = (g_1^\theta(\cdot), g_2^\theta(\cdot), \dots, g_m^\theta(\cdot))^{\mathrm{T}}$, $f_j^\theta(\cdot)\colon \mathbb{R} \to \mathbb{R}$, $g_i^\theta(\cdot) : \mathbb{R} \to \mathbb{R}$ for $i = 1,2,\dots,m$, $j = 1,2,\dots,n$, $\theta = R, I, J, K$.

The response system of QVBAMNN (2) is taken as

$$\begin{aligned}\dot{u}(t) &= -Cu(t) + Nf\big(v(t)\big) + Mf\big(v\big(t-\delta(t)\big)\big) + H + \lambda(t),\\ \dot{v}(t) &= -Dv(t) + Qg\big(u(t)\big) + Pg\big(u\big(t-\tau(t)\big)\big) + Z + \pi(t),\end{aligned} \tag{3}$$

where $\lambda(t)$, $\pi(t)$ are the control input to be designed. The initial conditions of are given by

$$\begin{aligned}&u(s) = \tilde{\phi}(s), \quad s \in \big[t_0 - \tau(t),\, t_0\big],\\ &\tilde{\phi}(s) = \tilde{\phi}^R(s) + \tilde{\phi}^I(s)\mathrm{i} + \tilde{\phi}^J(s)\mathrm{j} + \tilde{\phi}^K(s)\mathrm{k},\\ &y(s) = \tilde{\varphi}(s), \quad s \in \big[t_0 - \delta(t),\, t_0\big],\\ &\tilde{\varphi}(s) = \tilde{\varphi}^R(s) + \tilde{\varphi}^I(s)\mathrm{i} + \tilde{\varphi}^J(s)\mathrm{j} + \tilde{\varphi}^K(s)\mathrm{k}.\end{aligned}$$

Here the synchronization errors between the drive system (2) and the response system (3) are defined as

$$\varepsilon(t) = x(t) - u(t), \qquad e(t) = y(t) - v(t).$$

Then we obtain the error dynamics of (2) and the response system (3) with the following vector matrix form:

$$\begin{aligned}\dot{\varepsilon}(t) &= -C\varepsilon(t) + Nf\big(e(t)\big) + Mf\big(e\big(t-\delta(t)\big)\big) + \lambda(t),\\ \dot{e}(t) &= -De(t) + Qg\big(\varepsilon(t)\big) + Pg\big(\varepsilon\big(t-\tau(t)\big)\big) + \pi(t),\end{aligned} \tag{4}$$

where $f(e(t)) = f(y(t)) - f(v(t))$, $f(e(t-\delta(t))) = f(y(t-\delta(t))) - f(v(t-\delta(t)))$, $g(\varepsilon(t)) = g(x(t)) - g(u(t))$, and $g(\varepsilon(t-\tau(t))) = g(x(t-\tau(t))) - g(u(t-\tau(t)))$.

Using Hamilton's principle, the QVBAMNN (4) is decomposed into the following eight real-valued systems:

$$\begin{aligned}\dot{\varepsilon}^R(t) = {}&-C\varepsilon^R(t)\\ &+ \big(N^R f^R\big(e^R(t)\big) - N^I f^I\big(e^I(t)\big) - N^J f^J\big(e^J(t)\big) - N^K f^K\big(e^K(t)\big)\big)\\ &+ \big(M^R f^R\big(e^R\big(t-\delta(t)\big)\big) - M^I f^I\big(e^I\big(t-\delta(t)\big)\big) - M^J f^J\big(e^J\big(t-\delta(t)\big)\big)\\ &- M^K f^K\big(e^K\big(t-\delta(t)\big)\big)\big) + \lambda^R(t),\end{aligned} \tag{5}$$

$$\begin{aligned}\dot{\varepsilon}^I(t) = {}&-C\varepsilon^I(t)\\ &+ \big(N^R f^I\big(e^I(t)\big) + N^I f^R\big(e^R(t)\big) + N^J f^K\big(e^K(t)\big) - N^K f^J\big(e^J(t)\big)\big)\\ &+ \big(M^R f^I\big(e^I\big(t-\delta(t)\big)\big) + M^I f^R\big(e^R\big(t-\delta(t)\big)\big) + M^J f^K\big(e^K\big(t-\delta(t)\big)\big)\\ &- M^K f^J\big(e^J\big(t-\delta(t)\big)\big)\big) + \lambda^I(t),\end{aligned} \tag{6}$$

$$\begin{aligned}\dot{\varepsilon}^J(t) = {}&-C\varepsilon^J(t)\\ &+ \big(N^R f^J\big(e^J(t)\big) + N^J f^R\big(e^R(t)\big) - N^I f^K\big(e^K(t)\big) + N^K f^I\big(e^I(t)\big)\big)\\ &+ \big(M^R f^J\big(e^J\big(t-\delta(t)\big)\big) + M^J f^R\big(e^R\big(t-\delta(t)\big)\big) - M^I f^K\big(e^K\big(t-\delta(t)\big)\big)\\ &+ M^K f^I\big(e^I\big(t-\delta(t)\big)\big)\big) + \lambda^J(t),\end{aligned} \tag{7}$$

$$\begin{aligned}
\dot{\varepsilon}^K(t) &= -C\varepsilon^K(t) \\
&\quad + \big(N^R f^K\big(e^K(t)\big) + N^K f^R\big(e^R(t)\big) - N^J f^I\big(e^I(t)\big) + N^I f^J\big(e^J(t)\big)\big) \\
&\quad + \big(M^R f^K\big(e^K\big(t-\delta(t)\big)\big) + M^K f^R\big(e^R\big(t-\delta(t)\big)\big) - M^J f^I\big(e^I\big(t-\delta(t)\big)\big) \\
&\quad + M^I f^J\big(e^J\big(t-\delta(t)\big)\big)\big) + \lambda^K(t), \qquad (8) \\
\dot{e}^R(t) &= -De^R(t) \\
&\quad + \big(Q^R g^R\big(\varepsilon^R(t)\big) - Q^I g^I\big(\varepsilon^I(t)\big) - Q^J g^J\big(\varepsilon^J(t)\big) - Q^K g^K\big(\varepsilon^K(t)\big)\big) \\
&\quad + \big(P^R g^R\big(\varepsilon^R\big(t-\tau(t)\big)\big) - P^I g^I\big(\varepsilon^I\big(t-\tau(t)\big)\big) - P^J g^J\big(\varepsilon^J\big(t-\tau(t)\big)\big) \\
&\quad - P^K g^K\big(\varepsilon^K\big(t-\tau(t)\big)\big)\big) + \pi^R(t), \qquad (9) \\
\dot{e}^I(t) &= -De^I(t) \\
&\quad + \big(Q^R g^I\big(\varepsilon^I(t)\big) + Q^I g^R\big(\varepsilon^R(t)\big) + Q^J g^K\big(\varepsilon^K(t)\big) - Q^K g^J\big(\varepsilon^J(t)\big)\big) \\
&\quad + \big(P^R g^I\big(\varepsilon^I\big(t-\tau(t)\big)\big) + P^I g^R\big(\varepsilon^R\big(t-\tau(t)\big)\big) + P^J g^K\big(\varepsilon^K\big(t-\tau(t)\big)\big) \\
&\quad - P^K g^J\big(\varepsilon^J\big(t-\tau(t)\big)\big)\big) + \pi^I(t), \qquad (10) \\
\dot{e}^J(t) &= -De^J(t) \\
&\quad + \big(Q^R g^J\big(\varepsilon^J(t)\big) + Q^J g^R\big(\varepsilon^R(t)\big) - Q^I g^K\big(\varepsilon^K(t)\big) + Q^K g^I\big(\varepsilon^I(t)\big)\big) \\
&\quad + \big(P^R g^J\big(\varepsilon^J\big(t-\tau(t)\big)\big) + P^J g^R\big(\varepsilon^R\big(t-\tau(t)\big)\big) - P^I g^K\big(\varepsilon^K\big(t-\tau(t)\big)\big) \\
&\quad + P^K g^I\big(\varepsilon^I\big(t-\tau(t)\big)\big)\big) + \pi^J(t), \qquad (11) \\
\dot{e}^K(t) &= -De^K(t) \\
&\quad + \big(Q^R g^K\big(\varepsilon^K(t)\big) + Q^K g^R\big(\varepsilon^R(t)\big) - Q^J g^I\big(\varepsilon^I(t)\big) + Q^I g^J\big(\varepsilon^J(t)\big)\big) \\
&\quad + \big(P^R g^K\big(\varepsilon^K\big(t-\tau(t)\big)\big) + P^K g^R\big(\varepsilon^R\big(t-\tau(t)\big)\big) - P^J g^I\big(\varepsilon^I\big(t-\tau(t)\big)\big) \\
&\quad + P^I g^J\big(\varepsilon^J\big(t-\tau(t)\big)\big)\big) + \pi^K(t). \qquad (12)
\end{aligned}$$

where $\varepsilon^\theta(t) = x^\theta(t) - u^\theta(t)$, $e^\theta(t) = y^\theta(t) - v^\theta(t)$, $f^\theta(e(t)) = f^\theta(x(t)) - f^\theta(u(t))$, $g^\theta(\varepsilon(t)) = g^\theta(y(t)) - g^\theta(v(t))$, $f^\theta(e(t-\delta(t))) = f^\theta(x(t-\delta(t))) - f^\theta(u(t-\delta(t)))$, $g^\theta(\varepsilon(t-\tau(t))) = g^\theta(y(t-\tau(t))) - g^\theta(v(t-\tau(t)))$ ($\theta = R, I, J, K$).

