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Controllability and Ulam–Hyers stability analysis of conformable fractional differential systems with time delays and impulsive effects*

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Abstract. We investigate the controllability and Ulam–Hyers stability of a class of conformable fractional differential systems with time delays and impulsive effects. Specifically, we analyze the movement rules before and after the impulse, utilizing both the Banach and Schauder fixed point theorem to derive controllability. Furthermore, we employ nonlinear functional analysis methods to study Ulam–Hyers stability. To demonstrate the applicability and feasibility of our main conclusions, an illustrative example is provided.

Keywords: conformable fractional derivative, impulse, delay, controllability, Ulam–Hyers stability.

1 Introduction

In recent decades, the theories of fractional calculus have been widely used in engineering practice, which promotes the rapid development of science and technology and constantly pushes the boundaries of integer calculus theory. The definition of fractional derivatives has many forms, such as Riemann–Liouville fractional derivatives, Caputo fractional derivatives, Hadamard fractional derivatives, etc. However, some of these definitions cannot satisfy the traditional properties of classical calculus, especially in the case of higher order, and their expressions are more complicated, which makes it difficult to apply in engineering. For the theory and application of fractional calculus, we refer the reader to see [6, 12, 16, 18, 21, 29–31, 33] and the references therein.

In 2014, Khalil et al. proposed a new derivative called conformable fractional derivative in [11], which only relies on the definition of limits and satisfies almost all the properties of classical calculus. In 2015, Abdeljawad et al. further developed and improved the basic theory of conformable fractional calculus [1]. In the past decade, many scholars

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have been interested in differential systems with conformable fractional derivatives and have made rich achievements (see [17, 26, 32, 34] and the references therein). These research shows that the research progress and development of conformable fractional differential systems are very rapid, but there are few achievements in the controllability and stability of such system.

The controllability of differential system has long been an indispensable topic in system theory, and in practical problems, only controllable systems have application value. In 1987, Balachandran et al. reviewed the use of fixed point theorems to study the controllability of differential systems since 1967 (see [5]). The approaches have been widely used by scholars to this day (see [3, 4, 10, 19, 26, 27] and references therein). In [26], the authors applied Krasnoselskii's fixed point theorem to derive a complete controllability result for a semilinear conformable differential system. In [3], the controllability of a semilinear impulsive Atangana–Baleanu fractional differential system with delay was discussed by means of the fixed point theorem. In [19], Radhakrishnan and Sathya obtained the sufficient conditions for controllability of Hilfer fractional Langevin dynamical system with impulse by using the generalized fractional calculus and fixed point theory. In [4], the controllability of a Caputo fractional stochastic system with delay and impulse was investigated using Mittag-Leffler functions. In [27], by using fixed point theorem, the result of relatively exact controllability of fractional stochastic differential systems was obtained. From the existing literature it can be seen that the study of approximate controllability of initial valued differential systems is relatively common. In contrast, research on boundary value differential systems has progressed slowly, especially in the case of boundary value fractional differential systems, which have great research value and application prospects.

In many application scenarios, stability plays a very important role and is the primary condition for the normal operation of the system. In control system, stability is the basic requirement and important index of system design. In [23], Ulam proposed the Ulam stability problem for the first time in 1940, and Hyers answered the question the following year (see [9]). After that, many scholars widely promoted this stability problem and collectively referred to it as the Ulam–Hyers stability problem. In 1978, Rassias further generalized the concept of Ulam–Hyers stability, forming the Rassias–Ulam–Hyers stability problem (see [20]). After decades of development, scholars have conducted in-depth research on the Ulam stability of differential systems and achieved rich results. In [24], Fan Wan et al. studied the stability for a class of impulsive conformable fractional integro-differential equations with antiperiodic boundary conditions. In [8], the Ulam–Hyers stability was studied for a new type of fractional system with Caputo–Hadamard derivative under nonlocal boundary conditions. In [35], Zhao studied the solvability and generalized Ulam–Hyers stability of a nonlinear Atangana–Baleanu–Caputo fractional coupled system with a Laplacian operator and impulses. In [14], Luo et al. focused on the relative controllability and Ulam–Hyers stability of Riemann–Liouville fractional delay differential system of order $\alpha \in (1, 2)$. For more information, we refer the reader to see [2, 15, 25, 28] and references therein.

