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Existence, uniqueness and averaging principle for fractional Dirichlet boundary value problems driven by Rosenblatt process and Lévy jumps

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Abstract. We address the existence, uniqueness, and averaging principle for Caputo–Hadamard fractional dynamic systems with Dirichlet boundary conditions driven by Rosenblatt process and pure Lévy jumps. First, Lemma 4 establishes the equivalent integral equation representation of our system. Using this foundation, existence and uniqueness are proved by Banach's contraction principle under stochastic calculus, Lipschitz and finite energy conditions. Subsequently, under appropriate averaging assumptions, the system is averaged out with time scale ϵ. Mean-square convergence between original solution and its counterpart is verified by employing tools such as Wiener–Itô double integral, Cauchy–Schwarz, Doob's martingale, and Gronwall–Bellman inequalities. Eventually, computational example with numerical simulations is provided to support the theoretical results.

Keywords: Caputo–Hadamard derivative, stochastic differential equation, averaging principle, Lévy jumps, Rosenblatt process.

1 Introduction

We aim to discuss the fractional stochastic differential equation with Rosenblatt process subject to small Lévy jumps as follows:

$$\begin{aligned} &{}^{CH}D_t^{\vartheta} u(t) + \lambda\, {}^{CH}D_t^{\vartheta-1} u(t) \\ &\quad = X\bigl(t, u(t)\bigr) + \sigma\bigl(t, u(t)\bigr)\frac{\mathrm{d}R_H(t)}{\mathrm{d}t} + \int\limits_{|y|<d} P\bigl(t, y, u(t)\bigr)\, \tilde{N}(\mathrm{d}t, \mathrm{d}y), \quad t \in \mathbb{J}, \end{aligned} \tag{1}$$

with Dirichlet boundary conditions

$$u(1) = \phi_1, \quad u(b) = \phi_b.$$

Here ${}^{CH}D_t^{\vartheta}$ and ${}^{CH}D_t^{\vartheta-1}$ denote the Caputo–Hadamard fractional derivatives of order $1 < \vartheta \leqslant 2$, $\lambda, \phi_1, \phi_b \in \mathbb{R}$, and $\mathbb{J} = [1, b]$ is a time interval. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space with a normal filtration $\{\mathcal{F}_t\}_{t\in[1,b]}$. Consider two real separable Hilbert spaces $\mathbb{Z}$ and $\mathbb{U}$. Let $R_H = \{R_H(t),\ t \geqslant 0\}$ be a $\mathbb{U}$-valued Rosenblatt process with Hurst parameter $H \in (1/2, 1)$. The state $u(\cdot)$ takes values in a real separable Hilbert space $\mathbb{Z}$ with norm $\|\cdot\|$. Let N be an $\mathcal{F}_t$-adapted Poisson random measure on $\mathbb{J} \times (\mathbb{R}^n \setminus \{0\})$, and let $\mathbb{Y} = \{y\colon\ |y| < d,\ d \in (0, \infty)\} \subset \mathbb{R}^n \setminus \{0\}$ denote the set of admissible jumps. Assume that $(\mathbb{Y}, \psi, \mu)$ is a σ-finite measurable space. The mappings $X : \mathbb{J} \times \mathbb{Z} \to \mathbb{Z}$, $\sigma : \mathbb{J} \times \mathbb{Z} \to L_0^2(\mathbb{U}, \mathbb{Z})$, and $P : \mathbb{J} \times \mathbb{Y} \times \mathbb{Z} \to \mathbb{Z}$ are measurable and continuous.

Fractional calculus has garnered significant interest for its ability to effectively model the dynamic behavior of systems exhibiting memory and hereditary characteristics. It is significantly effective in characterising memory-based models, which has led to a growing body of research [8, 16, 22, 27, 33]. Stochastic differential equations (SDEs) are powerful tools for addressing system problems influenced by random processes [14, 20]. Fractional systems with boundary constraints facilitate more precise simulations of real-world physical processes, encompassing systems with irregular geometries or complex boundary conditions. This advancement enhances the understanding of stress distribution and heat transfer in intricate, noisy environments. Additionally, such models are used in fluid dynamics, where fractional derivatives are important for capturing the memory effects seen in turbulent flows or in porous media, where randomness and jumps in the flow behavior are crucial to the model's accuracy [5, 7, 12, 19, 23, 26].

The averaging principle is extensively utilized to examine the behavior of dynamical systems and aids in comprehending their long-term by averaging fast oscillations. However, Khasminskii [15] was the first to introduce it in SDEs. His work made significant contributions to solving uncertain problems in stochastic analysis, particularly regarding the convergence of idle systems on the slow time scale as $\epsilon \to 0$. Following his pioneering work, Cerrai et al. [4] discussed the application to stochastic reaction–diffusion equations. [31] investigated the approximation properties of solutions to SDEs with non-Lipschitz coefficients. Later, Xu et al. [30] explored the averaging principle for SDEs involving the Caputo derivative, Cui et al. [6] examined averaging results for SDEs with impulses and non-Lipschitz coefficients.

On the other hand, the Rosenblatt process, a self-similar, non-Gaussian stochastic, is particularly effective for simulating situations characterized by a significant correlation between current and remote past values, owing to its long-range dependence [24, 28]. Lévy jumps are part of Lévy noise. Lévy noise is a subclass of non-Gaussian noises. Many fields, including physics, biology, and economics, exhibit random fluctuations with breaks [3]. The averaging principle for fractional SDEs has evolved across several dimensions: foundational work on fractional Brownian motion-driven SDEs [13] was extended to time-delayed systems [10], impulsive ψ-Hilfer systems with Poisson jumps [11], and Lévy noise [18, 29]. Recent advances address Poisson jumps [2], Rosenblatt process [9], and stability with Lévy noise [25].

The Caputo–Hadamard derivative, known by its logarithmic kernel, is specifically suited for modeling extremely slow diffusion process. Despite its importance, research on this operator is limited. Notably, Abbas et al. [1] established existence results for

Caputo–Hadamard differential equations, while Mouy et al. [21] studied the same systems with pantograph-type delay. Subsequently, Liu et al. [17] introduced a novel estimation technique to address the challenges posed by the logarithmic term.

The existing averaging results for fractional stochastic systems primarily cover Caputo-type derivatives driven by Brownian motion or Lévy noise [6, 18, 25, 29, 30]. Similarly, studies involving the Rosenblatt process have focused on evolution equations without boundary constraints [9]. For Caputo–Hadamard operators, the available works establish only existence, uniqueness, or stability for deterministic or Brownian-driven systems [1, 17, 21], and none of them incorporate non-Gaussian long-memory noise or jump discontinuities. To the best of our knowledge, there is no averaging principle available for fractional dynamics that simultaneously involve: (i) the Caputo–Hadamard derivative, (ii) Rosenblatt process and pure Lévy jumps, and (iii) Dirichlet boundary conditions. Therefore, the framework developed in this manuscript fills a clear gap in the literature and extends the scope of averaging theory to a class of non-Gaussian, memory-dependent boundary value problems that were previously untreated.

The primary contributions of this manuscript are as follows: A novel boundary value framework for fractional dynamic system, delivering slow diffusion process, is developed. The main results of this manuscript are the establishment of existence and uniqueness results in Lemma 4 via the Banach fixed point theorem, as well as the agreement between the original solution and its averaged counterpart in the mean-square sense in Theorem 2. Finally, numerical simulations are performed for better understanding of theoretical results.

2 Preliminaries

Let N be a $\mathcal{F}_t$-adapted Poisson random measure that is independent of Rosenblatt process R_H, and let $\tilde{N}$ denote its compensated Poisson random measure defined as

$$\tilde{N}(\mathrm{d}t, \mathrm{d}y) := N(\mathrm{d}t, \mathrm{d}y) - \mu(\mathrm{d}y)\mathrm{d}t,$$

where μ is a Lévy measure satisfying $\int_{\mathbb{R}^n\setminus\{0\}}(|y|^2 \wedge 1)\mu\,\mathrm{d}y < \infty$, i.e.

$$\int\limits_{\mathbb{R}^n\setminus\{0\}} \frac{y^2}{1+y^2}\mu\,\mathrm{d}y < \infty,$$

and $\int_{|y|<d} P(t, y, u(t))\,\tilde{N}(\mathrm{d}t, \mathrm{d}y)$ is called small jumps. Let $(\mathbb{U}, \|\cdot\|)$ and $(\mathbb{Z}, \|\cdot\|)$ be two real separable Hilbert spaces. Let $L(\mathbb{U}, \mathbb{Z})$ represent the family of all bounded linear operators from $\mathbb{U}$ to $\mathbb{Z}$, equipped with the operator norm. Furthermore, let $L_0^2(Q^{1/2}\mathbb{U}, \mathbb{Z})$ signify the separable Hilbert space of all Hilbert–Schmidt operators from $Q^{1/2}\mathbb{U}$ into $\mathbb{Z}$.

Definition 1. (See [28].) Let $\eta : \mathbb{J} \to L_0^2(Q^{1/2}\mathbb{U}, \mathbb{Z})$ be such that

$$\sum_{j=1}^{\infty} \left\| K_H^*\left(\eta\,Q^{1/2}e_j\right)\right\|_{L^2(\mathbb{J},\mathbb{Z})} < \infty.$$

It's stochastic integral depending upon the Rosenblatt process is defined as

$$\begin{aligned}\int_0^t \eta(\omega)\,\mathrm{d}R_Q(\omega) &= \sum_{j=1}^{\infty}\int_0^t \eta(\omega)Q^{1/2}e_j\,\mathrm{d}r_j(p)\\ &= \sum_{j=1}^{\infty}\int_0^t\int_0^t \bigl(K_H^*\bigl(\eta\,Q^{1/2}e_j\bigr)\bigr)(y_1,y_2)\,\mathrm{d}\mathcal{W}_1(y_1)\,\mathrm{d}\mathcal{W}_1(y_2),\end{aligned}$$

where $K_H^*:\mathbb{J}\to L^2([1,b])$ is an operator, $r_j(t)$ – sequence of two-sided, one-dimensional Rosenblatt process, and $\{e_j,\ j=1,2,\dots\}$ is a complete orthonormal basis.