Assumption 1. The activation functions $f_j^\theta(\cdot)$, $g_i^\theta(\cdot)$ ($i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$, $\theta = R, I, J, K$), where $f_j(x) = f_j^R(x^R) + f_j^I(x^I)\mathrm{i} + f_j^J(x^J)\mathrm{j} + f_j^K(x^K)\mathrm{k}$, $g_i(x) = g_i^R(x^R) + g_i^I(x^I)\mathrm{i} + g_i^J(x^J)\mathrm{j} + g_i^K(x^K)\mathrm{k}$, satisfy the following Lipschitz criteria for every $a, b \in \mathbb{R}$:

$$\big\|f_j^\theta(a) - f_j^\theta(b)\big\|_\varphi \leqslant l_j^\theta \|a-b\|_\varphi, \qquad \big\|g_i^\theta(a) - g_i^\theta(b)\big\|_\varphi \leqslant \eta_i^\theta \|a-b\|_\varphi,$$

where $l_j^\theta, \eta_i^\theta > 0$ are real constants, and $\varphi = 1, 2, \infty$.

Assumption 2. Assume that the time delays in system (2) satisfy $0 \leqslant \delta(t), \tau(t)$ and the following conditions:

$$\lim_{t\to\infty} \big(t - \delta(t)\big) = +\infty, \qquad \lim_{t\to\infty} \big(t - \tau(t)\big) = +\infty.$$

Assumption 3. Assume that the time delays in system (2) satisfy the following conditions:

$$0 \leqslant \delta(t) \leqslant \delta, \qquad 0 \leqslant \tau(t) \leqslant \tau.$$

Definition 1. The drive system (2) and the response system (3) are said to be globally asymptotically synchronized if

$$\lim_{t\to\infty}\left(\left\|x^\theta(t)-u^\theta(t)\right\|_\varphi+\left\|y^\theta(t)-v^\theta(t)\right\|_\varphi\right)=0,$$

where $\theta = R, I, J, K$ and $\varphi = 1, 2, \infty$.

Definition 2. The drive system (2) and the response system (3) are said to be exponentially synchronized if there exist constants $k > 1$ and $\lambda > 0$ such that

$$\begin{aligned}&\sum_{\theta\in\{R,I,J,K\}}\left(\left\|x^\theta(t)-u^\theta(t)\right\|_\varphi+\left\|y^\theta(t)-v^\theta(t)\right\|_\varphi\right)\\&\quad\leqslant k\sup_{t_0-\sigma\leqslant t\leqslant t_0}\left(\sum_{\theta\in\{R,I,J,K\}}\left(\left\|\phi^\theta(s)-\tilde{\phi}^\theta(s)\right\|_\varphi+\left\|\varphi^\theta(s)-\tilde{\varphi}^\theta(s)\right\|_\varphi\right)\right)\mathrm{e}^{-\lambda(t-t_0)}\end{aligned}$$

under the condition $t \geqslant t_0$, where $\varphi = 1, 2, \infty$.

Definition 3. (See [13].) The matrix measure of a real square matrix $A = (a_{ij})_{n\times n}$ is as follows:

$$\mu_p(A)=\lim_{\alpha\to 0^+}\frac{\|I+\alpha A\|_\varphi-1}{\alpha},$$

where $\|\cdot\|_\varphi$ is an induced matrix norm on $\mathbb{R}^{n\times n}$, I is the identity matrix, and $\varphi = 1, 2, \infty$.

When the matrix norm $\|A\|_1 = \max_j \sum_{i=1}^n |a_{ij}|$, $\|A\|_2 = \sqrt{\lambda_{\max}(A^\mathrm{T}A)}$, $\|A\|_\infty = \max_i \sum_{j=1}^n |a_{ij}|$, we can obtain the matrix measure

$$\mu_1(A)=\left\{\max_j a_{jj}+\sum_{i=1,\,i\neq j}^n|a_{ij}|\right\},\qquad \mu_2(A)=\frac{1}{2}\lambda_{\max}\left(A^\mathrm{T}+A\right),$$
$$\mu_\infty(A)=\max_i\left\{a_{ii}+\sum_{j=1,\,j\neq i}^n a_{ij}|\right\}.$$

Remark 1. A comparison of the definitions and properties of matrix norms and matrix measures reveals significant differences between the two. Matrix norms always yield nonnegative values and satisfy $\|-A\|_\varphi = \|A\|_\varphi$, meaning they are insensitive to the sign of a matrix. In contrast, the matrix measure $\mu_\omega(A)$ can take on either positive or negative values, and generally satisfies $\mu_\omega(A) \neq \mu_\omega(-A)$. This sign sensitivity endows the matrix measure with stronger adaptability in the synchronization analysis of systems, enabling it to yield more precise synchronization criteria.

The following lemmas are necessary for the main results.

Lemma 1. (*See* [11].) *Let* $\xi(t) \geqslant 0$ *be a continuous function defined for* $t \in \mathbb{R}$ *that satisfies*

$$\dot{\xi}(t) \leqslant -\alpha\xi(t) + \beta \sup_{t-\delta(t)\leqslant s\leqslant t} \xi(s), \quad t \geqslant t_0,$$

where α *and* β *are constants satisfying* $\alpha > \beta > 0$, *and the time-varying* $\delta(t) \geqslant 0$ *satisfies* $\lim_{t\to\infty}(t - \delta(t)) = +\infty$. *Then*

$$\lim_{t\to\infty} \xi(t) = 0.$$

Lemma 2. (*See* [11].) *Consider the nonnegative continuous function* $\Phi(t)$, *which is defined on* $[t_0 - \sigma(t), +\infty]$. *Assume that there exist constants* $k_1 > k_2 > 0$ *such that for* $t \geqslant t_0$,

$$D^+\bigl(\Phi(t)\bigr) \leqslant -k_1\Phi(t) + k_2\bar{\Phi}(t),$$

where $\bar{\Phi}(t) \triangleq \sup_{t-t_0\leqslant m\leqslant t} \Phi(m)$. *Then the inequality*

$$\Phi(t) \leqslant \bar{\Phi}(t)\mathrm{e}^{-q(t-t_0)}, \quad t \geqslant t_0,$$

holds. Here $q > 0$ *denotes the unique solution of* $q = k_1 - k_2\mathrm{e}^{q\sigma}$, *and* $D^+\Phi(t)$ *is defined as*

$$D^+\Phi(t) = \overline{\lim_{\alpha\to 0}} \frac{\Phi(t+\alpha) - \Phi(t)}{\alpha}.$$

Lemma 3. (*See* [6].) *The matrix measure* $\mu_\varphi(\cdot)$ *defined in Definition* 3 *has the following properties*:

(i) $-\|A\|_\varphi \leqslant \mu_\varphi(A) \leqslant \|A\|_\varphi$ *for all* $A \in \mathbb{R}^{n\times n}$;
(ii) $\mu_\varphi(\alpha A) = \alpha\mu_\varphi(A)$ *for all* $\alpha > 0$ *and all* $A \in \mathbb{R}^{n\times n}$;
(iii) $\mu_\varphi(A + B) \leqslant \mu_\varphi(A) + \mu_\varphi(B)$ *for all* $A, B \in \mathbb{R}^{n\times n}$.

3 Main results

The control input vectors are designed as

$$\lambda^\theta(t) = \omega\varepsilon^\theta(t), \quad \pi^\theta(t) = \tilde{\omega}e^\theta(t), \quad \theta = R, I, J, K, \tag{13}$$

where $\lambda^\theta(t) = [\lambda_1^\theta(t), \lambda_2^\theta(t), \ldots, \lambda_i^\theta(t)]^{\mathrm{T}}$, $\pi^\theta(t) = [\pi_1^\theta(t), \pi_2^\theta(t), \ldots, \pi_j^\theta(t)]^{\mathrm{T}}$,

$$\omega = \begin{pmatrix} \omega_{11} & \cdots & \omega_{1i} \\ \vdots & \ddots & \vdots \\ \omega_{i1} & \cdots & \omega_{ii} \end{pmatrix} \quad \text{and} \quad \tilde{\omega} = \begin{pmatrix} \tilde{\omega}_{11} & \cdots & \tilde{\omega}_{1j} \\ \vdots & \ddots & \vdots \\ \tilde{\omega}_{j1} & \cdots & \tilde{\omega}_{jj} \end{pmatrix}$$

are controller gain matrices.

Theorem 1. *Under Assumptions* 1 *and* 2, *if controller gain matrices satisfy*

$$0 \leqslant 2\max(\bar{L}\tilde{M}, \bar{\Gamma}\tilde{P}) < -\max\bigl(\mu_\varphi(\omega - C) + \bar{\Gamma}\tilde{Q},\ \mu_\varphi(\tilde{\omega} - D) + \bar{L}\tilde{N}\bigr), \tag{14}$$

then the QVBAMNN (3) *will be GAS with the response system* (2).

Proof. Let us define

$$L\big(\varepsilon(t), e(t)\big) = L\big(\varepsilon(t)\big) + L\big(e(t)\big),$$

where

$$\begin{aligned} L\big(\varepsilon(t)\big) &= \big\|\varepsilon^R(t)\big\|_\varphi + \big\|\varepsilon^I(t)\big\|_\varphi + \big\|\varepsilon^J(t)\big\|_\varphi + \big\|\varepsilon^K(t)\big\|_\varphi, \\ L\big(e(t)\big) &= \big\|e^R(t)\big\|_\varphi + \big\|e^I(t)\big\|_\varphi + \big\|e^J(t)\big\|_\varphi + \big\|e^K(t)\big\|_\varphi. \end{aligned}$$