Meanwhile, delay and impulse phenomena are widespread in various engineering systems, such as automatic control. In light of this, we are highly interested in the controllability and stability of conformable fractional impulsive differential systems with delay. Given the scarcity of research results for reference, this work is quite challenging. In this paper, we propose a new model for impulsive boundary systems with time delay and conformable fractional derivatives (abbreviated as SYS) as follows:

$$\begin{aligned} &\mathfrak{D}_0^\alpha x(t) + f\bigl(t, x(t), x(t+\sigma)\bigr) + Au(t) = 0, \quad t \in (0,\xi), \\ &\mathfrak{D}_\xi^\alpha x(t) + f\bigl(t, x(t), x(t-\sigma)\bigr) = 0, \quad t \in (\xi,1), \\ &\Delta x|_{t=\xi} = P\bigl(x(\xi^-)\bigr), \qquad \Delta x'|_{t=\xi} = Q\bigl(x(\xi^+)\bigr), \\ &x(0) = 0, \qquad x'(1) = kx'(0), \end{aligned} \tag{1}$$

where $1 < \alpha \leqslant 2$. $\mathfrak{D}_{t_i}^\alpha$ are the conformable fractional derivatives of order α starting from $t_i = 0, \xi$. $\xi \in (0,1)$, $I = [0,1]$; $I_0 = [0,\xi]$, $I_1 = [\xi,1]$, and $I' = I \setminus \{\xi\}$. $f \in C(I \times \mathbb{R}^2, \mathbb{R})$, $P, Q \in C(\mathbb{R},\mathbb{R})$, $\Delta x|_{t=\xi} = x(\xi^+) - x(\xi^-)$, $\Delta x'|_{t=\xi} = x'(\xi^+) - x'(\xi^-)$, where $x(\xi^+)$, $x'(\xi^+)$ and $x(\xi^-)$, $x'(\xi^-)$ represent the right and left limits of $x(t)$ and $x'(t)$ at $t = \xi$, respectively. $0 < \sigma \leqslant \min\{\xi, 1-\xi\}$, $k \in [0,1)$, and $A \in \mathbb{R}$ are constants, $u \in C(I_0 \times \mathbb{R}, \mathbb{R})$ is a control function. Compared with the references we mentioned, the innovative points are as follows:

(i) SYS (1) is a piecewise conformable fractional impulsive delay differential system with boundary conditions. Piecewise segments and boundary constraints make the system much more complex, and these are the most significant differences from our previous studies [29–31].

(ii) We define a linear operator $\mathcal{W}$ and prove its invertibility and boundedness. The control function u can be determined by means of $\mathcal{W}$.

(iii) By means of the Banach and Schauder fixed point theorem, sufficient conditions for controllability and Ulam–Hyers stability are obtained.

This article is organized as follows. In Section 2, we introduce the basic definitions and properties of the conformable derivative and controllability. In Section 3, we consider the controllability of the solutions of SYS (1) by using the fixed point theorems. In Section 4, the Ulam–Hyers stability of the system is studied. In Section 5, an example is given to illustrate the effectiveness of the conclusions.

2 Preliminaries

In this section, we will introduce some definitions and useful lemmas to help readers better understand this article. First, we define a Banach space, and then we will begin our discussion in this space. Denote

$$PC(I,\mathbb{R}) = \bigl\{x : I \to \mathbb{R} \bigm| x \in C(I', \mathbb{R}),\ x(\xi^+) \text{ and } x(\xi^-) \text{ exist},\ x(\xi^-) = x(\xi)\bigr\}$$

with the norm

$$\|x\|_{PC} = \sup_{t\in I} \bigl|x(t)\bigr|.$$

Then $PC(I,\mathbb{R})$ is Banach space.

Now we present the definition of the conformable fractional derivative.