Lemma 1. (*See* [28].) *Let* $\eta:\mathbb{J}\to L_0^2(Q^{1/2}\mathbb{U},\mathbb{Z})$ *be such that*

$$\sum_{j=1}^{\infty}\bigl\|\eta Q^{1/2}e_j\bigr\|_{L^{1/H}(\mathbb{J},\mathbb{Z})}<\infty.$$

Then, for any $r,s\in\mathbb{J}$ *with* $s>r$*, we have*

$$\mathbf{E}\Biggl\|\int_r^s \eta(\omega)\,\mathrm{d}R_Q(\omega)\Biggr\|^2 \leqslant C_H(s-r)^{2H-1}\sum_{j=1}^{\infty}\int_r^s \bigl\|\eta(\omega)Q^{1/2}e_j\bigr\|^2\,\mathrm{d}\omega.$$

If, in addition, $\sum_{j=1}^{\infty}\|\eta Q^{1/2}e_j\|$ *converges uniformly for* $t\in\mathbb{J}$*, then it fulfils*

$$\mathbf{E}\Biggl\|\int_r^s \eta(\omega)\,\mathrm{d}R_Q(\omega)\Biggr\|^2 \leqslant C_H(s-r)^{2H-1}\int_r^s \bigl\|\eta(\omega)\bigr\|_{L_0^2}^2\,\mathrm{d}\omega.$$

Definition 2. (See [1].) The Hadamard fractional integral of order $\vartheta>0$ of a function u is expressed as

$$I^{\vartheta}u(t)=\frac{1}{\Gamma(\vartheta)}\int_1^t\biggl(\ln\frac{t}{s}\biggr)^{\vartheta-1}\frac{u(s)}{s}\,\mathrm{d}s,\quad t>1.$$

Definition 3. (See [1].) The Hadamard fractional derivative of order $m-1<\vartheta\leqslant m$ for u is expressed as

$$\begin{aligned}{}^{H}D_t^{\vartheta}u(t) &= \biggl(t\frac{\mathrm{d}}{\mathrm{d}t}\biggr)^{m}\,{}^{H}I^{m-\vartheta}u(t)\\ &= \frac{1}{\Gamma(m-\vartheta)}\biggl(t\frac{\mathrm{d}}{\mathrm{d}t}\biggr)^{m}\int_1^t\biggl(\ln\frac{t}{s}\biggr)^{m-\vartheta-1}\frac{u(s)}{s}\,\mathrm{d}s,\end{aligned}$$

where $m=[\vartheta]+1$, $[\vartheta]$ – integer part of ϑ.

Definition 4. (See [1].) The Caputo–Hadamard fractional derivative of order $m-1<\vartheta\leqslant m$ of u is expressed as

$$^{CH}D_t^{\vartheta}u(t)=\frac{1}{\Gamma(m-\vartheta)}\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^m\int\limits_1^t\left(\ln\frac{t}{s}\right)^{m-\vartheta-1}\frac{u(s)}{s}\,\mathrm{d}s.$$

Lemma 2. (*See* [16, 21].) *Let* $m-1<\vartheta\leqslant m$, $m\in\mathbb{N}$ *and* $u\in C^{m-1}[1,\infty)$. *Then*

$$^{H}I^{\vartheta}\left[^{CH}D_t^{\vartheta}u(s)\right]=u(s)-\sum_{k=0}^{m-1}c_k\ln^k s,\quad c_k=\frac{1}{\Gamma(k+1)}\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^k u(t)\bigg|_{t=1}\in\mathbb{R}.$$

Lemma 3. (*See* [16, 21].) *Let* $1<\vartheta<2$. *Then*

$$^{H}I_{1^+}^{\vartheta}\,{}^{CH}D_{1^+}^{\vartheta-1}u(s)={}^{H}I_{1^+}u(s)-\frac{c_0\ln s}{\Gamma(2)}.$$

3 Existence and uniqueness results

The following lemma serves as a foundational step toward the subsequent existence and uniqueness analysis for system (1).

Lemma 4. *Let* $u\in\mathbb{Z}$. *The Hadamard fractional integral equation of* (1) *is given by*

$$\begin{aligned}
u(t)={}&(\lambda\ln t+1)\phi_1+\frac{\ln t}{\ln b}\Bigg[\phi_b-(\lambda\ln b+1)\phi_1+\lambda\int\limits_1^b\frac{u(s)}{s}\,\mathrm{d}s\\
&-\frac{1}{\Gamma(\vartheta)}\int\limits_1^b\left(\ln\frac{b}{s}\right)^{\vartheta-1}\frac{X(s,u(s))}{s}\,\mathrm{d}s-\frac{1}{\Gamma(\vartheta)}\int\limits_1^b\left(\ln\frac{b}{s}\right)^{\vartheta-1}\frac{\sigma(s,u(s))}{s}\,\mathrm{d}R_H(s)\\
&-\frac{1}{\Gamma(\vartheta)}\int\limits_1^b\left(\ln\frac{b}{s}\right)^{\vartheta-1}\int\limits_{\mathbb{Y}}\frac{P(s,y,u(s))}{s}\,\tilde{N}(\mathrm{d}s,\mathrm{d}y)\Bigg]-\lambda\int\limits_1^t\frac{u(s)}{s}\,\mathrm{d}s\\
&+\frac{1}{\Gamma(\vartheta)}\int\limits_1^t\left(\ln\frac{t}{s}\right)^{\vartheta-1}\frac{X(s,u(s))}{s}\,\mathrm{d}s+\frac{1}{\Gamma(\vartheta)}\int\limits_1^t\left(\ln\frac{t}{s}\right)^{\vartheta-1}\frac{\sigma(s,u(s))}{s}\,\mathrm{d}R_H(s)\\
&+\frac{1}{\Gamma(\vartheta)}\int\limits_1^t\left(\ln\frac{t}{s}\right)^{\vartheta-1}\int\limits_{\mathbb{Y}}\frac{P(s,y,u(s))}{s}\,\tilde{N}(\mathrm{d}s,\mathrm{d}y).
\end{aligned}$$

Proof. Multiplying both sides of Eq. (1) by $^{H}I_{1^+}^{\vartheta}$, we obtain

$$\begin{aligned}
&^{H}I_{1^+}^{\vartheta}\left(^{CH}D_{1^+}^{\vartheta}u\right)(t)+\lambda\,{}^{H}I_{1^+}^{\vartheta}\left(^{CH}D_{1^+}^{\vartheta-1}u\right)(t)\\
&\quad={}^{H}I_{1^+}^{\vartheta}\left(X\bigl(t,u(t)\bigr)+\sigma\bigl(t,u(t)\bigr)\frac{\mathrm{d}R_H(t)}{\mathrm{d}t}+\int\limits_{|y|<d}P\bigl(t,y,u(t)\bigr)\,\tilde{N}(\mathrm{d}t,\mathrm{d}y)\right).
\end{aligned}$$

By using Lemma 2, we have

$$
\begin{aligned}
&u(t) - c_0 - c_1 \log t + \lambda\, {}^{H}I_{1^+}^{\vartheta}\bigl({}^{CH}D_{1^+}^{\vartheta-1}u\bigr)(t) \\
&\quad = {}^{H}I_{1^+}^{\vartheta}\biggl(X\bigl(t,u(t)\bigr) + \sigma\bigl(t,u(t)\bigr)\frac{\mathrm{d}R_H(t)}{\mathrm{d}t} + \int\limits_{|y|<d} P\bigl(t,y,u(t)\bigr)\,\tilde{N}(\mathrm{d}t,\mathrm{d}y)\biggr).
\end{aligned}
$$

Now, to calculate ${}^{H}I_{1^+}^{\vartheta}({}^{CH}D_{1^+}^{\vartheta-1}u)(t)$, we use Lemma 3.

$$
\begin{aligned}
u(t) &= (\lambda \ln t + 1)c_0 + c_1 \ln t - \lambda \int\limits_1^t \frac{u(s)}{s}\,\mathrm{d}s + \frac{1}{\Gamma(\vartheta)}\int\limits_1^t \left(\ln\frac{t}{s}\right)^{\vartheta-1}\frac{X(s,u(s))}{s}\,\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\vartheta)}\int\limits_1^t \left(\ln\frac{t}{s}\right)^{\vartheta-1}\frac{\sigma(s,u(s))}{s}\,\mathrm{d}R_H(s) \\
&\quad + \frac{1}{\Gamma(\vartheta)}\int\limits_1^t \left(\ln\frac{t}{s}\right)^{\vartheta-1}\int\limits_{\mathbb{Y}}\frac{P(s,y,u(s))}{s}\,\tilde{N}(\mathrm{d}s,\mathrm{d}y). \qquad (2)
\end{aligned}
$$