Based on the Dini derivative and the Taylor theorem of Peano's reminder, we can calculate the derivative of $L(\varepsilon(t), e(t))$ as follows:

$$\begin{aligned} D^+L\big(\varepsilon(t), e(t)\big) = \overline{\lim_{\alpha\to 0^+}} &\bigg[\frac{\|\varepsilon^R(t+\alpha)\|_\varphi + \|\varepsilon^I(t+\alpha)\|_\varphi + \|\varepsilon^J(t+\alpha)\|_\varphi + \|\varepsilon^K(t+\alpha)\|_\varphi}{\alpha} \\ &+ \frac{\|e^R(t+\alpha)\|_\varphi + \|e^I(t+\alpha)\|_\varphi + \|e^J(t+\alpha)\|_\varphi + \|e^K(t+\alpha)\|_\varphi}{\alpha} \\ &- \frac{\|\varepsilon^R(t)\|_\varphi + \|\varepsilon^I(t)\|_\varphi + \|\varepsilon^J(t)\|_\varphi + \|\varepsilon^K(t)\|_\varphi}{\alpha} \\ &- \frac{\|e^R(t)\|_\varphi + \|e^I(t)\|_\varphi + \|e^J(t)\|_\varphi + \|e^K(t)\|_\varphi}{\alpha}\bigg] \\ = \overline{\lim_{\alpha\to 0^+}} &\bigg[\frac{\|\varepsilon^R(t) + \alpha\dot\varepsilon^R(t) + \phi(\alpha)\|_\varphi + \|\varepsilon^I(t) + \alpha\dot\varepsilon^I(t) + \phi(\alpha)\|_\varphi}{\alpha} \\ &+ \frac{\|\varepsilon^J(t) + \alpha\dot\varepsilon^J(t) + \phi(\alpha)\|_\varphi + \|\varepsilon^K(t) + \alpha\dot\varepsilon^K(t) + \phi(\alpha)\|_\varphi}{\alpha} \\ &+ \frac{\|e^R(t) + \alpha\dot e^R(t) + \phi(\alpha)\|_\varphi + \|e^I(t) + \alpha\dot e^I(t) + \phi(\alpha)\|_\varphi}{\alpha} \\ &+ \frac{\|e^J(t) + \alpha\dot e^J(t) + \phi(\alpha)\|_\varphi + \|e^K(t) + \alpha\dot e^K(t) + \phi(\alpha)\|_\varphi}{\alpha} \\ &- \frac{\|\varepsilon^R(t)\|_\varphi + \|\varepsilon^I(t)\|_\varphi + \|\varepsilon^J(t)\|_\varphi + \|\varepsilon^K(t)\|_\varphi}{\alpha} \\ &- \frac{\|e^R(t)\|_\varphi + \|e^I(t)\|_\varphi + \|e^J(t)\|_\varphi + \|e^K(t)\|_\varphi}{\alpha}\bigg]. \end{aligned}$$

Now from (5)–(12) and (13) we get

$$\begin{aligned} &\overline{\lim_{\alpha\to 0^+}} \frac{\|\varepsilon^R(t) + \alpha\dot\varepsilon^R(t) + \phi(\alpha)\|_\varphi - \|\varepsilon^R(t)\|_\varphi}{\alpha} \\ &\quad = \overline{\lim_{\alpha\to 0^+}} \bigg[\frac{1}{\alpha}\Big(\big\|\varepsilon^R(t) + \alpha\big(-C\varepsilon^R(t) + \big(N^R f^R\big(e^R(t)\big) - N^I f^I\big(e^I(t)\big) \\ &\qquad - N^J f^J\big(e^J(t)\big) - N^K f^K\big(e^K(t)\big)\big) + \big(M^R f^R\big(e^R\big(t-\delta(t)\big)\big) \\ &\qquad - M^I f^I\big(e^I\big(t-\delta(t)\big)\big) - M^J f^J\big(e^J\big(t-\delta(t)\big)\big) - M^K f^K\big(e^K\big(t-\delta(t)\big)\big)\big) \\ &\qquad + \lambda^R(t)\big) + \phi(\alpha)\|_\varphi - \big\|\varepsilon^R(t)\big\|_\varphi\Big)\bigg] \end{aligned}$$

$$\begin{aligned}
&\leqslant \overline{\lim_{\alpha\to 0^+}}\biggl[\frac{\|I+\alpha(\omega-C)\|_\varphi - 1}{\alpha}\|\varepsilon^R(t)\|_\varphi + \|N^R\|_\varphi\|f^R(e^R(t))\|_\varphi\\
&\quad + \|N^I\|_\varphi\|f^I(e^I(t))\|_\varphi + \|N^J\|_\varphi\|f^J(e^J(t))\|_\varphi + \|N^K\|_\varphi\|f^K(e^K(t))\|_\varphi\\
&\quad + \|M^R\|_\varphi\|f^R(e^R(t-\delta(t)))\|_\varphi + \|M^I\|_\varphi\|f^I(e^I(t-\delta(t)))\|_\varphi\\
&\quad + \|M^J\|_\varphi\|f^J(e^J(t-\delta(t)))\|_\varphi + \|M^K\|_\varphi\|f^K(e^K(t-\delta(t)))\|_\varphi\biggr]. \qquad (15)
\end{aligned}$$

Using Assumption 1, we have

$$\|f^R(e^R(t))\|_\varphi \leqslant l^R(\|e^R(t)\|_\varphi + \|e^I(t)\|_\varphi + \|e^J(t)\|_\varphi + \|e^K(t)\|_\varphi). \qquad (16)$$

Similarly, one can obtain that

$$\begin{aligned}
\|f^I(e^I(t))\|_\varphi &\leqslant l^I(\|e^R(t)\|_\varphi + \|e^I(t)\|_\varphi + \|e^J(t)\|_\varphi + \|e^K(t)\|_\varphi),\\
\|f^J(e^J(t))\|_\varphi &\leqslant l^J(\|e^R(t)\|_\varphi + \|e^I(t)\|_\varphi + \|e^J(t)\|_\varphi + \|e^K(t)\|_\varphi),\\
\|f^K(e^K(t))\|_\varphi &\leqslant l^K(\|e^R(t)\|_\varphi + \|e^I(t)\|_\varphi + \|e^J(t)\|_\varphi + \|e^K(t)\|_\varphi), \qquad (17)
\end{aligned}$$

$$\begin{aligned}
\|f^R(e^R(t-\delta(t)))\|_\varphi &\leqslant l^R(\|e^R(t-\delta(t))\|_\varphi + \|e^I(t-\delta(t))\|_\varphi\\
&\quad + \|e^J(t-\delta(t))\|_\varphi + \|e^K(t-\delta(t))\|_\varphi),\\
\|f^I(e^I(t-\delta(t)))\|_\varphi &\leqslant l^I(\|e^R(t-\delta(t))\|_\varphi + \|e^I(t-\delta(t))\|_\varphi\\
&\quad + \|e^J(t-\delta(t))\|_\varphi + \|e^K(t-\delta(t))\|_\varphi),\\
\|f^J(e^J(t-\delta(t)))\|_\varphi &\leqslant l^J(\|e^R(t-\delta(t))\|_\varphi + \|e^I(t-\delta(t))\|_\varphi\\
&\quad + \|e^J(t-\delta(t))\|_\varphi + \|e^K(t-\delta(t))\|_\varphi),\\
\|f^K(e^K(t-\delta(t)))\|_\varphi &\leqslant l^K(\|e^R(t-\delta(t))\|_\varphi + \|e^I(t-\delta(t))\|_\varphi\\
&\quad + \|e^J(t-\delta(t))\|_\varphi + \|e^K(t-\delta(t))\|_\varphi). \qquad (18)
\end{aligned}$$

For function $g^\theta(\cdot)$, the following inequalities hold:

$$\begin{aligned}
\|g^R(\varepsilon^R(t))\|_\varphi &\leqslant \eta^R(\|\varepsilon^R(t)\|_\varphi + \|\varepsilon^I(t)\|_\varphi + \|\varepsilon^J(t)\|_\varphi + \|\varepsilon^K(t)\|_\varphi),\\
\|g^I(\varepsilon^I(t))\|_\varphi &\leqslant \eta^I(\|\varepsilon^R(t)\|_\varphi + \|\varepsilon^I(t)\|_\varphi + \|\varepsilon^J(t)\|_\varphi + \|\varepsilon^K(t)\|_\varphi),\\
\|g^J(\varepsilon^J(t))\|_\varphi &\leqslant \eta^J(\|\varepsilon^R(t)\|_\varphi + \|\varepsilon^I(t)\|_\varphi + \|\varepsilon^J(t)\|_\varphi + \|\varepsilon^K(t)\|_\varphi),\\
\|g^K(\varepsilon^K(t))\|_\varphi &\leqslant \eta^K(\|\varepsilon^R(t)\|_\varphi + \|\varepsilon^I(t)\|_\varphi + \|\varepsilon^J(t)\|_\varphi + \|\varepsilon^K(t)\|_\varphi),
\end{aligned}$$

$$\begin{aligned}
\|g^R(\varepsilon^R(t-\tau(t)))\|_\varphi &\leqslant \eta^R(\|\varepsilon^R(t-\tau(t))\|_\varphi + \|\varepsilon^I(t-\tau(t))\|_\varphi\\
&\quad + \|\varepsilon^J(t-\tau(t))\|_\varphi + \|\varepsilon^K(t-\tau(t))\|_\varphi),\\
\|g^I(\varepsilon^I(t-\tau(t)))\|_\varphi &\leqslant \eta^I(\|\varepsilon^R(t-\tau(t))\|_\varphi + \|\varepsilon^I(t-\tau(t))\|_\varphi\\
&\quad + \|\varepsilon^J(t-\tau(t))\|_\varphi + \|\varepsilon^K(t-\tau(t))\|_\varphi),
\end{aligned}$$

$$\begin{aligned}\left\|g^J\left(\varepsilon^J\left(t-\tau(t)\right)\right)\right\|_\varphi \leqslant \eta^J\big(&\left\|\varepsilon^R\left(t-\tau(t)\right)\right\|_\varphi+\left\|\varepsilon^I\left(t-\tau(t)\right)\right\|_\varphi\\&+\left\|\varepsilon^J\left(t-\tau(t)\right)\right\|_\varphi+\left\|\varepsilon^K\left(t-\tau(t)\right)\right\|_\varphi\big),\\\left\|g^K\left(\varepsilon^K\left(t-\tau(t)\right)\right)\right\|_\varphi \leqslant \eta^K\big(&\left\|\varepsilon^R\left(t-\tau(t)\right)\right\|_\varphi+\left\|\varepsilon^I\left(t-\tau(t)\right)\right\|_\varphi\\&+\left\|\varepsilon^J\left(t-\tau(t)\right)\right\|_\varphi+\left\|\varepsilon^K\left(t-\tau(t)\right)\right\|_\varphi\big).\end{aligned}$$