Definition 1. (See [1].) Let $0 < \alpha \leqslant 1$ and let $f : [a, +\infty) \to \mathbb{R}$. The (left) conformable fractional derivative of order α starting from a is defined by

$$\mathfrak{D}_a^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f\big(t + \varepsilon (t-a)^{1-\alpha}\big) - f(t)}{\varepsilon}, \quad t > a.$$

When $a = 0$, we write $\mathfrak{D}^\alpha f(t)$. If $\mathfrak{D}_a^\alpha f(t)$ exists on (a, b), $\mathfrak{D}_a^\alpha f(a) = \lim_{t \to a+} \mathfrak{D}_a^\alpha f(t)$.

Definition 2. (See [1].) Let $n \in \mathbb{N}$, $\alpha \in (n,\, n+1]$, and $\beta = \alpha - n$. If $f \in C^n([a, +\infty))$ and $\mathfrak{D}_a^\beta f^{(n)}(t)$ exists, we define

$$\mathfrak{D}_a^\alpha f(t) = \mathfrak{D}_a^\beta f^{(n)}(t), \quad t > a.$$

When $a = 0$, we write $\mathfrak{D}^\alpha f(t)$.

Remark 1. Given $\alpha \in (0, 1]$, if f is differentiable, then $\mathfrak{D}_a^\alpha f(t) = (t-a)^{1-\alpha} f'(t)$. If $\alpha \in (n, n+1]$, $k = 0, 1, 2, \dots, n$, then $\mathfrak{D}_a^\alpha (t-a)^k = 0$.

Definition 3. (See [1].) Let $n \in \mathbb{N}$ and $\alpha \in (n,\, n+1]$. Then the conformable fractional integral of order α starting at a is defined by

$$\mathfrak{I}_a^\alpha f(t) = \frac{1}{n!} \int_a^t (t-s)^n s^{\alpha - n - 1} f(s) \, \mathrm{d}s.$$

Lemma 1. (*See* [1].) *Assume that* $f : [a, +\infty) \to \mathbb{R}$, $n \in \mathbb{N}$, *and* $\alpha \in (n,\, n+1]$ *such that* $f^{(n)}(t)$ *is continuous. Then we have*

$$\mathfrak{D}_a^\alpha \mathfrak{I}_a^\alpha f(t) = f(t), \quad t > a.$$

Lemma 2. (*See* [1].) *Let* $n \in \mathbb{N}$, $\alpha \in (n,\, n+1]$ *and* $f : [a, +\infty) \to \mathbb{R}$ *be* $n+1$ *times differentiable for* $t > a$. *Then we have*

$$\mathfrak{I}_a^\alpha \mathfrak{D}_a^\alpha f(t) = f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(a)(t-a)^k}{k!}, \quad t > a.$$

Definition 4. The SYS (1) is said to be controllable if, for a given $\eta \in (0, \xi) \cup (\xi, 1)$ and $\overline{x} \in \mathbb{R}$, there exists a control function $u \in C(I_0 \times \mathbb{R}, \mathbb{R})$ such that the solution x of the SYS (1) satisfies $x(\eta) = \overline{x}$.

Lemma 3. *Let* $1 < \alpha \leqslant 2$, $y_1, y_2 \in PC(I, \mathbb{R})$, $p, q \in \mathbb{R}$, *and* $k \neq 1$. *Then the SYS*

$$\begin{aligned} &\mathfrak{D}_0^\alpha x(t) + y_1(t) = 0, \quad t \in (0, \xi), \\ &\mathfrak{D}_\xi^\alpha x(t) + y_2(t) = 0, \quad t \in (\xi, 1), \\ &\Delta x|_{t=\xi} = p, \qquad \Delta x'|_{t=\xi} = q, \\ &x(0) = 0, \qquad x'(1) = kx'(0) \end{aligned} \tag{2}$$

is equivalent to the following integral equation:

$$x(t) = \int_0^{\xi} G(t,s) s^{\alpha-2} y_1(s)\,\mathrm{d}s + \int_{\xi}^{1} G(t,s)(s-\xi)^{\alpha-2} y_2(s)\,\mathrm{d}s + \varphi(t,p,q), \quad (3)$$

where

$$G(t,s) = \frac{1}{1-k}\begin{cases} kt + (1-k)s, & 0 \leqslant s \leqslant t \leqslant 1, \\ t, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \quad (4)$$

$$\varphi(t,p,q) = \frac{1}{1-k}\begin{cases} -qt, & t \in [0,\xi]; \\ (1-k)p - q(\xi + k(t-\xi)), & t \in (\xi,1]. \end{cases} \quad (5)$$