By using the boundary condition $u(1) = \phi_1$, we get $c_0 = \phi_1$, and by using $u(b) = \phi_b$, we deduce that

$$
\begin{aligned}
\phi_b &= \bigl(\lambda(\ln b) + 1\bigr)\phi_1 + c_1 \ln b - \lambda \int\limits_1^b \frac{u(s)}{s}\,\mathrm{d}s + \frac{1}{\Gamma(\vartheta)}\int\limits_1^b \left(\ln\frac{b}{s}\right)^{\vartheta-1}\frac{X(s,u(s))}{s}\,\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\vartheta)}\int\limits_1^b \left(\ln\frac{b}{s}\right)^{\vartheta-1}\frac{\sigma(s,u(s))}{s}\,\mathrm{d}R_H(s) \\
&\quad + \frac{1}{\Gamma(\vartheta)}\int\limits_1^b \left(\ln\frac{b}{s}\right)^{\vartheta-1}\int\limits_{\mathbb{Y}}\frac{P(s,y,u(s))}{s}\,\tilde{N}(\mathrm{d}s,\mathrm{d}y).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
c_1 &= \frac{1}{\ln b}\Biggl[\phi_b - \bigl(\lambda(\ln b) + 1\bigr)\phi_1 + \lambda \int\limits_1^b \frac{u(s)}{s}\,\mathrm{d}s - \frac{1}{\Gamma(\vartheta)}\int\limits_1^b \left(\ln\frac{b}{s}\right)^{\vartheta-1}\frac{X(s,u(s))}{s}\,\mathrm{d}s \\
&\quad - \frac{1}{\Gamma(\vartheta)}\int\limits_1^b \left(\ln\frac{b}{s}\right)^{\vartheta-1}\frac{\sigma(s,u(s))}{s}\,\mathrm{d}R_H(s) \\
&\quad - \frac{1}{\Gamma(\vartheta)}\int\limits_1^b \left(\ln\frac{b}{s}\right)^{\vartheta-1}\int\limits_{\mathbb{Y}}\frac{P(s,y,u(s))}{s}\,\tilde{N}(\mathrm{d}s,\mathrm{d}y)\Biggr].
\end{aligned}
$$

By substituting the values of c_0 and c_1 in Eq. (2), we get

$$\begin{aligned}
u(t) &= (\lambda \ln t + 1)\phi_1 + \frac{\ln t}{\ln b}\Biggl[\phi_b - (\lambda \ln b + 1)\phi_1 + \lambda \int_1^b \frac{u(s)}{s}\,\mathrm{d}s \\
&\quad - \frac{1}{\Gamma(\vartheta)}\int_1^b \left(\ln\frac{b}{s}\right)^{\vartheta-1}\frac{X(s,u(s))}{s}\,\mathrm{d}s - \frac{1}{\Gamma(\vartheta)}\int_1^b \left(\ln\frac{b}{s}\right)^{\vartheta-1}\frac{\sigma(s,u(s))}{s}\,\mathrm{d}R_H(s) \\
&\quad - \frac{1}{\Gamma(\vartheta)}\int_1^b \left(\ln\frac{b}{s}\right)^{\vartheta-1}\int_{\mathbb{Y}}\frac{P(s,y,u(s))}{s}\,\tilde{N}(\mathrm{d}s,\mathrm{d}y)\Biggr] - \lambda\int_1^t \frac{u(s)}{s}\,\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\vartheta)}\int_1^t \left(\ln\frac{t}{s}\right)^{\vartheta-1}\frac{X(s,u(s))}{s}\,\mathrm{d}s + \frac{1}{\Gamma(\vartheta)}\int_1^t \left(\ln\frac{t}{s}\right)^{\vartheta-1}\frac{\sigma(s,u(s))}{s}\,\mathrm{d}R_H(s) \\
&\quad + \frac{1}{\Gamma(\vartheta)}\int_1^t \left(\ln\frac{t}{s}\right)^{\vartheta-1}\int_{\mathbb{Y}}\frac{P(s,y,u(s))}{s}\,\tilde{N}(\mathrm{d}s,\mathrm{d}y).
\end{aligned}$$

Hence the desired result follows. □

The solution of system (1) can be defined as follows.

Definition 5. An $\mathcal{F}_t$-valued stochastic process $\{u(t),\, t \in [1,b]\}$ is called a unique solution of system (1) if $u(t)$ satisfies the following integral equation:

$$\begin{aligned}
u(t) &= (\lambda \ln t + 1)\phi_1 + \frac{\ln t}{\ln b}\Biggl[\phi_b - (\lambda \ln b + 1)\phi_1 + \lambda \int_1^b \frac{u(s)}{s}\,\mathrm{d}s \\
&\quad - \frac{1}{\Gamma(\vartheta)}\int_1^b \left(\ln\frac{b}{s}\right)^{\vartheta-1}\frac{X(s,u(s))}{s}\,\mathrm{d}s - \frac{1}{\Gamma(\vartheta)}\int_1^b \left(\ln\frac{b}{s}\right)^{\vartheta-1}\frac{\sigma(s,u(s))}{s}\,\mathrm{d}R_H(s) \\
&\quad - \frac{1}{\Gamma(\vartheta)}\int_1^b \left(\ln\frac{b}{s}\right)^{\vartheta-1}\int_{\mathbb{Y}}\frac{P(s,y,u(s))}{s}\,\tilde{N}(\mathrm{d}s,\mathrm{d}y)\Biggr] - \lambda\int_1^t \frac{u(s)}{s}\,\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\vartheta)}\int_1^t \left(\ln\frac{t}{s}\right)^{\vartheta-1}\frac{X(s,u(s))}{s}\,\mathrm{d}s + \frac{1}{\Gamma(\vartheta)}\int_1^t \left(\ln\frac{t}{s}\right)^{\vartheta-1}\frac{\sigma(s,u(s))}{s}\,\mathrm{d}R_H(s) \\
&\quad + \frac{1}{\Gamma(\vartheta)}\int_1^t \left(\ln\frac{t}{s}\right)^{\vartheta-1}\int_{\mathbb{Y}}\frac{P(s,y,u(s))}{s}\,\tilde{N}(\mathrm{d}s,\mathrm{d}y). \qquad (3)
\end{aligned}$$

We impose the necessary conditions to X, σ, and P.

Hypothesis 1. For any $u_1, u_2 \in \mathbb{Z}$ and $t \in \mathbb{J}$, there exists $C_1, C_2 > 0$ such that

(i) $\|X(t,u_1)\|^2 \vee \|\sigma(t,u_1)\|^2 \vee \|P(t,u_1)\|^2 \leqslant C_1(1+\|u_1\|^2)$,

(ii) $\|X(t,u_1)-X(t,u_2)\|^2 \vee \|\sigma(t,u_1)-\sigma(t,u_2)\|^2 \vee \int_{\mathbb{Y}} \|P(t,u_1)-P(t,u_2)\|^2 \mu\,\mathrm{d}y \leqslant C_2\|u_1-u_2\|^2$,

where $u_1 \vee u_2 = \max\{u_1, u_2\}$.

Hypothesis 2. For any $\mathcal{F}_t$-adapted solutions u and w, there exists a constant $C_e > 0$ such that

$$\mathbf{E}\left[\int_1^b \|u(s)-w(s)\|_{\mathbb{Z}}^2\,\mathrm{d}s\right] \leqslant C_e^2.$$

Define

$$\mathbb{B} := \left\{u\colon [1,b] \to L^2(\Omega;\mathbb{Z}) \;\Big|\; \sup_{t\in[1,b]} \mathbf{E}\|u(t)\|_{\mathbb{Z}}^2 < \infty\right\}$$

equipped with the norm

$$\|u\|_{\mathbb{B}} := \sup_{t\in[1,b]} \left(\mathbf{E}\|u(t)\|_{\mathbb{Z}}^2\right)^{1/2}.$$

Then $\mathbb{B}$ is a Banach space. Define an operator $\mathcal{L} : \mathbb{B} \to \mathbb{B}$ by

$$\begin{aligned}
&\mathcal{L}(u(t)) \\
&\quad= (\lambda \ln t + 1)\phi_1 + \frac{\ln t}{\ln b}\left[\phi_b - (\lambda \ln b + 1)\phi_1 + \lambda \int_1^b \frac{u(s)}{s}\,\mathrm{d}s\right. \\
&\qquad - \frac{1}{\Gamma(\vartheta)} \int_1^b \left(\ln\frac{b}{s}\right)^{\vartheta-1} \frac{X(s,u(s))}{s}\,\mathrm{d}s - \frac{1}{\Gamma(\vartheta)} \int_1^b \left(\ln\frac{b}{s}\right)^{\vartheta-1} \frac{\sigma(s,u(s))}{s}\,\mathrm{d}R_H(s) \\
&\qquad \left.- \frac{1}{\Gamma(\vartheta)} \int_1^b \left(\ln\frac{b}{s}\right)^{\vartheta-1} \int_{\mathbb{Y}} \frac{P(s,y,u(s))}{s}\,\tilde{N}(\mathrm{d}s,\mathrm{d}y)\right] - \lambda \int_1^t \frac{u(s)}{s}\,\mathrm{d}s \\
&\qquad + \frac{1}{\Gamma(\vartheta)} \int_1^t \left(\ln\frac{t}{s}\right)^{\vartheta-1} \frac{X(s,u(s))}{s}\,\mathrm{d}s + \frac{1}{\Gamma(\vartheta)} \int_1^t \left(\ln\frac{t}{s}\right)^{\vartheta-1} \frac{\sigma(s,u(s))}{s}\,\mathrm{d}R_H(s) \\
&\qquad + \frac{1}{\Gamma(\vartheta)} \int_1^t \left(\ln\frac{t}{s}\right)^{\vartheta-1} \int_{\mathbb{Y}} \frac{P(s,y,u(s))}{s}\,\tilde{N}(\mathrm{d}s,\mathrm{d}y).
\end{aligned}$$

We aim to show that the operator $\mathcal{L}$ is a contraction map.