By substituting (16)–(18) into inequality (15), we obtain

$$\begin{aligned}&\overline{\lim_{\alpha\to0^+}}\frac{\|\varepsilon^R(t)+\alpha\dot{\varepsilon}^R(t)+\phi(\alpha)\|_\varphi-\left\|\varepsilon^R(t)\right\|_\varphi}{\alpha}\\&\quad\leqslant\overline{\lim_{\alpha\to0^+}}\bigg[\frac{\|I+\alpha(\omega-C)\|_\varphi-1}{\alpha}\left\|\varepsilon^R(t)\right\|_\varphi+\left\|N^R\right\|_\varphi\big(l^R\big(\left\|e^R(t)\right\|_\varphi+\left\|e^I(t)\right\|_\varphi\\&\qquad+\left\|e^J(t)\right\|_\varphi+\left\|e^K(t)\right\|_\varphi\big)\big)+\left\|N^I\right\|_\varphi\big(l^I\big(\left\|e^R(t)\right\|_\varphi+\left\|e^I(t)\right\|_\varphi+\left\|e^J(t)\right\|_\varphi\\&\qquad+\left\|e^K(t)\right\|_\varphi\big)\big)+\left\|N^J\right\|_\varphi\big(l^J\big(\left\|e^R(t)\right\|_\varphi+\left\|e^I(t)\right\|_\varphi+\left\|e^J(t)\right\|_\varphi+\left\|e^K(t)\right\|_\varphi\big)\big)\\&\qquad+\left\|N^K\right\|_\varphi\big(l^K\big(\left\|e^R(t)\right\|_\varphi+\left\|e^I(t)\right\|_\varphi+\left\|e^J(t)\right\|_\varphi+\left\|e^K(t)\right\|_\varphi\big)\big)\\&\qquad+\left\|M^R\right\|_\varphi\big(l^R\big(\left\|e^R\left(t-\delta(t)\right)\right\|_\varphi+\left\|e^I\left(t-\delta(t)\right)\right\|_\varphi+\left\|e^J\left(t-\delta(t)\right)\right\|_\varphi\\&\qquad+\left\|e^K\left(t-\delta(t)\right)\right\|_\varphi\big)\big)+\left\|M^I\right\|_\varphi\big(l^I\big(\left\|e^R\left(t-\delta(t)\right)\right\|_\varphi+\left\|e^I\left(t-\delta(t)\right)\right\|_\varphi\\&\qquad+\left\|e^J\left(t-\delta(t)\right)\right\|_\varphi+\left\|e^K\left(t-\delta(t)\right)\right\|_\varphi\big)\big)+\left\|M^J\right\|_\varphi\big(l^J\big(\left\|e^R\left(t-\delta(t)\right)\right\|_\varphi\\&\qquad+\left\|e^I\left(t-\delta(t)\right)\right\|_\varphi+\left\|e^J\left(t-\delta(t)\right)\right\|_\varphi+\left\|e^K\left(t-\delta(t)\right)\right\|_\varphi\big)\big)\\&\qquad+\left\|M^K\right\|_\varphi\big(l^K\big(\left\|e^R\left(t-\delta(t)\right)\right\|_\varphi+\left\|e^I\left(t-\delta(t)\right)\right\|_\varphi+\left\|e^J\left(t-\delta(t)\right)\right\|_\varphi\\&\qquad+\left\|e^K\left(t-\delta(t)\right)\right\|_\varphi\big)\big)\bigg]\\&\quad=\overline{\lim_{\alpha\to0^+}}\bigg[\frac{\|I+\alpha(\omega-C)\|_\varphi-1}{\alpha}\left\|\varepsilon^R(t)\right\|_\varphi+\big(\left\|N^R\right\|_\varphi l^R+\left\|N^I\right\|_\varphi l^I\\&\qquad+\left\|N^J\right\|_\varphi l^J+\left\|N^K\right\|_\varphi l^K\big)\big(\left\|e^R(t)\right\|_\varphi+\left\|e^I(t)\right\|_\varphi+\left\|e^J(t)\right\|_\varphi+\left\|e^K(t)\right\|_\varphi\big)\\&\qquad+\big(\left\|M^R\right\|_\varphi l^R+\left\|M^I\right\|_\varphi l^I+\left\|M^J\right\|_\varphi l^J+\left\|M^K\right\|_\varphi l^K\big)\big(\left\|e^R\left(t-\delta(t)\right)\right\|_\varphi\\&\qquad+\left\|e^I\left(t-\delta(t)\right)\right\|_\varphi+\left\|e^J\left(t-\delta(t)\right)\right\|_\varphi+\left\|e^K\left(t-\delta(t)\right)\right\|_\varphi\big)\bigg].\end{aligned}\tag{19}$$

Likewise, we have

$$\begin{aligned}&\overline{\lim_{\alpha\to0^+}}\frac{\|\varepsilon^I(t)+\alpha\dot{\varepsilon}^I(t)+\phi(\alpha)\|_\varphi-\left\|\varepsilon^I(t)\right\|_\varphi}{\alpha}\\&\quad\leqslant\overline{\lim_{\alpha\to0^+}}\bigg[\frac{\|I+\alpha(\omega-C)\|_\varphi-1}{\alpha}\left\|\varepsilon^I(t)\right\|_\varphi+\big(\left\|N^R\right\|_\varphi l^I+\left\|N^I\right\|_\varphi l^R\\&\qquad+\left\|N^J\right\|_\varphi l^K+\left\|N^K\right\|_\varphi l^J\big)\big(\left\|e^R(t)\right\|_\varphi+\left\|e^I(t)\right\|_\varphi+\left\|e^J(t)\right\|_\varphi+\left\|e^K(t)\right\|_\varphi\big)\end{aligned}$$

$$
\begin{aligned}
&+ \left(\left\|M^R\right\|_\varphi l^I + \left\|M^I\right\|_\varphi l^R + \left\|M^J\right\|_\varphi l^K + \left\|M^K\right\|_\varphi l^J\right)\left(\left\|e^R\big(t-\delta(t)\big)\right\|_\varphi\right. \\
&\left.\left. + \left\|e^I\big(t-\delta(t)\big)\right\|_\varphi + \left\|e^J\big(t-\delta(t)\big)\right\|_\varphi + \left\|e^K\big(t-\delta(t)\big)\right\|_\varphi\right)\right],
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
&\overline{\lim_{\alpha\to 0^+}} \frac{\left\|\varepsilon^J(t) + \alpha\dot{\varepsilon}^J(t) + \phi(\alpha)\right\|_\varphi - \left\|\varepsilon^J(t)\right\|_\varphi}{\alpha} \\
&\quad\leqslant \overline{\lim_{\alpha\to 0^+}} \left[\frac{\left\|I + \alpha(\omega - C)\right\|_\varphi - 1}{\alpha}\left\|\varepsilon^J(t)\right\|_\varphi + \left(\left\|N^R\right\|_\varphi l^J + \left\|N^J\right\|_\varphi l^R\right.\right. \\
&\quad\quad \left. + \left\|N^I\right\|_\varphi l^K + \left\|N^K\right\|_\varphi l^I\right)\left(\left\|e^R(t)\right\|_\varphi + \left\|e^I(t)\right\|_\varphi + \left\|e^J(t)\right\|_\varphi + \left\|e^K(t)\right\|_\varphi\right) \\
&\quad\quad + \left(\left\|M^R\right\|_\varphi l^J + \left\|M^J\right\|_\varphi l^R + \left\|M^I\right\|_\varphi l^K + \left\|M^K\right\|_\varphi l^I\right)\left(\left\|e^R\big(t-\delta(t)\big)\right\|_\varphi\right. \\
&\quad\quad \left.\left. + \left\|e^I\big(t-\delta(t)\big)\right\|_\varphi + \left\|e^J\big(t-\delta(t)\big)\right\|_\varphi + \left\|e^K\big(t-\delta(t)\big)\right\|_\varphi\right)\right],
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
&\overline{\lim_{\alpha\to 0^+}} \frac{\left\|\varepsilon^K(t) + \alpha\dot{\varepsilon}^K(t) + \phi(\alpha)\right\|_\varphi - \left\|\varepsilon^K(t)\right\|_\varphi}{\alpha} \\
&\quad\leqslant \overline{\lim_{\alpha\to 0^+}} \left[\frac{\left\|I + \alpha(\omega - C)\right\|_\varphi - 1}{\alpha}\left\|\varepsilon^K(t)\right\|_\varphi + \left(\left\|N^R\right\|_\varphi l^K + \left\|N^K\right\|_\varphi l^R\right.\right. \\
&\quad\quad \left. + \left\|N^J\right\|_\varphi l^I + \left\|N^I\right\|_\varphi l^J\right)\left(\left\|e^R(t)\right\|_\varphi + \left\|e^I(t)\right\|_\varphi + \left\|e^J(t)\right\|_\varphi + \left\|e^K(t)\right\|_\varphi\right) \\
&\quad\quad + \left(\left\|M^R\right\|_\varphi l^K + \left\|M^K\right\|_\varphi l^R + \left\|M^J\right\|_\varphi l^I + \left\|M^I\right\|_\varphi l^J\right)\left(\left\|e^R\big(t-\delta(t)\big)\right\|_\varphi\right. \\
&\quad\quad \left.\left. + \left\|e^I\big(t-\delta(t)\big)\right\|_\varphi + \left\|e^J\big(t-\delta(t)\big)\right\|_\varphi + \left\|e^K\big(t-\delta(t)\big)\right\|_\varphi\right)\right].
\end{aligned}
\tag{22}
$$