Proof. Let $x = x(t)$ be the solution of SYS (2). Then, for $t \in I_0$, there exist constants $c_0, c_1 \in \mathbb{R}$ such that

$$x(t) = -\int_0^t (t-s)s^{\alpha-2} y_1(s)\,\mathrm{d}s + c_0 + c_1 t, \qquad x'(t) = -\int_0^t s^{\alpha-2} y_1(s)\,\mathrm{d}s + c_1.$$

Hence,

$$x(0) = c_0, \qquad x'(0) = c_1,$$

$$x(\xi^-) = -\int_0^{\xi} (\xi-s)s^{\alpha-2} y_1(s)\,\mathrm{d}s + c_0 + c_1\xi, \qquad x'(\xi^-) = -\int_0^{\xi} s^{\alpha-2} y_1(s)\,\mathrm{d}s + c_1.$$

For $t \in I_1$, there exist constants $d_0, d_1 \in \mathbb{R}$ such that

$$x(t) = -\int_{\xi}^{t} (t-s)(s-\xi)^{\alpha-2} y_2(s)\,\mathrm{d}s + d_0 + d_1(t-\xi),$$

$$x'(t) = -\int_{\xi}^{t} (s-\xi)^{\alpha-2} y_2(s)\,\mathrm{d}s + d_1.$$

Thus,

$$x(\xi^+) = d_0, \qquad x'(\xi^+) = d_1,$$

$$x(1) = -\int_{\xi}^{1} (1-s)(s-\xi)^{\alpha-2} y_2(s)\,\mathrm{d}s + d_0 + d_1(1-\xi),$$

$$x'(1) = -\int_{\xi}^{1} (s-\xi)^{\alpha-2} y_2(s)\,\mathrm{d}s + d_1.$$

Applying the boundary and impulsive conditions, we obtain

$$c_0 = 0,$$
$$c_1 = \frac{1}{1-k}\left(\int_0^{\xi} s^{\alpha-2}y_1(s)\,\mathrm{d}s + \int_{\xi}^{1}(s-\xi)^{\alpha-2}y_2(s)\,\mathrm{d}s - q\right),$$
$$d_0 = \frac{1}{1-k}\left(\int_0^{\xi}\big((1-k)s+k\xi\big)s^{\alpha-2}y_1(s)\,\mathrm{d}s+\xi\int_{\xi}^{1}(s-\xi)^{\alpha-2}y_2(s)\,\mathrm{d}s+(1-k)p - q\xi\right),$$
$$d_1 = \frac{1}{1-k}\left(k\int_0^{\xi} s^{\alpha-2}y_1(s)\,\mathrm{d}s + \int_{\xi}^{1}(s-\xi)^{\alpha-2}y_2(s)\,\mathrm{d}s - kq\right).$$

Substituting these parameters into the previous expression, we can easily obtain (3). It is not difficult to find that the above derivation process is reversible. Therefore, SYS (2) and the integral equation (3) are equivalent.

The proof is completed. □

From the definition of $G(t,s)$ and $\varphi(t,p,q)$ we can easily draw the conclusion.

Lemma 4. *Let* $0 \leqslant k < 1$*,* $p > 0$*, and* $q < 0$*. Assume that* $G \in C(I \times I)$ *and satisfies* $0 \leqslant G(t,s) \leqslant 1/(1-k)$*. Moreover, assume that the function* $\varphi(t,p,q)$ *satisfies* $0 \leqslant \varphi(t,p,q) \leqslant p - q/(1-k)$*.*

Proof. On the one hand, for $0 \leqslant s \leqslant t \leqslant 1$, we get $kt \geqslant 0$, $(1-k)s \geqslant 0$, and $(1-k)s \leqslant (1-k)t$, so

$$0 \leqslant G(t,s) = \frac{kt + (1-k)s}{1-k} \leqslant \frac{kt + (1-k)t}{1-k} = \frac{1}{1-k}.$$

For $0 \leqslant t \leqslant s \leqslant 1$, we get

$$0 \leqslant G(t,s) = \frac{t}{1-k} \leqslant \frac{1}{1-k}.$$

In particular, when $s = t$, $G(t,s) = s/(1-k) = t/(1-k)$.