Lemma 5. *Under Hypothesis* 1, *the operator* $\mathcal{L}$ *satisfies*

$$\left\|\mathcal{L}(u)-\mathcal{L}(w)\right\|_{\mathbb{B}} \leqslant \varkappa\|u-w\|_{\mathbb{B}} \quad \textit{for some } \varkappa \in (0,1),$$

where $\varkappa$ *is given by*

$$\varkappa := 4\left(\lambda^2 \ln^2 b + \frac{C_2 \ln b^{2\vartheta-1}}{(2\vartheta-1)\Gamma^2(\vartheta)}\left(C_H(b-1)^{2H-1}+2\right)\right)^{1/2}.$$

Proof. Using the inequality

$$\sup_{1\leqslant t\leqslant b}\left(\frac{\ln t}{\ln b}\right)^2 \leqslant 1 \tag{4}$$

and letting $\Delta u := u - w$, we estimate the difference $\mathcal{L}(u)-\mathcal{L}(w)$ by grouping similar terms:

$$\mathbf{E}\left\|\mathcal{L}(u)-\mathcal{L}(w)\right\|_{\mathbb{B}}^2 \leqslant 8\sum_{i=1}^{4}\mathbf{E}\|J_i\|_{\mathbb{B}}^2.$$

For

$$J_1 = -\lambda\int_1^t \frac{\Delta u(s)}{s}\,\mathrm{d}s + \frac{\lambda \ln t}{\ln b}\int_1^b \frac{\Delta u(s)}{s}\,\mathrm{d}s,$$

using the Cauchy–Schwarz inequality and inequality (4),

$$\mathbf{E}\|J_1\|^2 \leqslant \lambda^2\left[\ln^2 b + \ln^2 b\right]\|\Delta u\|_{\mathbb{B}}^2 = 2\lambda^2\ln^2 b\|\Delta u\|_{\mathbb{B}}^2.$$

For

$$J_2 = \frac{1}{\Gamma(\vartheta)}\int_1^t\left(\ln\frac{t}{s}\right)^{\vartheta-1}\frac{\Delta X}{s}\,\mathrm{d}s + \frac{\ln t}{\ln b}\frac{1}{\Gamma(\vartheta)}\int_1^b\left(\ln\frac{b}{s}\right)^{\vartheta-1}\frac{\Delta X}{s}\,\mathrm{d}s,$$

by Cauchy–Schwarz inequality,

$$\mathbf{E}\left\|\int_1^t\left(\ln\frac{t}{s}\right)^{\vartheta-1}\frac{\Delta X}{s}\,\mathrm{d}s\right\|^2 \leqslant \int_1^t\left(\ln\frac{t}{s}\right)^{2\vartheta-2}\frac{\mathrm{d}s}{s}\int_1^t\frac{\mathbf{E}\|\Delta X\|^2}{s}\,\mathrm{d}s.$$

Using Hypothesis 1(ii) and the change of variables $r = \ln(t/s)$,

$$\int_1^t\left(\ln\frac{t}{s}\right)^{2\vartheta-2}\frac{\mathrm{d}s}{s} = \int_0^{\ln t} r^{2\vartheta-2}\,\mathrm{d}r \leqslant \frac{(\ln b)^{2\vartheta-1}}{2\vartheta-1},$$

we get

$$\mathbf{E}\left\|\int_1^t\left(\ln\frac{t}{s}\right)^{\vartheta-1}\frac{\Delta X}{s}\,\mathrm{d}s\right\|^2 \leqslant \frac{C_2(\ln b)^{2\vartheta-1}}{2\vartheta-1}\|\Delta u\|_{\mathbb{B}}^2.$$

The same bound holds for the integral over $[1,b]$, and using inequality (4), we obtain

$$\mathbf{E}\|J_2\|^2 \leqslant \frac{C_2}{\Gamma^2(\vartheta)}\left[\frac{(\ln b)^{2\vartheta-1}}{2\vartheta-1}+\frac{(\ln b)^{2\vartheta-1}}{2\vartheta-1}\right]\|\Delta u\|_{\mathbb{B}}^2 = \frac{2C_2(\ln b)^{2\vartheta-1}}{\Gamma^2(\vartheta)(2\vartheta-1)}\|\Delta u\|_{\mathbb{B}}^2.$$

Thus,

$$\mathbf{E}\|J_2\|^2 \leqslant \frac{2C_2(\ln b)^{2\vartheta-1}}{\Gamma^2(\vartheta)(2\vartheta-1)}\|\Delta u\|_{\mathbb{B}}^2.$$

For

$$J_3 = \frac{1}{\Gamma(\vartheta)}\int_1^t\left(\ln\frac{t}{s}\right)^{\vartheta-1}\frac{\Delta\sigma}{s}\,\mathrm{d}R_H(s) + \frac{\ln t}{\ln b}\frac{1}{\Gamma(\vartheta)}\int_1^b\left(\ln\frac{b}{s}\right)^{\vartheta-1}\frac{\Delta\sigma}{s}\,\mathrm{d}R_H(s),$$

by Lemma 1,

$$\mathbf{E}\left\|\int_1^t\left(\ln\frac{t}{s}\right)^{\vartheta-1}\frac{\Delta\sigma}{s}\,\mathrm{d}R_H(s)\right\|^2 \leqslant C_H(b-1)^{2H-1}\int_1^t\left(\ln\frac{t}{s}\right)^{2\vartheta-2}\frac{\mathbf{E}\|\Delta\sigma\|^2}{s^2}\,\mathrm{d}s.$$

Using Hypothesis 1(ii) and the change of variables $r=\ln(t/s)$, we get

$$\int_1^t\left(\ln\frac{t}{s}\right)^{2\vartheta-2}\frac{\mathrm{d}s}{s^2} = \int_0^{\ln t} r^{2\vartheta-2}\mathrm{e}^{-r}\,\mathrm{d}r \leqslant \int_0^{\ln b} r^{2\vartheta-2}\,\mathrm{d}r \leqslant \frac{(\ln b)^{2\vartheta-1}}{2\vartheta-1}.$$

Hence,

$$\mathbf{E}\left\|\int_1^t\left(\ln\frac{t}{s}\right)^{\vartheta-1}\frac{\Delta\sigma}{s}\,\mathrm{d}R_H(s)\right\|^2 \leqslant \frac{C_H(b-1)^{2H-1}C_2(\ln b)^{2\vartheta-1}}{2\vartheta-1}\|\Delta u\|_{\mathbb{B}}^2.$$

An analogous argument for the integral over $[1,b]$, together with inequality (4), gives

$$\begin{aligned}
&\mathbf{E}\|J_3\|^2 \\
&\quad\leqslant \frac{C_2}{\Gamma^2(\vartheta)}\left[\frac{C_H(b-1)^{2H-1}(\ln b)^{2\vartheta-1}}{2\vartheta-1}+\frac{C_H(b-1)^{2H-1}(\ln b)^{2\vartheta-1}}{2\vartheta-1}\right]\|\Delta u\|_{\mathbb{B}}^2 \\
&\quad= \frac{2C_H(b-1)^{2H-1}C_2(\ln b)^{2\vartheta-1}}{\Gamma^2(\vartheta)(2\vartheta-1)}\|\Delta u\|_{\mathbb{B}}^2.
\end{aligned}$$

For

$$\begin{aligned}
J_4 &= \frac{1}{\Gamma(\vartheta)}\int_1^t\left(\ln\frac{t}{s}\right)^{\vartheta-1}\int_{\mathbb{Y}}\frac{\Delta P}{s}\,\tilde{N}(\mathrm{d}s,\mathrm{d}y) \\
&\quad+ \frac{\ln t}{\ln b}\frac{1}{\Gamma(\vartheta)}\int_1^b\left(\ln\frac{b}{s}\right)^{\vartheta-1}\int_{\mathbb{Y}}\frac{\Delta P}{s}\,\tilde{N}(\mathrm{d}s,\mathrm{d}y),
\end{aligned}$$

we first apply Itô isometry and inequality (4) to get

$$\mathbf{E}\|J_4\|^2 \leqslant \frac{2}{\Gamma^2(\vartheta)}\Biggl[\mathbf{E}\int_1^t \biggl(\ln\frac{t}{s}\biggr)^{2(\vartheta-1)} \int_{\mathbb{Y}} \frac{\|\Delta P\|^2}{s^2}\,\mu(\mathrm{d}y)\,\mathrm{d}s \\ + \sup_{1\leqslant t\leqslant b}\biggl(\frac{\ln t}{\ln b}\biggr)^2 \mathbf{E}\int_1^b \biggl(\ln\frac{b}{s}\biggr)^{2(\vartheta-1)} \int_{\mathbb{Y}} \frac{\|\Delta P\|^2}{s^2}\,\mu(\mathrm{d}y)\,\mathrm{d}s\Biggr].$$

By Hypothesis 1(ii),

$$\int_{\mathbb{Y}} \|\Delta P\|^2\,\mu(\mathrm{d}y) \leqslant C_2\bigl\|\Delta u(s)\bigr\|^2,$$

and since $s \in [1, b]$, we have $s^{-2} \leqslant 1$. Hence,

$$\mathbf{E}\|J_4\|^2 \leqslant \frac{2C_2}{\Gamma^2(\vartheta)}\Biggl[\int_1^b \biggl(\ln\frac{b}{s}\biggr)^{2(\vartheta-1)} \mathrm{d}s + \int_1^b \biggl(\ln\frac{b}{s}\biggr)^{2(\vartheta-1)} \mathrm{d}s\Biggr]\|\Delta u\|_{\mathbb{B}}^2 \\ = \frac{4C_2}{\Gamma^2(\vartheta)}\int_1^b \biggl(\ln\frac{b}{s}\biggr)^{2\vartheta-2} \mathrm{d}s\,\|\Delta u\|_{\mathbb{B}}^2.$$

Using the change of variables $r = \ln(b/s)$, which yields $r \in [0, \ln b]$,

$$\int_1^b \biggl(\ln\frac{b}{s}\biggr)^{2\vartheta-2} \mathrm{d}s = \int_0^{\ln b} r^{2\vartheta-2}\mathrm{e}^{-r}\,\mathrm{d}r \leqslant \int_0^{\ln b} r^{2\vartheta-2}\,\mathrm{d}r = \frac{(\ln b)^{2\vartheta-1}}{2\vartheta-1},$$

we finally obtain

$$\mathbf{E}\|J_4\|^2 \leqslant \frac{2C_2(\ln b)^{2\vartheta-1}}{\Gamma^2(\vartheta)(2\vartheta-1)}\|\Delta u\|_{\mathbb{B}}^2.$$