Similarly, for state vector $e(t)$, we can get

$$
\begin{aligned}
&\overline{\lim_{\alpha\to 0^+}} \frac{\left\|e^R(t) + \alpha\dot{e}^R(t) + \phi(\alpha)\right\|_\varphi - \left\|e^R(t)\right\|_\varphi}{\alpha} \\
&\quad\leqslant \overline{\lim_{\alpha\to 0^+}} \left[\frac{\left\|I + \alpha(\tilde{\omega} - D)\right\|_\varphi - 1}{\alpha}\left\|e^R(t)\right\|_\varphi + \left(\left\|Q^R\right\|_\varphi \eta^R + \left\|Q^I\right\|_\varphi \eta^I\right.\right. \\
&\quad\quad \left. + \left\|Q^J\right\|_\varphi \eta^J + \left\|Q^K\right\|_\varphi \eta^K\right)\left(\left\|\varepsilon^R(t)\right\|_\varphi + \left\|\varepsilon^I(t)\right\|_\varphi + \left\|\varepsilon^J(t)\right\|_\varphi + \left\|\varepsilon^K(t)\right\|_\varphi\right) \\
&\quad\quad + \left(\left\|P^R\right\|_\varphi \eta^R + \left\|P^I\right\|_\varphi \eta^I + \left\|P^J\right\|_\varphi \eta^J + \left\|P^K\right\|_\varphi \eta^K\right)\left(\left\|\varepsilon^R\big(t-\tau(t)\big)\right\|_\varphi\right. \\
&\quad\quad \left.\left. + \left\|\varepsilon^I\big(t-\tau(t)\big)\right\|_\varphi + \left\|\varepsilon^J\big(t-\tau(t)\big)\right\|_\varphi + \left\|\varepsilon^K\big(t-\tau(t)\big)\right\|_\varphi\right)\right],
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
&\overline{\lim_{\alpha\to 0^+}} \frac{\left\|e^I(t) + \alpha\dot{e}^I(t) + \phi(\alpha)\right\|_\varphi - \left\|e^I(t)\right\|_\varphi}{\alpha} \\
&\quad\leqslant \overline{\lim_{\alpha\to 0^+}} \left[\frac{\left\|I + \alpha(\tilde{\omega} - D)\right\|_\varphi - 1}{\alpha}\left\|e^I(t)\right\|_\varphi + \left(\left\|Q^R\right\|_\varphi \eta^I + \left\|Q^I\right\|_\varphi \eta^R\right.\right. \\
&\quad\quad \left. + \left\|Q^J\right\|_\varphi \eta^K + \left\|Q^K\right\|_\varphi \eta^J\right)\left(\left\|\varepsilon^R(t)\right\|_\varphi + \left\|\varepsilon^I(t)\right\|_\varphi + \left\|\varepsilon^J(t)\right\|_\varphi + \left\|\varepsilon^K(t)\right\|_\varphi\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(\|P^R\|_\varphi \eta^I + \|P^I\|_\varphi \eta^R + \|P^J\|_\varphi \eta^K + \|P^K\|_\varphi \eta^J\right)\left(\left\|\varepsilon^R\left(t-\tau(t)\right)\right\|_\varphi \right.\\
&\left.\left. + \left\|\varepsilon^I\left(t-\tau(t)\right)\right\|_\varphi + \left\|\varepsilon^J\left(t-\tau(t)\right)\right\|_\varphi + \left\|\varepsilon^K\left(t-\tau(t)\right)\right\|_\varphi\right)\right], \qquad (24)
\end{aligned}
$$

$$
\begin{aligned}
&\overline{\lim_{\alpha\to 0^+}} \frac{\|e^J(t) + \alpha\dot{e}^J(t) + \phi(\alpha)\|_\varphi - \|e^J(t)\|_\varphi}{\alpha} \\
&\quad \leqslant \overline{\lim_{\alpha\to 0^+}} \left[\frac{\|I + \alpha(\tilde{\omega} - D)\|_\varphi - 1}{\alpha}\|e^J(t)\|_\varphi + \left(\|Q^R\|_\varphi \eta^J + \|Q^J\|_\varphi \eta^R \right.\right.\\
&\quad + \left.\|Q^I\|_\varphi \eta^K + \|Q^K\|_\varphi \eta^I\right)\left(\|\varepsilon^R(t)\|_\varphi + \|\varepsilon^I(t)\|_\varphi + \|\varepsilon^J(t)\|_\varphi + \|\varepsilon^K(t)\|_\varphi\right)\\
&\quad + \left(\|P^R\|_\varphi \eta^J + \|P^J\|_\varphi \eta^R + \|P^I\|_\varphi \eta^K + \|P^K\|_\varphi \eta^I\right)\left(\left\|\varepsilon^R\left(t-\tau(t)\right)\right\|_\varphi \right.\\
&\quad \left.\left. + \left\|\varepsilon^I\left(t-\tau(t)\right)\right\|_\varphi + \left\|\varepsilon^J\left(t-\tau(t)\right)\right\|_\varphi + \left\|\varepsilon^K\left(t-\tau(t)\right)\right\|_\varphi\right)\right], \qquad (25)
\end{aligned}
$$

$$
\begin{aligned}
&\overline{\lim_{\alpha\to 0^+}} \frac{\|e^K(t) + \alpha\dot{e}^K(t) + \phi(\alpha)\|_\varphi - \|e^K(t)\|_\varphi}{\alpha} \\
&\quad \leqslant \overline{\lim_{\alpha\to 0^+}} \left[\frac{\|I + \alpha(\tilde{\omega} - D)\|_\varphi - 1}{\alpha}\|e^K(t)\|_\varphi + \left(\|Q^R\|_\varphi \eta^K + \|Q^K\|_\varphi \eta^R \right.\right.\\
&\quad + \left.\|Q^J\|_\varphi \eta^I + \|Q^I\|_\varphi \eta^J\right)\left(\|\varepsilon^R(t)\|_\varphi + \|\varepsilon^I(t)\|_\varphi + \|\varepsilon^J(t)\|_\varphi + \|\varepsilon^K(t)\|_\varphi\right)\\
&\quad + \left(\|P^R\|_\varphi \eta^K + \|P^K\|_\varphi \eta^R + \|P^J\|_\varphi \eta^I + \|P^I\|_\varphi \eta^J\right)\left(\left\|\varepsilon^R\left(t-\tau(t)\right)\right\|_\varphi \right.\\
&\quad \left.\left. + \left\|\varepsilon^I\left(t-\tau(t)\right)\right\|_\varphi + \left\|\varepsilon^J\left(t-\tau(t)\right)\right\|_\varphi + \left\|\varepsilon^K\left(t-\tau(t)\right)\right\|_\varphi\right)\right]. \qquad (26)
\end{aligned}
$$

Then using Definition 3 and Lemma 3 and adding (19)–(26) yields

$$
\begin{aligned}
&D^+ L\left(\varepsilon(t), e(t)\right)\\
&\quad \leqslant \mu_\varphi(\omega - C)\left(\|\varepsilon^R(t)\|_\varphi + \|\varepsilon^I(t)\|_\varphi + \|\varepsilon^J(t)\|_\varphi + \|\varepsilon^K(t)\|_\varphi\right)\\
&\qquad + \left(l^R + l^I + l^J + l^K\right)\left(\|N^R\|_\varphi + \|N^I\|_\varphi + \|N^J\|_\varphi + \|N^K\|_\varphi\right)\\
&\qquad \times \left(\|e^R(t)\|_\varphi + \|e^I(t)\|_\varphi + \|e^J(t)\|_\varphi + \|e^K(t)\|_\varphi\right)\\
&\qquad + \left(l^R + l^I + l^J + l^K\right)\left(\|M^R\|_\varphi + \|M^I\|_\varphi + \|M^J\|_\varphi + \|M^K\|_\varphi\right)\\
&\qquad \times \left(\left\|e^R\left(t-\delta(t)\right)\right\|_\varphi + \left\|e^I\left(t-\delta(t)\right)\right\|_\varphi + \left\|e^J\left(t-\delta(t)\right)\right\|_\varphi + \left\|e^K\left(t-\delta(t)\right)\right\|_\varphi\right)\\
&\qquad + \mu_\varphi(\tilde{\omega} - D)\left(\|e^R(t)\|_\varphi + \|e^I(t)\|_\varphi + \|e^J(t)\|_\varphi + \|e^K(t)\|_\varphi\right)\\
&\qquad + \left(\eta^R + \eta^I + \eta^J + \eta^K\right)\left(\|Q^R\|_\varphi + \|Q^I\|_\varphi + \|Q^J\|_\varphi + \|Q^K\|_\varphi\right)\\
&\qquad \times \left(\|\varepsilon^R(t)\|_\varphi + \|\varepsilon^I(t)\|_\varphi + \|\varepsilon^J(t)\|_\varphi + \|\varepsilon^K(t)\|_\varphi\right)\\
&\qquad + \left(\eta^R + \eta^I + \eta^J + \eta^K\right)\left(\|P^R\|_\varphi + \|P^I\|_\varphi + \|P^J\|_\varphi + \|P^K\|_\varphi\right)\\
&\qquad \times \left(\left\|\varepsilon^R\left(t-\tau(t)\right)\right\|_\varphi + \left\|\varepsilon^I\left(t-\tau(t)\right)\right\|_\varphi + \left\|\varepsilon^J\left(t-\tau(t)\right)\right\|_\varphi + \left\|\varepsilon^K\left(t-\tau(t)\right)\right\|_\varphi\right)
\end{aligned}
$$

$$\leqslant \max\bigl(\mu_\varphi(\omega - C) + \bar{\Gamma}\tilde{Q}, \mu_\varphi(\tilde{\omega} - D) + \bar{L}\tilde{N}\bigr)\bigl(L\bigl(\varepsilon(t)\bigr) + L\bigl(e(t)\bigr)\bigr)$$
$$+ \max\bigl(\bar{L}\tilde{M}, \bar{\Gamma}\bar{P}\bigr)\bigl(\bigl(L\bigl(\varepsilon\bigl(t - \tau(t)\bigr)\bigr) + L\bigl(e\bigl(t - \delta(t)\bigr)\bigr)\bigr)\bigr), \tag{27}$$

where

$$\bar{L}\tilde{N} = \bigl(l^R + l^I + l^J + l^K\bigr)\bigl(\|N^R\|_\varphi + \|N^I\|_\varphi + \|N^J\|_\varphi + \|N^K\|_\varphi\bigr),$$
$$\bar{\Gamma}\tilde{Q} = \bigl(\eta^R + \eta^I + \eta^J + \eta^K\bigr)\bigl(\|Q^R\|_\varphi + \|Q^I\|_\varphi + \|Q^J\|_\varphi + \|Q^K\|_\varphi\bigr),$$
$$\bar{L}\tilde{M} = \bigl(l^R + l^I + l^J + l^K\bigr)\bigl(\|M^R\|_\varphi + \|M^I\|_\varphi + \|M^J\|_\varphi + \|M^K\|_\varphi\bigr),$$
$$\bar{\Gamma}\tilde{P} = \bigl(\eta^R + \eta^I + \eta^J + \eta^K\bigr)\bigl(\|P^R\|_\varphi + \|P^I\|_\varphi + \|P^J\|_\varphi + \|P^K\|_\varphi\bigr).$$

Define time delay $\rho(t) = \max(\delta(t), \tau(t))$.