Obviously, $G \in C(I \times I)$ and $0 \leqslant G(t,s) \leqslant 1/(1-k)$.

On the other hand, noting that $q < 0$, we can conclude that for $t \in [0,\xi]$,

$$0 \leqslant \varphi(t,p,q) = \frac{-qt}{1-k} \leqslant \frac{-q}{1-k},$$

and for $t \in (\xi, 1]$,

$$\begin{aligned}0 \leqslant \varphi(t,p,q) &= p + \frac{-q(\xi + k(t-\xi))}{1-k} \leqslant p + \frac{-q(\xi + t - \xi)}{1-k}\\ &= p + \frac{-qt}{1-k} \leqslant p + \frac{-q}{1-k}.\end{aligned}$$

So, $0 \leqslant \varphi(t,p,q) \leqslant p - q/(1-k)$.

The proof is completed. □

For $x \in PC(I, \mathbb{R})$, we define operator $\mathfrak{T} : PC(I, \mathbb{R}) \to PC(I, \mathbb{R})$ as follows:

$$\begin{aligned}\mathfrak{T}x(t) = &\int_0^\xi G(t,s)s^{\alpha-2} f\bigl(s, x(s), x(s+\sigma)\bigr)\,\mathrm{d}s + A\int_0^\xi G(t,s)s^{\alpha-2}u(s)\,\mathrm{d}s \\ &+ \int_\xi^1 G(t,s)(s-\xi)^{\alpha-2} f\bigl(s, x(s), x(s-\sigma)\bigr)\,\mathrm{d}s \\ &+ \varphi\bigl(t, P\bigl(x(\xi^-)\bigr), Q\bigl(x(\xi^+)\bigr)\bigr). \end{aligned} \tag{6}$$

Obviously, SYS (1) is equivalent to integral equation (6), which means that $x = x(t)$ is a solution of SYS (1) if and only if $x = x(t)$ is a fixed point of operator $\mathfrak{T}$.

To study the controllability of the system, fix $\eta \in (0, \xi) \cup (\xi, 1)$ and define

$$N = \left\{ x \in L\bigl([0,\xi], \mathbb{R}\bigr) \;\Big|\; \int_0^\xi G(\eta, s)s^{\alpha-2}x(s)\,\mathrm{d}s = 0 \right\}.$$

Then N is a closed subspace on $L([0,\xi], \mathbb{R})$. For quotient space $L([0,\xi], \mathbb{R})/N$, define norm

$$\|x\|_N = \inf_{y \in L([0,\xi],\mathbb{R})/N} \|x - y\|,$$

$L([0,\xi], \mathbb{R})/N$ is a Banach space.

Define linear operator $\mathcal{W} : L([0,\xi], \mathbb{R})/N \to \mathbb{R}$ as follows:

$$\mathcal{W}u = A\int_0^\xi G(\eta, s)s^{\alpha-2}u(s)\,\mathrm{d}s, \quad u \in L\bigl([0,\xi], \mathbb{R}\bigr)/N. \tag{7}$$

Let

$$\beta = \int_0^\xi G(\eta, s)s^{\alpha-2}\,\mathrm{d}s = \begin{cases} \frac{\eta(\alpha\xi^{\alpha-1} - (1-k)\eta^{\alpha-1})}{\alpha(1-k)(\alpha-1)} := r_1, & \eta \in (0,\xi), \\ \frac{\xi^{\alpha-1}((\alpha-1)(1-k)\xi + \alpha k\eta)}{\alpha(1-k)(\alpha-1)} := r_2, & \eta \in (\xi, 1). \end{cases}$$

Obviously, $\beta > 0$ if $0 \leqslant k < 1$.