Combining all estimates, we have

$$\|\mathcal{L}(u) - \mathcal{L}(w)\|_{\mathbb{B}} \\ \leqslant 4\biggl(\lambda^2\ln^2 b + \frac{C_2\ln b^{2\vartheta-1}}{(2\vartheta-1)\Gamma^2(\vartheta)}\bigl(C_H(b-1)^{2H-1} + 2\bigr)\biggr)^{1/2}\|\Delta u\|_{\mathbb{B}} \\ = \varkappa\|\Delta u\|_{\mathbb{B}}.$$

Hence the desired result follows. □

Theorem 1. *Assume that the operator $\mathcal{L} : \mathbb{B} \to \mathbb{B}$ is well-defined and $\varkappa < 1$. Then*:

(i) *there exists a unique $u^* \in \mathbb{B}$ such that $u^* = \mathcal{L}(u^*)$;*

(ii) *for any initial element $u_0 \in \mathbb{B}$, the sequence $u_{n+1} := \mathcal{L}(u_n)$, $n \geqslant 0$, converges in $\mathbb{B}$ to u^*.*

Proof. Let $u_{n+1} := \mathcal{L}(u_n)$ be defined recursively. By Lemma 5,

$$\|u_{n+1} - u_n\|_{\mathbb{B}} \leqslant \varkappa\,\|u_n - u_{n-1}\|_{\mathbb{B}} \leqslant \cdots \leqslant \varkappa^n\|\mathcal{L}(u_0) - u_0\|_{\mathbb{B}}.$$

Hence, $\{u_n\}$ is a Cauchy sequence in the Banach space $\mathbb{B}$ and converges to some $u^* \in \mathbb{B}$.

Passing to the limit in $u_{n+1} = \mathcal{L}(u_n)$ gives the fixed-point identity $u^* = \mathcal{L}(u^*)$.

To prove uniqueness, let u^* and w^* be fixed points. Then

$$\|u^* - w^*\|_{\mathbb{B}} = \big\|\mathcal{L}(u^*) - \mathcal{L}(w^*)\big\|_{\mathbb{B}} \leqslant \varkappa\|u^* - w^*\|_{\mathbb{B}}.$$

Since $\varkappa < 1$, this implies $u^* = w^*$. □

4 Averaging principle

In this section, we will study the averaging result for our system (1). First, we reformulate the integral equation (3) in the following more convenient form with time scale $\epsilon \in (0, 1]$:

$$\begin{aligned} u_\epsilon(t) &= (\lambda \ln t + 1)\phi_1 + \frac{\ln t}{\ln b}\Bigg[\phi_b - (\lambda \ln b + 1)\phi_1 + \lambda \int_1^b \frac{u(s)}{s}\,\mathrm{d}s \\ &\quad - \frac{1}{\Gamma(\vartheta)} \int_1^b \left(\ln\frac{b}{s}\right)^{\vartheta-1} \frac{X(s, u(s))}{s}\,\mathrm{d}s - \frac{1}{\Gamma(\vartheta)} \int_1^b \left(\ln\frac{b}{s}\right)^{\vartheta-1} \frac{\sigma(s, u(s))}{s}\,\mathrm{d}R_H(s) \\ &\quad - \frac{1}{\Gamma(\vartheta)} \int_1^b \left(\ln\frac{b}{s}\right)^{\vartheta-1} \int_{\mathbb{Y}} \frac{P(s, y, u(s))}{s}\,\tilde{N}(\mathrm{d}s, \mathrm{d}y)\Bigg] - \lambda \int_1^t \frac{u(s)}{s}\,\mathrm{d}s \\ &\quad + \frac{\epsilon^\vartheta}{\Gamma(\vartheta)} \int_1^t \left(\ln\frac{t}{s}\right)^{\vartheta-1} \frac{X(s, u_\epsilon(s))}{s}\,\mathrm{d}s + \frac{\epsilon^{\vartheta-H}}{\Gamma(\vartheta)} \int_1^t \left(\ln\frac{t}{s}\right)^{\vartheta-1} \frac{\sigma(s, u_\epsilon(s))}{s}\,\mathrm{d}R_H(s) \\ &\quad + \frac{\epsilon^{\vartheta-\frac{1}{2}}}{\Gamma(\vartheta)} \int_1^t \left(\ln\frac{t}{s}\right)^{\vartheta-1} \int_{\mathbb{Y}} \frac{P(s, y, u_\epsilon(s))}{s}\,\tilde{N}(\mathrm{d}s, \mathrm{d}y). \end{aligned}$$

Let $\bar{X} : \mathbb{Z} \to \mathbb{Z}$, $\bar{\sigma} : \mathbb{Z} \to L_0^2(\mathbb{U}, \mathbb{Z})$, and $\bar{P} : \mathbb{Y} \times \mathbb{Z} \to \mathbb{Z}$ be continuous and measurable functions representing the averaged forms of $X : \mathbb{J} \times \mathbb{Z} \to \mathbb{Z}$, $\sigma : \mathbb{J} \times \mathbb{Z} \to L_0^2(\mathbb{U}, \mathbb{Z})$, and $P : \mathbb{J} \times \mathbb{Y} \times \mathbb{Z} \to \mathbb{Z}$, respectively.

The averaged counterpart $w_\epsilon(t)$ is defined as

$$\begin{aligned} w_\epsilon(t) &= (\lambda \ln t + 1)\phi_1 + \frac{\ln t}{\ln b}\Bigg[\phi_b - (\lambda \ln b + 1)\phi_1 + \lambda \int_1^b \frac{w(s)}{s}\,\mathrm{d}s \\ &\quad - \frac{1}{\Gamma(\vartheta)} \int_1^b \left(\ln\frac{b}{s}\right)^{\vartheta-1} \frac{\bar{X}(w(s))}{s}\,\mathrm{d}s - \frac{1}{\Gamma(\vartheta)} \int_1^b \left(\ln\frac{b}{s}\right)^{\vartheta-1} \frac{\bar{\sigma}(w(s))}{s}\,\mathrm{d}R_H(s) \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\Gamma(\vartheta)} \int\limits_1^b \left(\ln \frac{b}{s}\right)^{\vartheta-1} \int\limits_{\mathbb{Y}} \frac{\bar{P}(y, w_\epsilon(s))}{s}\, \tilde{N}(\mathrm{d}s, \mathrm{d}y)\biggr] - \lambda \int\limits_1^t \frac{w(s)}{s}\,\mathrm{d}s \\
&+ \frac{\epsilon^{\vartheta}}{\Gamma(\vartheta)} \int\limits_1^t \left(\ln \frac{t}{s}\right)^{\vartheta-1} \frac{\bar{X}(w_\epsilon(s))}{s}\,\mathrm{d}s + \frac{\epsilon^{\vartheta-H}}{\Gamma(\vartheta)} \int\limits_1^t \left(\ln \frac{t}{s}\right)^{\vartheta-1} \frac{\bar{\sigma}(w_\epsilon(s))}{s}\,\mathrm{d}R_H(s) \\
&+ \frac{\epsilon^{\vartheta-\frac{1}{2}}}{\Gamma(\vartheta)} \int\limits_1^t \left(\ln \frac{t}{s}\right)^{\vartheta-1} \int\limits_{\mathbb{Y}} \frac{\bar{P}(y, w_\epsilon(s))}{s}\, \tilde{N}(\mathrm{d}s, \mathrm{d}y).
\end{aligned}
$$

Hypothesis 3. For any period $T \in \mathbb{J}$ and $u, v \in \mathbb{Z}$, there exist positive bounded functions $\gamma_i(t)$, $i = 1, 2, 3$, such that

(i) $$\frac{1}{T}\int\limits_1^T \left\|X(\omega, u) - \bar{X}(u)\right\|^2 \mathrm{d}\omega \leqslant \gamma_1(T)(1 + \|u\|^2),$$

(ii) $$\frac{1}{T}\int\limits_1^T \left\|\sigma(\omega, u) - \bar{\sigma}(u)\right\|^2 \mathrm{d}\omega \leqslant \gamma_2(T)(1 + \|u\|^2),$$

(iii) $$\frac{1}{T}\int\limits_1^T \biggl(\int\limits_{\mathbb{Y}} \left\|P(\omega, y, u) - \bar{P}(y, u)\right\|^2 \mu(\mathrm{d}y)\biggr) \mathrm{d}\omega \leqslant \gamma_3(T)\bigl(1 + \|u\|^2\bigr),$$

where $\lim_{T\to\infty} \gamma_i(T) = 0$, $i = 1, 2, 3$.

Theorem 2. *If Hypotheses* 1–3 *hold, then for a given* $\delta > 0$*, there exist* $C > 0$*,* $\epsilon_1 \in (0, \epsilon_0]$*, and* $\alpha \in (0,\, 2(\vartheta - H))$ *such that for all* $\epsilon \in (0, \epsilon_1]$*,*

$$\mathbf{E}\Bigl(\sup_{t\in[1, C\epsilon^{-\alpha}]} \left\|u_\epsilon(t) - w_\epsilon(t)\right\|^2\Bigr) \leqslant \delta.$$