From Assumption 2 we can obtain $\lim_{t\to\infty}(t - \rho(t)) = \infty$. Then for (27), we have

$$\begin{aligned}
&D^+ L\bigl(\varepsilon(t), e(t)\bigr)\\
&\quad\leqslant \max\bigl(\mu_\varphi(\omega - C) + \bar{\Gamma}\tilde{Q},\ \mu_\varphi(\tilde{\omega} - D) + \bar{L}\tilde{N}\bigr)\bigl(L\bigl(\varepsilon(t)\bigr) + L\bigl(e(t)\bigr)\bigr)\\
&\qquad + \max\bigl(\bar{L}\tilde{M}, \bar{\Gamma}\bar{P}\bigr)\bigl(\bigl(L\bigl(\varepsilon\bigl(t - \tau(t)\bigr), e\bigl(t - \tau(t)\bigr)\bigr) + L\bigl(\varepsilon\bigl(t - \delta(t)\bigr), e\bigl(t - \delta(t)\bigr)\bigr)\bigr)\bigr)\\
&\quad\leqslant \max\bigl(\mu_\varphi(\omega - C) + \bar{\Gamma}\tilde{Q},\ \mu_\varphi(\tilde{\omega} - D) + \bar{L}\tilde{N}\bigr)\bigl(L\bigl(\varepsilon(t)\bigr) + L\bigl(e(t)\bigr)\bigr)\\
&\qquad + \max\bigl(\bar{L}\tilde{M}, \bar{\Gamma}\bar{P}\bigr)\Bigl(\sup_{s\in[t-\rho(t),\,t]} L\bigl(\varepsilon(s), e(s)\bigr) + \sup_{s\in[t-\rho(t),\,t]} L\bigl(\varepsilon(s), e(s)\bigr)\Bigr)\\
&\quad= \max\bigl(\mu_\varphi(\omega - C) + \bar{\Gamma}\tilde{Q},\ \mu_\varphi(\tilde{\omega} - D) + \bar{L}\tilde{N}\bigr)\bigl(L\bigl(\varepsilon(t)\bigr) + L\bigl(e(t)\bigr)\bigr)\\
&\qquad + 2\max\bigl(\bar{L}\tilde{M}, \bar{\Gamma}\bar{P}\bigr) \sup_{s\in[t-\rho(t),\,t]} L\bigl(\varepsilon(s), e(s)\bigr).
\end{aligned}$$

Let

$$\alpha = -\max\bigl(\mu_\varphi(\omega - C) + \bar{\Gamma}\tilde{Q},\ \mu_\varphi(\tilde{\omega} - D) + \bar{L}\tilde{N}\bigr),$$
$$\beta = 2\max(\bar{L}\tilde{M}, \bar{\Gamma}\tilde{P}).$$

From condition (14) we have $\alpha > \beta > 0$. By Lemma 1, it can be shown that $\|\varepsilon^\theta(t)\|_\varphi$ $(\theta = R, I, J, K)$ and $\|e^\theta(t)\|_\varphi(\theta = R, I, J, K)$ converge globally asymptotically to zero, leading to the conclusion that the master system in (2) and the response system in (3) are GAS. The proof is complete. □

Remark 2. At present, a large number of literatures have utilized the Lyapunov function method to study the dynamics of neural networks [1, 21, 28]. However, in practical applications, selecting an appropriate Lyapunov function to prove the stability or synchronization of neural networks is highly challenging and often involves complicated integral computations. In this paper, we utilize the matrix measure approach and Halanay inequality approach to analyze the synchronization conditions for QVBAMNNs. The obtained synchronization results, expressed in terms of norms and measures, are more concise compared to the matrix-form results derived through Lyapunov function methods.

Remark 3. We know that there is a noncommutative problem with quaternions. To overcome this problem, the quaternion decomposition strategy was adopted, which decomposed the original system into equivalent real-valued subsystems. Although this method increased the dimension of the system in a formal sense, it enabled us to avoid the direct handling difficulties brought about by noncommutativity and ultimately derived easily verifiable synchronization conditions.

Remark 4. In the analysis of neural network synchronization, the boundedness of time delay has a decisive influence on the convergence performance of the system. According to [11], when the delay is unbounded, the generalized Halanay inequality can only guarantee the asymptotic synchronization of the system. When the time delay is bounded, the exponential synchronization of the system can be established based on the Halanay inequality. Accordingly, Theorem 1 establishes GAS for QVBAMNN (2) under unbounded delays, while Assumption 2 provides the bounded delay condition required for GES.

Theorem 2. *Under Assumptions* 1 *and* 3*, if controller gain matrices satisfy*

$$0 < \max(\bar{L}\tilde{M}, \bar{\varGamma}\tilde{P}) < -\max\bigl(\mu_\varphi(\omega - C) + \bar{\varGamma}\tilde{Q},\, \mu_\varphi(\tilde{\omega} - D) + \bar{L}\tilde{N}\bigr), \tag{28}$$

then the QVBAMNN (3) *will be GES with the response system* (2)*.*

Proof. The proof is similar to that of Theorem 1, except that the scaling of (27) is different. Here Lemma 2 is employed to obtain the GES of QVBAMNN (2).

Under Assumption 3, (27) is deflated as follows:

$$\begin{aligned}
&D^+L\bigl(\varepsilon(t), e(t)\bigr)\\
&\quad\leqslant \max\bigl(\mu_\varphi(\omega - C) + \bar{\varGamma}\tilde{Q},\, \mu_\varphi(\tilde{\omega} - D) + \bar{L}\tilde{N}\bigr)\bigl(L\bigl(\varepsilon(t)\bigr) + L\bigl(e(t)\bigr)\bigr)\\
&\qquad + \max\bigl(\bar{L}\tilde{M}, \bar{\varGamma}\tilde{P}\bigr)\bigl(\bigl(L\bigl(\varepsilon\bigl(t - \tau(t)\bigr)\bigr) + L\bigl(e\bigl(t - \delta(t)\bigr)\bigr)\bigr)\bigr)\\
&\quad\leqslant \max\bigl(\mu_\varphi(\omega - C) + \bar{\varGamma}\tilde{Q},\, \mu_\varphi(\tilde{\omega} - D) + \bar{L}\tilde{N}\bigr)L\bigl(\varepsilon(t), e(t)\bigr)\\
&\qquad + \max(\bar{L}\tilde{M}, \bar{\varGamma}\tilde{P}) \sup_{t-\max(\delta,\tau)\leqslant s\leqslant t} L\bigl(\varepsilon(s), e(s)\bigr).
\end{aligned}$$

Let

$$\begin{aligned}
k_1 &= -\max\bigl(\mu_\varphi(\omega - C) + \bar{\varGamma}\tilde{Q},\, \mu_\varphi(\tilde{\omega} - D) + \bar{L}\tilde{N}\bigr),\\
k_2 &= \max(\bar{L}\tilde{M}, \bar{\varGamma}\tilde{P}).
\end{aligned}$$

From condition (28) we have $k_1 > k_2 > 0$. By Lemma 2, it can be shown that

$$L\bigl(\varepsilon(t), e(t)\bigr) \leqslant \sup_{t_0-\max(\delta,\tau)\leqslant s\leqslant t_0} L\bigl(\varepsilon(s), e(s)\bigr)\mathrm{e}^{-\lambda(t-t_0)},$$

where λ is the unique positive solution of

$$\begin{aligned}
\lambda &= k_1 - k_2\mathrm{e}^{\lambda\max(\delta,\tau)}\\
&= -\max\bigl(\mu_\varphi(\omega - C) + \bar{\varGamma}\tilde{Q},\, \mu_\varphi(\tilde{\omega} - D) + \bar{\varGamma}\tilde{N}\bigr)\\
&\quad - \max(\bar{L}\tilde{M}, \bar{\varGamma}\tilde{P})\mathrm{e}^{\lambda\max(\delta,\tau)}.
\end{aligned}$$

According to Definition 2, we obtain that $L(\varepsilon(t), e(t))$ decays exponentially to zero at the rate of $\max(\delta, \tau)$. By Lemma 3 it follows that $\|\varepsilon^\theta(t)\|_\varphi$ ($\theta = R, I, J, K$) and $\|\mathrm{e}^\theta(t)\|_\varphi$ ($\theta = R, I, J, K$) converge globally and exponentially to zero, leading to the conclusion that the master system in (2) is GES with respect to the response system in (3). The proof is complete. □