Lemma 5. *Suppose $A \neq 0$, $0 \leqslant k < 1$, and $\eta \in (0,\xi) \cup (\xi, 1)$. Then the linear operator $\mathcal{W}$ defined by* (7) *is invertible, and $\|\mathcal{W}^{-1}\| \leqslant |A\beta|^{-1}$.*

Proof. First, we prove that $\mathcal{W}$ is an injective. Let $u_1, u_2 \in L([0,\xi], \mathbb{R})/N$ and $u_1 \neq u_2$. If

$$\mathcal{W}u_1 - \mathcal{W}u_2 = \mathcal{W}(u_1 - u_2) = A\int_0^\xi G(\eta, s)s^{\alpha-2}\bigl(u_1(s) - u_2(s)\bigr)\,\mathrm{d}s = 0,$$

then

$$u_1 - u_2 \in N.$$

Contradiction, therefore $\mathcal{W}$ is injective.

Next, we prove that $\mathcal{W}$ is surjective. For $r \in \mathbb{R}$, there exists a function

$$u_0 = \begin{cases} \frac{r}{Ar_1}, & \eta \in (0, \xi), \\ \frac{r}{Ar_2}, & \eta \in (\xi, 1), \end{cases}$$

that satisfies

$$\mathcal{W}u_0 = A \int_0^\xi G(\eta, s) s^{\alpha-2} u_0(s)\, \mathrm{d}s = r.$$

Hence, $\mathcal{W}$ is surjective. To sum up, $\mathcal{W}$ is invertible.

At last, we prove $\|\mathcal{W}^{-1}\| \leqslant |A\beta|^{-1}$. For any $\gamma \in \mathbb{R}$, take

$$u = A\beta\gamma = \gamma A \int_0^\xi G(\eta, s) s^{\alpha-2}\, \mathrm{d}s.$$

Then

$$v = \mathcal{W}u = A \int_0^\xi G(\eta, s) s^{\alpha-2} u(s)\, \mathrm{d}s = \left(A \int_0^\xi G(\eta, s) s^{\alpha-2}\, \mathrm{d}s \right)^2 \gamma,$$

so $\gamma = (A \int_0^\xi G(\eta, s) s^{\alpha-2}\, \mathrm{d}s)^{-2} v$.

As a result,

$$u = A\beta\gamma = \left(A \int_0^\xi G(\eta, s) s^{\alpha-2}\, \mathrm{d}s \right)^{-1} v,$$

thus

$$\begin{aligned} \left\|\mathcal{W}^{-1}v\right\|_N &= \|u\|_N \\ &= \left\| \left(A \int_0^\xi G(\eta, s) s^{\alpha-2}\, \mathrm{d}s \right)^{-1} v \right\|_N \leqslant \left| A \int_0^\xi G(\eta, s) s^{\alpha-2}\, \mathrm{d}s \right|^{-1} |v|. \end{aligned}$$

Hence,

$$\left\|\mathcal{W}^{-1}\right\| = \sup_{v \neq 0} \frac{\|\mathcal{W}^{-1}v\|_N}{|v|} \leqslant \sup_{v \neq 0} \frac{|A \int_0^\xi G(\eta, s) s^{\alpha-2}\, \mathrm{d}s|^{-1} |v|}{|v|} = |A\beta|^{-1}.$$

The proof is completed. □

In the next section, we will use the Banach fixed point theorem to discuss the controllability of the solution of SYS (1). We present the theorem as follows.

Theorem 1 [Banach fixed point theorem]. (*See* [7].) *Let X be a nonempty closed subset of a Banach space E, and let $S : X \to X$ be a contraction. Then there is a unique $u^* \in X$ with $Su^* = u^*$.*

Theorem 2 [Schauder fixed point theorem]. (*See* [22].) *Let X be a nonempty bounded closed convex subset of a Banach space E, and let $K : X \to X$ be a completely continuous operator. Then K has a fixed point in X.*

3 Controllability

In this section, we apply the two fixed point theorems to study the controllability of solutions of SYS (1) in the Banach space $PC(I, \mathbb{R})$. For convenience, we first introduce some notations and assumptions.

Denote $\mathbb{R}^+ := [0, +\infty)$, $\lambda = \tau^{\alpha-1}/((1-k)(\alpha-1))$, and $\tau = \max\{\xi,\, 1-\xi\}$.