Proof. For any $t \in [1, a] \subseteq [1, b]$, we have

$$
\begin{aligned}
&\mathbf{E}\Bigl(\sup_{1\leqslant t\leqslant a} \left\|u_\epsilon(t) - w_\epsilon(t)\right\|^2\Bigr) \\
&\quad = \biggl[8\lambda^2 \mathbf{E} \sup_{1\leqslant t\leqslant a} \biggl\|\frac{\ln t}{\ln b}\int\limits_1^b \frac{u(s) - w(s)}{s}\,\mathrm{d}s\biggr\|^2 + 8\lambda^2 \mathbf{E} \sup_{1\leqslant t\leqslant a} \biggl\|\int\limits_1^t \frac{u(s) - w(s)}{s}\,\mathrm{d}s\biggr\|^2\biggr] \\
&\qquad + \frac{8}{\Gamma^2(\vartheta)} \mathbf{E} \sup_{1\leqslant t\leqslant a} \biggl\|\frac{\ln t}{\ln b}\int\limits_1^b \left(\ln \frac{b}{s}\right)^{\vartheta-1} \frac{X(s, u(s)) - \bar{X}(w(s))}{s}\,\mathrm{d}s\biggr\|^2 \\
&\qquad + \frac{8}{\Gamma^2(\vartheta)} \mathbf{E} \sup_{1\leqslant t\leqslant a} \biggl\|\frac{\ln t}{\ln b}\int\limits_1^b \left(\ln \frac{b}{s}\right)^{\vartheta-1} \frac{\sigma(s, u(s)) - \bar{\sigma}(w(s))}{s}\,\mathrm{d}R_H(s)\biggr\|^2
\end{aligned}
$$

$$+\frac{8}{\Gamma^2(\vartheta)}\mathbf{E}\sup_{1\leqslant t\leqslant a}\left\|\frac{\ln t}{\ln b}\int_1^b\left(\ln\frac{b}{s}\right)^{\vartheta-1}\int_{\mathbb{Y}}\frac{P(s,y,u(s))-\bar{P}(y,w(s))}{s}\,\tilde{N}(\mathrm{d}s,\mathrm{d}y)\right\|^2$$
$$+\frac{8\epsilon^{2\vartheta}}{\Gamma^2(\vartheta)}\mathbf{E}\sup_{1\leqslant t\leqslant a}\left\|\int_1^t\left(\ln\frac{t}{s}\right)^{\vartheta-1}\frac{X(s,u_\epsilon(s))-\bar{X}(w_\epsilon(s))}{s}\,\mathrm{d}s\right\|^2$$
$$+\frac{8\epsilon^{2(\vartheta-H)}}{\Gamma^2(\vartheta)}\mathbf{E}\sup_{1\leqslant t\leqslant a}\left\|\int_1^t\left(\ln\frac{t}{s}\right)^{\vartheta-1}\frac{\sigma(s,u_\epsilon(s))-\bar{\sigma}(w_\epsilon(s))}{s}\,\mathrm{d}R_H(s)\right\|^2$$
$$+\frac{8\epsilon^{2\vartheta-1}}{\Gamma^2(\vartheta)}\mathbf{E}\sup_{1\leqslant t\leqslant a}\left\|\int_1^t\left(\ln\frac{t}{s}\right)^{\vartheta-1}\int_{\mathbb{Y}}\frac{P(s,y,u_\epsilon(s))-\bar{P}(y,w_\epsilon(s))}{s}\,\tilde{N}(\mathrm{d}s,\mathrm{d}y)\right\|^2$$
$$:=\varPi_1+\varPi_2+\varPi_3+\varPi_4+\varPi_5+\varPi_6+\varPi_7. \tag{5}$$

We know that

$$\sup_{1\leqslant t\leqslant a}\left(\frac{\ln t}{\ln b}\right)^2\leqslant\frac{\ln^2 a}{\ln^2 b}\leqslant\ell^2,\quad \ell>0, \tag{6}$$

and let $z(s)=(\ln(b/s))^{2\vartheta-2}/s^2$, obviously $\max_{1\leqslant s\leqslant b}z(s)=(\ln b)^{2\vartheta-2}$, therefore

$$\sup_{1\leqslant t\leqslant a}\Bigl(\max_{1\leqslant s\leqslant b}z(s)\Bigr)=(\ln b)^{2\vartheta-2}. \tag{7}$$

Estimation of $\varPi_1$. Using inequality (6) and Hypothesis 2, we have

$$\varPi_1=8\lambda^2\mathbf{E}\left[\ell^2\int_1^b\|u-w\|^2\,\mathrm{d}s+\int_1^a\|u-w\|^2\,\mathrm{d}s\right]$$
$$\leqslant 8\lambda^2\bigl(\ell^2+1\bigr)C_e^2:=\mathcal{K}_1. \tag{8}$$

Estimation of $\varPi_2$. Using Cauchy–Schwarz's inequality and inequality (6), we get

$$\varPi_2\leqslant\frac{16\ell^2}{\Gamma^2(\vartheta)}\int_1^b\frac{(\ln\frac{b}{s})^{2\vartheta-2}}{s^2}\,\mathrm{d}s\int_1^b\mathbf{E}\bigl\|X\bigl(s,u(s)\bigr)-X\bigl(s,w(s)\bigr)\bigr\|^2\,\mathrm{d}s$$
$$+\frac{16\ell^2}{\Gamma^2(\vartheta)}\int_1^b\frac{(\ln\frac{b}{s})^{2\vartheta-2}}{s^2}\,\mathrm{d}s\int_1^b\mathbf{E}\bigl\|X\bigl(s,w(s)\bigr)-\bar{X}\bigl(w(s)\bigr)\bigr\|^2\,\mathrm{d}s.$$

Applying Hypotheses 1–3, we obtain

$$\varPi_2\leqslant\frac{16\ell^2(\ln b)^{2\vartheta-1}}{(2\vartheta-1)\Gamma^2(\vartheta)}\Bigl(C_2C_e^2+b\sup_{1\leqslant t\leqslant b}\gamma_1(t)\Bigl(1+\mathbf{E}\sup_{1\leqslant t\leqslant b}\bigl\|w(t)\bigr\|^2\Bigr)\Bigr)$$
$$:=\mathcal{K}_2. \tag{9}$$

Estimation of Π_3. Using inequalities (6), (7), Lemma 1, and Hypotheses 1, 3, it follows

$$\Pi_3 \leqslant \frac{16\ell^2(\ln b)^{2\vartheta-2}C_2C_H(b-1)^{2H-1}}{\Gamma^2(\vartheta)}\int_1^b \mathbf{E}\big\|u(s)-w(s)\big\|^2\,\mathrm{d}s + \frac{16\ell^2(\ln b)^{2\vartheta-2}C_H(b-1)^{2H-1}}{\Gamma^2(\vartheta)} b \sup_{1\leqslant t\leqslant b}\gamma_2(t)\Big(1+\mathbf{E}\sup_{1\leqslant t\leqslant b}\big\|w(t)\big\|^2\Big).$$

Exploiting Hypothesis 2 yields

$$\Pi_3 \leqslant \frac{16\ell^2(\ln b)^{2\vartheta-2}C_H(b-1)^{2H-1}}{\Gamma^2(\vartheta)}\Big(C_2C_e^2 + b\sup_{1\leqslant t\leqslant b}\gamma_2(t)\Big(1+\mathbf{E}\sup_{1\leqslant t\leqslant b}\big\|w(t)\big\|^2\Big)\Big) := \mathcal{K}_3. \tag{10}$$

Estimation of Π_4. Noting Doob's martingale inequality, Itô isometry, and Hypotheses 1–3, we obtain

$$\begin{aligned}\Pi_4 &\leqslant \frac{64\ell^2(\ln b)^{2\vartheta-1}}{(2\vartheta-1)\Gamma^2(\vartheta)}\mathbf{E}\int_1^b\Big(\int_{\mathbb{Y}}\big\|P\big(s,y,u(s)\big)-P\big(s,y,w(s)\big)\big\|^2\mu\,\mathrm{d}y\Big)\,\mathrm{d}s \\ &\quad + \frac{64\ell^2(\ln b)^{2\vartheta-1}}{(2\vartheta-1)\Gamma^2(\vartheta)}\mathbf{E}\int_1^b\Big(\int_{\mathbb{Y}}\big\|P\big(s,y,w(s)\big)-\bar{P}\big(y,w(s)\big)\big\|^2\mu\,\mathrm{d}y\Big)\,\mathrm{d}s \\ &\leqslant \frac{64\ell^2(\ln b)^{2\vartheta-1}}{(2\vartheta-1)\Gamma^2(\vartheta)}\Big(C_2C_e^2 + b\sup_{1\leqslant t\leqslant b}\gamma_3(t)\Big(1+\mathbf{E}\sup_{1\leqslant t\leqslant b}\big\|w(t)\big\|^2\Big)\Big) := \mathcal{K}_4.\end{aligned} \tag{11}$$

Estimation of Π_5. Using Hypotheses 1 and 3, we get

$$\Pi_5 \leqslant \frac{16\epsilon^{2\vartheta}}{\Gamma^2(\vartheta)}C_2\int_1^a \frac{(\ln\frac{a}{s})^{2\vartheta-2}}{s^2}\mathbf{E}\sup_{1\leqslant s_1\leqslant s}\big\|u_\epsilon(s_1)-w_\epsilon(s_1)\big\|^2\,\mathrm{d}s + \mathcal{M}_1 a\epsilon^{2\vartheta}(\ln a)^{2\vartheta-1}, \tag{12}$$

where

$$\mathcal{M}_1 = \frac{16}{(2\vartheta-1)\Gamma^2(\vartheta)}\sup_{1\leqslant t\leqslant a}\gamma_1(t)\Big(1+\mathbf{E}\sup_{1\leqslant t\leqslant a}\|w_\epsilon(t)\|^2\Big).$$

Estimation of Π_6. By Lemma 1 and Hypotheses 1, 3, we get

$$\Pi_6 \leqslant \frac{16\epsilon^{2(\vartheta-H)}C_2C_H(a-1)^{2H-1}}{\Gamma^2(\vartheta)}\int_1^a \frac{(\ln\frac{a}{s})^{2\vartheta-2}}{s^2}\mathbf{E}\sup_{1\leqslant s_1\leqslant s}\big\|u_\epsilon(s_1)-w_\epsilon(s_1)\big\|^2\,\mathrm{d}s + \mathcal{M}_2 a\epsilon^{2(\vartheta-H)}(\ln a)^{2\vartheta-2}(a-1)^{2H-1}, \tag{13}$$

where

$$\mathcal{M}_2 = \frac{16C_H}{\Gamma^2(\vartheta)}\sup_{1\leqslant t\leqslant a}\gamma_2(t)\Big(1+\mathbf{E}\sup_{1\leqslant t\leqslant a}\big\|w_\epsilon(t)\big\|^2\Big).$$