4 Numerical examples

Example 1. Consider the master–response systems of 2-D QVBAMNN as

$$\begin{aligned}\dot{x}(t) &= -Cx(t) + Nf\big(y(t)\big) + Mf\big(y\big(t-\delta(t)\big)\big) + H,\\ \dot{y}(t) &= -Dy(t) + Qg\big(x(t)\big) + Pg\big(x\big(t-\tau(t)\big)\big) + Z,\end{aligned} \tag{29}$$

$$\begin{aligned}\dot{u}(t) &= -Cu(t) + Nf\big(v(t)\big) + Mf\big(v\big(t-\delta(t)\big)\big) + H + \lambda(t),\\ \dot{v}(t) &= -Dv(t) + Qg\big(u(t)\big) + Pg\big(u\big(t-\tau(t)\big)\big) + Z + \pi(t),\end{aligned} \tag{30}$$

where

$$C = \begin{bmatrix} 6.0 & 0\\ 0 & 7.0\end{bmatrix}, \qquad N = \begin{bmatrix} 0.6+1.0\mathrm{i}+0.3\mathrm{j}+0.2\mathrm{k} & -2.0-1.5\mathrm{i}-0.8\mathrm{j}-0.8\mathrm{k}\\ 0.2+0.5\mathrm{i}+0.2\mathrm{j}+0.3\mathrm{k} & 0.5+0.2\mathrm{i}+0.1\mathrm{j}+0.2\mathrm{k}\end{bmatrix},$$

$$D = \begin{bmatrix} 5.0 & 0\\ 0 & 6.0\end{bmatrix}, \qquad Q = \begin{bmatrix} 1.2+1.1\mathrm{i}+0.5\mathrm{j}+0.3\mathrm{k} & -1.2-1.3\mathrm{i}-1.0\mathrm{j}-1.0\mathrm{k}\\ 0.2+0.6\mathrm{i}+0.3\mathrm{j}+0.2\mathrm{k} & 0.3+0.2\mathrm{i}+0.1\mathrm{j}+0.3\mathrm{k}\end{bmatrix},$$

$$M = \begin{bmatrix} 0.4-1.3\mathrm{i}+0.8\mathrm{j}+1.1\mathrm{k} & 1.2+0.3\mathrm{i}-1.5\mathrm{j}+0.6\mathrm{k}\\ 1.6-1.2\mathrm{i}+0.8\mathrm{j}+0.8\mathrm{k} & 0.5-0.8\mathrm{i}+1.0\mathrm{j}+1.0\mathrm{k}\end{bmatrix},$$

$$P = \begin{bmatrix} 1.1-1.6\mathrm{i}+0.7\mathrm{j}+1.3\mathrm{k} & 1.2+1.0\mathrm{i}-1.6\mathrm{j}+0.8\mathrm{k}\\ 0.6-2.2\mathrm{i}+0.7\mathrm{j}+1.1\mathrm{k} & 0.3-1.6\mathrm{i}+1.2\mathrm{j}+0.8\mathrm{k}\end{bmatrix},$$

$$H = \begin{bmatrix} 0.8-0.3\mathrm{i}+0.5\mathrm{j}-0.8\mathrm{k}\\ -0.5+0.7\mathrm{i}-0.9\mathrm{j}+0.3\mathrm{k}\end{bmatrix} \qquad Z = \begin{bmatrix} 0.9-0.5\mathrm{i}+0.6\mathrm{j}-0.6\mathrm{k}\\ -0.6+0.6\mathrm{i}-1.0\mathrm{j}+0.2\mathrm{k}\end{bmatrix},$$

and activation functions are $f(y(t)) = \tanh(y(t))$, $g(x(t)) = \tanh(x(t))$. The initial value of QVBAMNN (29) is given by $x_1(0) = 1.5 + 0.5\mathrm{i} - 1.2\mathrm{j} - 4.0\mathrm{k}$, $y_1(0) = 2.2 + 1.2\mathrm{i} - 1.1\mathrm{j} - 4.2\mathrm{k}$, $x_2(0) = 5.0 + 0.3\mathrm{i} + 0.6\mathrm{j} - 1.4\mathrm{k}$, $y_2(0) = 5.5 + 0.1\mathrm{i} + 0.2\mathrm{j} - 1.6\mathrm{k}$, and the initial value of QVBAMNN (30) is given by $u_1(0) = 5.0 - 2.0\mathrm{i} + 3.0\mathrm{j} - 4.0\mathrm{k}$, $v_1(0) = 4.8 - 2.2\mathrm{i} + 4.0\mathrm{j} - 3.2\mathrm{k}$, $u_2(0) = -3.2 + 4.0\mathrm{i} - 5.2\mathrm{j} + 1.6\mathrm{k}$, $v_2(0) = 1 - 2\mathrm{i} + 1.5\mathrm{j} - 1\mathrm{k}$, and $\delta(t) = 0.2\sqrt{t}$, $\tau(t) = 0.1\sqrt{t}$. Control gain matrix $\omega = \left[\begin{smallmatrix} -125 & 2\\ 3 & -118\end{smallmatrix}\right]$, $\tilde{\omega} = \left[\begin{smallmatrix} -106 & 2\\ 1 & -105\end{smallmatrix}\right]$. When norm $\varphi = 2$, system (29) meets Assumption 1. The other parameters can be calculated as $l^\theta = \eta^\theta = 1$, $\theta = R, I, J, K$, and $\|N^R\|_2 = 2.04$, $\|N^I\|_2 = 1.73$, $\|N^J\|_2 = \|N^K\|_2 = 0.9$, $\|P^R\|_2 = \|P^I\|_2 = \|P^J\|_2 = 1.84$, $\|P^K\|_2 = 1.49$, $\|M^R\|_2 = 1.8$, $\|M^I\|_2 = 1.7$, $\|M^J\|_2 = \|M^K\|_2 = 1.2$, $\|Q^R\|_2 = 1.49$, $\|Q^I\|_2 = 2.8$, $\|Q^J\|_2 = 2.65$, $\|Q^K\|_2 = 1.73$, $\mu_2(\omega - C) \approx -124$, $\mu_2(\tilde{\omega} - D) = -109.5$. We can verify thar condition (14) of Theorem 1 is satisfied under the above parameters since

$$\begin{aligned} 0 \leqslant 2\max(\bar{L}\tilde{M}, \bar{\varGamma}\tilde{P}) &= 56\\ &< -\max\big(\mu_2(\omega - C) + \bar{\varGamma}\tilde{Q},\ \mu_2(\tilde{\omega} - D) + \bar{L}\tilde{N}\big) = 87.22,\end{aligned}$$

i.e., $\alpha > \beta > 0$. According to Theorem 1, QVBAMNNs (29) and (30) are GAS. Based on the data mentioned above, state trajectories of variables $x_i^\theta(t)$, $u_i^\theta(t)$, $y_i^\theta(t)$, $v_i^\theta(t)$ ($i = 1, 2$) of the QVBAMNNs (29) and (30) without controller and with controller are shown in Figs. 1 and 2, respectively. The results shown in Fig. 1 indicate that without controllers ω and $\tilde{\omega}$, the QVBAMNNs (29) and (30) failed to achieve synchronization. While from the results in Fig. 2 it can be seen that after introducing thecontrollers, the QVBAMNNs (29)

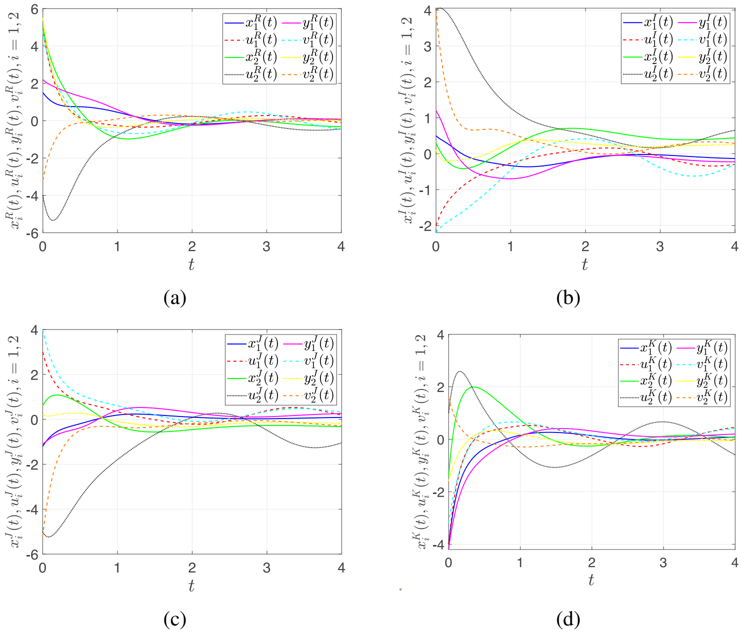

Figure 1. State trajectories of (29)–(30) without controller for $x_i^\theta(t)$, $u_i^\theta(t)$, $y_i^\theta(t)$, $v_i^\theta(t)$.

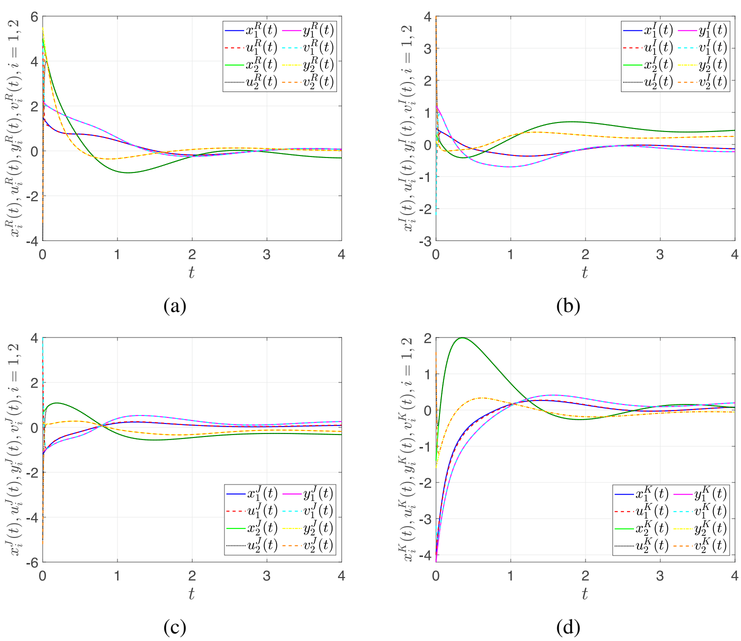

Figure 2. State trajectories of (29)–(30) with controller for $x_i^\theta(t)$, $u_i^\theta(t)$, $y_i^\theta(t)$, $v_i^\theta(t)$.

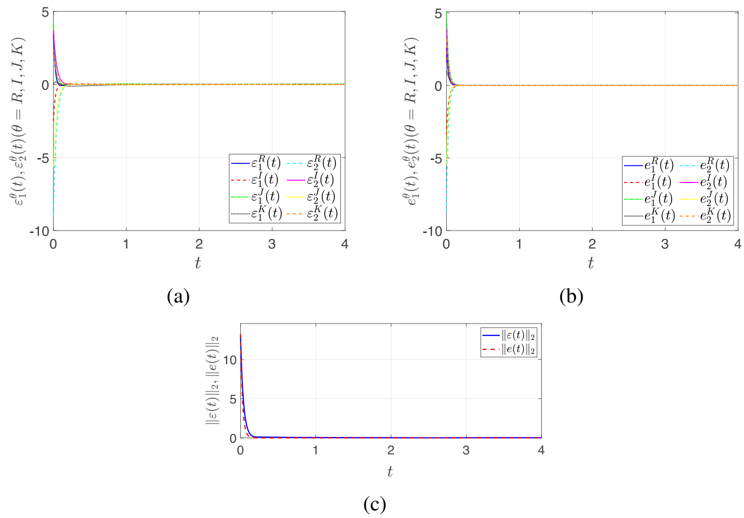

Figure 3. (a)–(b) Synchronization error of QVBAMNN. (c) Synchronization error for norm $\varphi = 2$.

and (30) achieved synchronization. The synchronization error is depicted in Fig. 3(a)–(b). Similarly, when $\varphi = 2$, the error norm is shown in Fig. 3(c).

Example 2. Consider model (29)–(30), as well as the parameters of Example 1. Here we simply change $\delta(t) = 0.2\sqrt{t}$ to $\delta(t) = 0.2\sin^2 t$ and $\tau(t) = 0.1\sqrt{t}$ to $\tau(t) = 0.1\sin^2 t$, thereby converting the unbounded time-varying delay to a bounded time-varying delay. Controller gain matrix $\omega = \left[\begin{smallmatrix} -75 & 2 \\ 3 & -78 \end{smallmatrix}\right], \tilde{\omega} = \left[\begin{smallmatrix} -76 & 2 \\ 1 & -75 \end{smallmatrix}\right]$. When $\varphi = 2$, system (29) satisfies Assumption 1. The other parameters can be calculated as $\mu_2(\omega - C) \approx -79.8$, $\mu_2(\tilde{\omega} - D) = -79.5$. From Example 1 we know that $k_1 = -\max(\mu_2(\omega - C) + \bar{\Gamma}\tilde{Q}, \mu_2(\tilde{\omega} - D) + \bar{L}\tilde{N}) = 45.12$, and $k_2 = \max(\bar{L}\tilde{M}, \bar{\Gamma}\tilde{P}) = 28$, i.e., $k_1 > k_2 > 0$, and we have calculated that $q = 2.142$ is the unique solution to equation $q = 45.12 - 28\mathrm{e}^{0.2q}$. According to Theorem 2, QVBAMNNs (29) and (30) are GES. State trajectories of variables $x_i^\theta(t), u_i^\theta(t)$, $y_i^\theta(t), v_i^\theta(t)$ $(i = 1, 2)$ without controller and with controller are shown in Figs. 4–5, respectively. Without controllers ω and $\tilde{\omega}$, the system does not synchronize (Fig. 4), whereas synchronization is achieved once control is applied (Fig. 5). The synchronization error is shown in Fig. 6(a)–(b), and the error norm for $\varphi = 2$ is given in Fig. 6(c).

Remark 5. When the norm $\varphi = 1$, based on Example 1, condition (14) satisfies

$$\begin{aligned} 0 &< 2\max(\bar{L}\tilde{M}, \bar{\Gamma}\tilde{P}) = 70.4 \\ &< -\max\bigl(\mu_1(\omega - C) + \bar{\Gamma}\tilde{Q},\ \mu_1(\tilde{\omega} - D) + \bar{L}\tilde{N}\bigr) = 77.4. \end{aligned}$$

In the case where the norm $\varphi = \infty$, condition (14) satisfies

$$\begin{aligned} 0 &< 2\max(\bar{L}\tilde{M}, \bar{\Gamma}\tilde{P}) = 72.8 \\ &< -\max\bigl(\mu_\infty(\omega - C) + \bar{\Gamma}\tilde{Q},\ \mu_\infty(\tilde{\omega} - D) + \bar{L}\tilde{N}\bigr) = 80. \end{aligned}$$

Similarly, condition (28) also holds true.

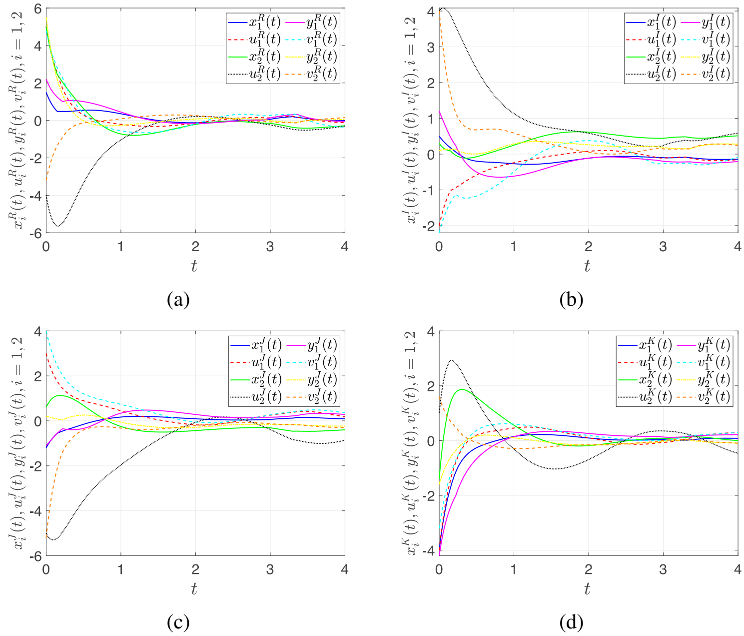

Figure 4. State trajectories of (29)–(30) without controller for $x_i^\theta(t)$, $u_i^\theta(t)$, $y_i^\theta(t)$, $v_i^\theta(t)$.

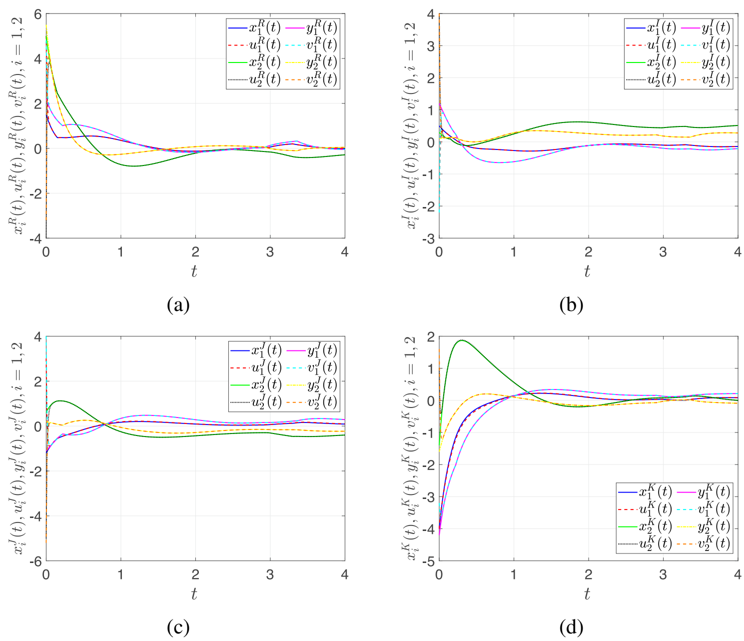

Figure 5. State trajectories of (29)–(30) with controller for $x_i^\theta(t)$, $u_i^\theta(t)$, $y_i^\theta(t)$, $v_i^\theta(t)$.

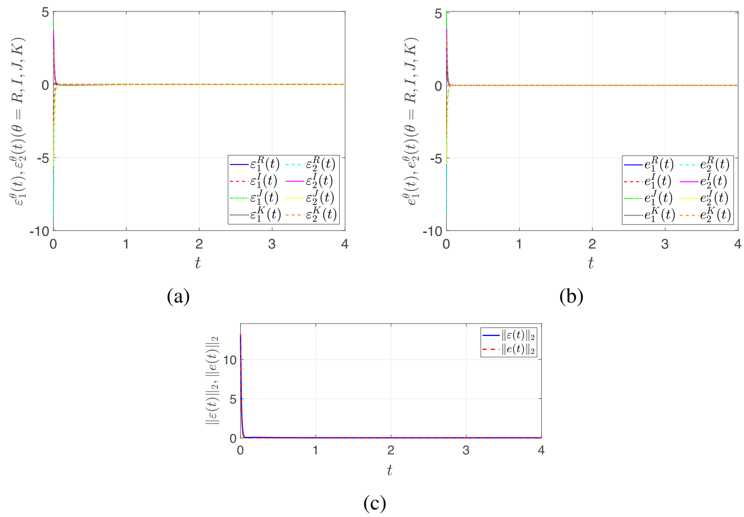

Figure 6. (a)–(b) Synchronization error of QVBAMNN. (c) Synchronization error for norm $\varphi = 2$.

5 Conclusions

This paper investigates the synchronization control problem for QVBAMNNs with time-varying delays. By decomposing the original system into equivalent RVNNs and designing state feedback controllers, based on both unbounded and bounded time-varying delay, using matrix measure approach and two forms of Halanay inequalities, sufficient conditions are established to ensure GAS and GES of the system. Finally, numerical simulations are used to verify the validity of the proposed theoretical results and the feasibility of the control strategy. Next, we will consider adopting methods such as adaptive controllers and sliding mode controllers to further explore the synchronization problem of QVNNs.

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