Estimation of Π_7. Exploiting Doob's martingale inequality, Itô isometry and applying Hypotheses 1, 3, we obtain

$$\Pi_7 \leqslant \frac{64\epsilon^{2\vartheta-1}}{\Gamma^2(\vartheta)} C_2 \int_1^a \frac{(\ln\frac{a}{s})^{2\vartheta-2}}{s^2} \mathbf{E} \sup_{1\leqslant s_1\leqslant s} \|u_\epsilon(s_1) - w_\epsilon(s_1)\|^2 \, \mathrm{d}s + \mathcal{M}_3 a\epsilon^{2\vartheta-1}(\ln a)^{2\vartheta-1}, \tag{14}$$

where

$$\mathcal{M}_3 = \frac{64}{(2\vartheta-1)\Gamma^2(\vartheta)} \sup_{1\leqslant t\leqslant a} \gamma_3(t)\Big(1 + \mathbf{E}\sup_{1\leqslant t\leqslant a}\|w_\epsilon(t)\|^2\Big).$$

Substituting Eqs. (8)–(14) into inequality (5) yields

$$\mathbf{E}\Big(\sup_{1\leqslant t\leqslant a}\|u_\epsilon(t) - w_\epsilon(t)\|^2\Big) \leqslant \mathcal{L}_1 + \mathcal{L}_2 \int_1^a \frac{(\ln\frac{a}{s})^{2\vartheta-2}}{s^2} \mathbf{E} \sup_{1\leqslant s_1\leqslant s} \|u_\epsilon(s_1) - w_\epsilon(s_1)\|^2 \, \mathrm{d}s, \tag{15}$$

where

$$\mathcal{L}_1 = \mathcal{K} + \mathcal{M}_1 a\epsilon^{2\vartheta}(\ln a)^{2\vartheta-1} + \mathcal{M}_2 a\epsilon^{2(\vartheta-H)}(\ln a)^{2\vartheta-2}(a-1)^{2H-1} + \mathcal{M}_3 a\epsilon^{2\vartheta-1}(\ln a)^{2\vartheta-1}, \quad \mathcal{K} = \max\{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4\}$$

and

$$\mathcal{L}_2 = \frac{16C_2\epsilon^{2\vartheta}}{\Gamma^2(\vartheta)} + \frac{64C_2\epsilon^{2\vartheta-1}}{\Gamma^2(\vartheta)} + \frac{16\epsilon^{2(\vartheta-H)}C_2C_H(a-1)^{2H-1}}{\Gamma^2(\vartheta)}.$$

Due to Gronwall–Bellman's inequality [32], inequality (15) becomes

$$\mathbf{E}\Big(\sup_{1\leqslant t\leqslant a}\|u_\epsilon(t) - w_\epsilon(t)\|^2\Big) \leqslant \mathcal{L}_1 E_{2\vartheta-1,1}\big(\mathcal{L}_2\Gamma(2\vartheta-1)(\ln a)^{2\vartheta-1}\big). \tag{16}$$

Then there exist $Q > 0$ and $\alpha \in (0, 2(\vartheta - H))$, where $H \in (1/2, 1)$, such that for all $t \in [1, Q\epsilon^{-\alpha}] \subset [1, b]$,

$$\mathbf{E}\Big(\sup_{1\leqslant t\leqslant Q\epsilon^{-\alpha}}\|u_\epsilon(t) - w_\epsilon(t)\|^2\Big) \leqslant L_3\epsilon^{2\vartheta-H-\alpha},$$

where

$$\begin{aligned} L_3 = {}& \mathcal{K} + \mathcal{M}_1 Q\epsilon^H\big(\ln(Q\epsilon^{-\alpha})\big)^{2\vartheta-1} + \mathcal{M}_2 Q\epsilon^{-H}\big(\ln(Q\epsilon^{-\alpha})\big)^{2\vartheta-2}\big(Q\epsilon^{-\alpha}-1\big)^{2H-1} \\ & + \mathcal{M}_3 Q\epsilon^{H-1}\big(\ln(Q\epsilon^{-\alpha})\big)^{2\vartheta-1} \\ & + E_{2\vartheta-1,1}\bigg(\bigg[\frac{16C_2\epsilon^{2\vartheta}}{\Gamma^2(\vartheta)} + \frac{64C_2\epsilon^{2\vartheta-1}}{\Gamma^2(\vartheta)} + \frac{16\epsilon^{2(\vartheta-H)}C_2C_H(Q\epsilon^{-\alpha}-1)^{2H-1}}{\Gamma^2(\vartheta)}\bigg] \\ & \times \Gamma(2\vartheta-1)\big(\ln(Q\epsilon^{-\alpha})\big)^{2\vartheta-1}\bigg). \end{aligned}$$

Thus, the desired conclusion follows naturally. □

5 Illustration

Consider the nonlinear fractional stochastic differential equation with Dirichlet boundary conditions driven by a Rosenblatt process and an α-stable Lévy jumps

$$ {}^{CH}D_t^{\vartheta} u(t) + \frac{1}{20}\, {}^{CH}D_t^{\vartheta-1} u(t) = X\bigl(t, u(t)\bigr) + \frac{\mathrm{d}R_H(t)}{\mathrm{d}t} + \int\limits_{0<y<1/2} y^2\, \mu_\alpha(\mathrm{d}y)\,\mathrm{d}t \qquad (17) $$

subject to the Dirichlet boundary conditions

$$ u(1) = u(2) = 0, $$

where $1 < \vartheta \leqslant 2$, the Hurst parameter $H \in (1/2, 1)$, and

$$ \mu_\alpha(\mathrm{d}y) = \frac{c}{y^{1+\alpha}}\,\mathrm{d}y, \quad 0 < \alpha < 2,\ c > 0, $$

is the α-stable Lévy measure on $(0, \infty)$ restricted to $\{0 < y < 1/2\}$. Note that, for $0 < \alpha < 2$,

$$ \int\limits_0^{1/2} y^2\, \mu_\alpha(\mathrm{d}y) = \int\limits_0^{1/2} y^2 \frac{c}{y^{1+\alpha}}\,\mathrm{d}y = c\int\limits_0^{1/2} y^{1-\alpha}\,\mathrm{d}y = \frac{c}{2-\alpha} 0.5^{2-\alpha} < \infty. $$

We choose the coefficients in the form

$$ X(t, u) := \frac{1}{2}\bigl(1 - \mathrm{e}^{-(t-1)}\bigr)\tanh(u), \qquad \sigma(t, u) = 1, \qquad P(t, y, u) = y^2. $$

For any $u_1, u_2 \in \mathbb{Z}$ and $t \in [1, 2]$, we have

$$ X(t, u_1) - X(t, u_2) = \frac{1}{2}\bigl(1 - \mathrm{e}^{-(t-1)}\bigr)(\tanh u_1 - \tanh u_2). $$

Using the mean-value theorem and the fact that $|\tanh' \xi| = \operatorname{sech}^2 \xi \leqslant 1$, we obtain

$$ \bigl\|X(t, u_1) - X(t, u_2)\bigr\| \leqslant \frac{1}{2}\bigl|1 - \mathrm{e}^{-(t-1)}\bigr|\|u_1 - u_2\| \leqslant \frac{1}{2}\|u_1 - u_2\|. $$

Similarly, growth is controlled since $|\tanh(u)| \leqslant 1$, so

$$ \bigl\|X(t, u)\bigr\|^2 \leqslant \frac{1}{4}\bigl|1 - \mathrm{e}^{-(t-1)}\bigr|^2 \leqslant \frac{1}{4} \leqslant C_X\bigl(1 + \|u\|^2\bigr) $$

for a suitable constant $C_X > 0$. $\sigma(t, u) \equiv 1$, hence $C_\sigma = 0$ and satisfies the required linear growth bound. Similarly, $P(t, y, u) = y^2$ is independent of u, so $C_P = 0$ and

$$ \int\limits_{0<y<1/2} \bigl\|P(t, y, u)\bigr\|^2\, \mu_\alpha(\mathrm{d}y) = \int\limits_0^{1/2} y^4\, \mu_\alpha(\mathrm{d}y) < \infty. $$

Thus Hypothesis 1(i), (ii) are satisfied with

$$C_2 = \max\{C_X^2, C_\sigma^2, C_P^2\} \leqslant \frac{1}{4}.$$

For any $u, w \in \mathbb{B}$, we have

$$\mathbf{E}\int_1^b \|u(s) - w(s)\|_{\mathbb{Z}}^2 \,\mathrm{d}s \leqslant b\|u - w\|_{\mathbb{B}}^2 < \infty,$$

so Hypothesis 2 is satisfied on $\mathbb{B}$. Under Hypothesis 1, Lemma 5 yields

$$\|\mathcal{L}(u) - \mathcal{L}(w)\|_{\mathbb{B}} \leqslant \varkappa\|u - w\|_{\mathbb{B}}$$

with

$$\varkappa := 4\biggl(\lambda^2 \ln^2 b + \frac{C_2(\ln b)^{2\vartheta-1}}{(2\vartheta - 1)\Gamma^2(\vartheta)}\bigl(C_H(b-1)^{2H-1} + 2\bigr)\biggr)^{1/2}.$$

For instance, taking $\lambda = 0.05$, $\vartheta = 1.8$, $b = 2$, $\ln b = \ln 2 \approx 0.6931$, $\Gamma(1.8) \approx 0.918$, $C_H = 1$, and $C_2 \approx 0.0441$, we find that $\varkappa \approx 0.51 < 1$. Hence, by the Banach contraction principle, system (17) admits a unique solution on $[1, 2]$.

Averaging principle

We now introduce the ϵ-scaled version of system (17)

$$\begin{aligned} &{}^{CH}D_t^{\vartheta} u_\epsilon(t) + \frac{1}{20}\,{}^{CH}D_t^{\vartheta-1} u_\epsilon(t) \\ &\quad = \epsilon^{\vartheta} X\bigl(t, u_\epsilon(t)\bigr) + \epsilon^{\vartheta-H}\frac{\mathrm{d}R_H(t)}{\mathrm{d}t} + \epsilon^{\vartheta-1/2}\int_{0<y<1/2} y^2\,\mu_\alpha(\mathrm{d}y)\,\mathrm{d}t \end{aligned} \tag{18}$$

with the same boundary conditions

$$u_\epsilon(1) = u_\epsilon(2) = 0.$$

The averaged coefficients are defined as

$$\bar{X}(u) = \lim_{T\to\infty}\frac{1}{T}\int_1^T X(\omega, u)\,\mathrm{d}\omega, \qquad \bar{\sigma}(u) = 1, \qquad \bar{P}(y, u) = y^2.$$

Thus

$$\bar{X}(u) = \frac{1}{2}\tanh u.$$

Moreover,

$$X(\omega, u) - \bar{X}(u) = -\frac{1}{2}\mathrm{e}^{-(\omega-1)}\tanh u,$$

and therefore

$$\|X(\omega, u) - \bar{X}(u)\|^2 \leqslant \frac{1}{4}\mathrm{e}^{-2(\omega-1)}\bigl(1 + \|u\|^2\bigr).$$

This yields

$$\frac{1}{T}\int_{1}^{T}\big\|X(\omega,u)-\bar{X}(u)\big\|^{2}\,\mathrm{d}\omega\leqslant\frac{C}{T}\big(1+\|u\|^{2}\big)=\gamma_{1}(T)\big(1+\|u\|^{2}\big)$$

with $\gamma_1(T)=C/T\to 0$ as $T\to\infty$. For σ and P, we have

$$\sigma(\omega,u)\equiv\bar{\sigma}(u),\qquad P(\omega,y,u)\equiv\bar{P}(y,u),$$

hence Hypothesis 3(ii), (iii) are satisfied with $\gamma_2(T)=\gamma_3(T)\equiv 0$.

The corresponding averaged equation associated with (18) reads

$$\begin{aligned}&{}^{CH}D_t^{\vartheta}w_\epsilon(t)+\frac{1}{20}\,{}^{CH}D_t^{\vartheta-1}w_\epsilon(t)\\&\quad=\epsilon^{\vartheta}\frac{1}{2}\tanh\big(w_\epsilon(t)\big)+\epsilon^{\vartheta-H}\frac{\mathrm{d}R_H(t)}{\mathrm{d}t}+\epsilon^{\vartheta-1/2}\int\limits_{0<y<1/2}y^2\,\mu_\alpha(\mathrm{d}y)\,\mathrm{d}t.\end{aligned}\tag{19}$$

Using the explicit computation

$$\int\limits_{0<y<1/2}y^2\,\mu_\alpha(\mathrm{d}y)=\frac{c}{2-\alpha}0.5^{2-\alpha},$$

we can rewrite (19) as

$$\begin{aligned}&{}^{CH}D_t^{\vartheta}w_\epsilon(t)+\frac{1}{20}\,{}^{CH}D_t^{\vartheta-1}w_\epsilon(t)\\&\quad=\epsilon^{\vartheta}\frac{1}{2}\tanh\big(w_\epsilon(t)\big)+\epsilon^{\vartheta-H}\frac{\mathrm{d}R_H(t)}{\mathrm{d}t}+\epsilon^{\vartheta-1/2}\frac{c(0.5)^{2-\alpha}}{2-\alpha}.\end{aligned}$$

Therefore, on any fixed finite interval $[1,2]$, we have $u_\epsilon\to w_\epsilon$ in mean-square sense as $\epsilon\to 0$, showing that the original and averaged systems are equivalent.

In Figs. 1, 2, we present three distinct solution trajectories for both the original and average systems. The initial trajectories display significant stochastic roughness stemming from long-range dependence and heavy-tailed Lévy jumps, while the averaged version adheres to smoother trajectories, confirming the theoretical averaging principle amidst the nonlinear drift and boundary constraints by letting $t=1.8$, $H=0.7$, $\alpha=1.8$, and $c=0.3$.

Figures 3, 4 illustrate the probability density functions (PDF) of the solutions at various time points. The original method shows pronounced, non-Gaussian peaks and significant tails, arising from the combined influences of Rosenblatt process and Lévy jumps, while the averaged approach produces more continuous distributions without losing essential stochastic characteristics.

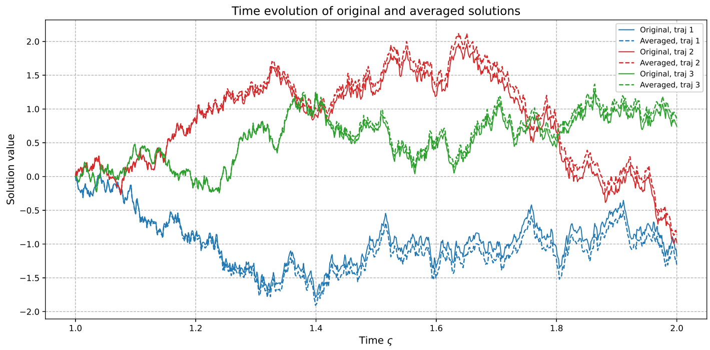

Figure 1. Time-domain evolution of sample trajectories of the original and averaged solutions for $\epsilon = 0.1$.

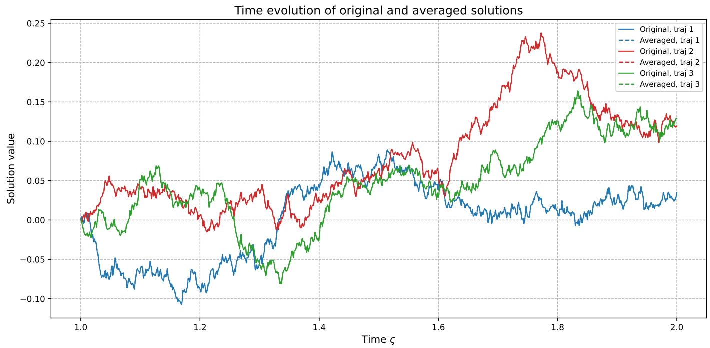

Figure 2. Time-domain evolution of sample trajectories of the original and averaged solutions for $\epsilon = 0.01$.

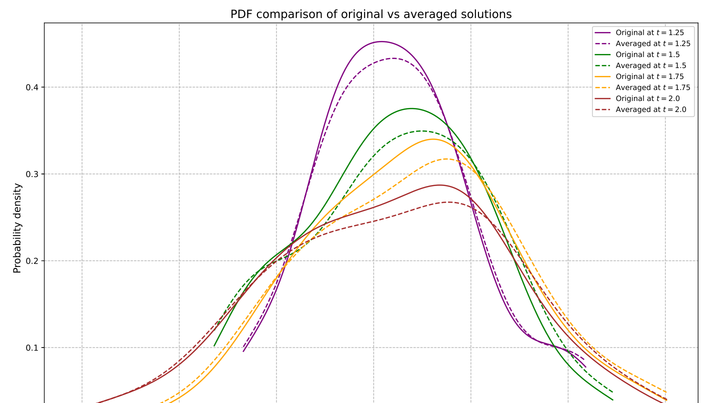

Figure 3. PDF comparison of the original and averaged solutions at selected time instants $t = 1.3$, 1.5, 1.7, and 1.9 for $\epsilon = 0.1$.

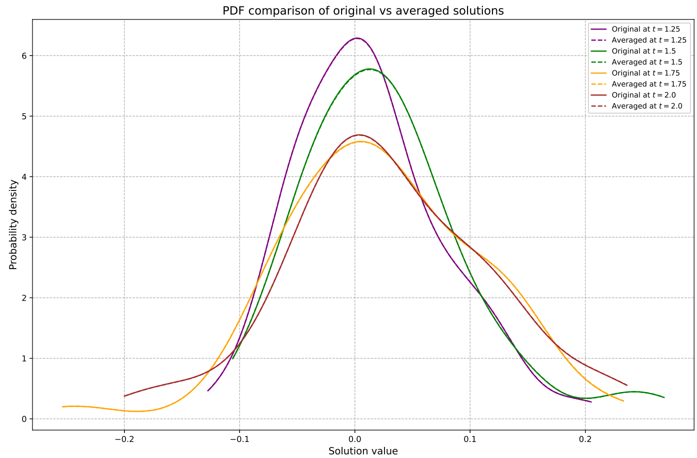

Figure 4. PDF comparison of the original and averaged solutions at selected time instants $t = 1.3$, 1.5, 1.7, and 1.9 for $\epsilon = 0.01$.

Remark. We used fBm-type increments with the same Hurst parameter H due to technical difficulty of Rosenblatt process implementation. This preserves long-range dependence and roughness, and the theoretical results remain valid because the approximation is used only for numerical illustration.

6 Conclusion

We have addressed the existence, uniqueness, and averaging principle for Caputo–Hadamard fractional stochastic differential equations with Dirichlet boundary conditions driven by Rosenblatt process and Lévy jumps. Subsequently, under appropriate averaging conditions, the system was averaged out, and the resulting solution approximates its nonautonomous counterpart. The mean-square convergence is profoundly verified with the rate $2\vartheta - H - \alpha$.

Limitations. The analysis is carried out exclusively for Dirichlet boundary conditions; extensions to Neumann, Robin, mixed, or nonlocal boundary conditions are not covered, and the averaging principle is established only in the mean-square sense and only for coefficients satisfying global Lipschitz and finite-energy assumptions, which excludes systems with superlinear growth and non-Lipschitz coefficients.

Practical implications. The framework we propose in (1), together with the illustrative example (17), encapsulates situations in which long-range memory and sudden jumps coexist. These combined effects are crucial for modeling viscoelastic materials subjected to random loads, climate systems affected by ongoing fluctuations and extreme occurrences, and financial processes that show heavy-tailed characteristics and long-range dependence in physics, engineering, and biological contexts.

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