(H1) There exist functions $\varpi_1, \varpi_2 \in C(I, \mathbb{R}^+)$ such that for $t \in I$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$,

$$\begin{aligned} 0 &\leqslant \bigl|f(t, x_2, y_2) - f(t, x_1, y_1)\bigr| \\ &\leqslant \varpi_1(t)|x_2 - x_1| + \varpi_2(t)|y_2 - y_1| \\ &\leqslant \varpi^*\bigl(|x_2 - x_1| + |y_2 - y_1|\bigr), \end{aligned}$$

where

$$\varpi^* = \max\Bigl\{\sup_{t\in I} \varpi_1(t),\, \sup_{t\in I} \varpi_2(t)\Bigr\}.$$

(H2) There exist constants $p_0, q_0 \geqslant 0$ such that for $x_1, x_2 \in \mathbb{R}^+$,

$$\begin{aligned} 0 &\leqslant \bigl|P(x_2) - P(x_1)\bigr| \leqslant p_0|x_2 - x_1|, \\ 0 &\leqslant \bigl|Q(x_2) - Q(x_1)\bigr| \leqslant q_0|x_2 - x_1|. \end{aligned}$$

(H3) There exist nonnegative functions $\rho_1, \rho_2, \rho_3 \in C(I, \mathbb{R}^+)$ such that for $t \in I$ and x,$y \in \mathbb{R}$,

$$0 < \bigl|f(t, x, y)\bigr| \leqslant \rho_1(t) + \rho_2(t)|x| + \rho_3(t)|y| \leqslant \rho_1^* + \rho_2^*|x| + \rho_3^*|y|,$$

where

$$\rho_1^* = \max_{t\in I} \bigl|\rho_1(t)\bigr|, \qquad \rho_2^* = \max_{t\in I} \bigl|\rho_2(t)\bigr|, \qquad \rho_3^* = \max_{t\in I} \bigl|\rho_3(t)\bigr|.$$

(H4) There exist nonnegative constants M_P, N_P, M_Q, N_Q such that for $x \in \mathbb{R}$,

$$0 \leqslant \bigl|P(x)\bigr| \leqslant M_P|x| + N_P, \qquad 0 \leqslant \bigl|Q(x)\bigr| \leqslant M_Q|x| + N_Q.$$

Theorem 3. *Suppose* (H1) *and* (H2) *hold. If*

$$0 < \bigl(1 + \lambda\beta^{-1}\bigr)\biggl(4\lambda\varpi^* + p_0 + \frac{q_0}{1-k}\biggr) = l < 1, \tag{8}$$

then SYS (1) *is controllable.*

Proof. If it can be shown that the operator $\mathfrak{T}$ has a fixed point x, then this fixed point x is the solution of SYS (1) satisfing $x(\eta) = \overline{x}$, which means that SYS (1) is controllable.

First, we prove the existence and uniqueness of the solution to SYS (1).

For $x_1, x_2 \in PC(I, \mathbb{R})$, $t \in [0,1]$,

$$\begin{aligned}
&\bigl|\mathfrak{T}x_2(t) - \mathfrak{T}x_1(t)\bigr| \\
&\quad\leqslant \int_0^{\xi} G(t,s)s^{\alpha-2}\bigl|f\bigl(s,x_2(s),x_2(s+\sigma)\bigr) - f\bigl(s,x_1(s),x_1(s+\sigma)\bigr)\bigr|\,\mathrm{d}s \\
&\qquad + \int_{\xi}^{1} G(t,s)(s-\xi)^{\alpha-2}\bigl|f\bigl(s,x_2(s),x_2(s-\sigma)\bigr) - f\bigl(s,x_1(s),x_1(s-\sigma)\bigr)\bigr|\,\mathrm{d}s \\
&\qquad + \bigl|\varphi\bigl(t,P\bigl(x_2(\xi^-)\bigr),Q\bigl(x_2(\xi^+)\bigr)\bigr) - \varphi\bigl(t,P\bigl(x_1(\xi^-)\bigr),Q\bigl(x_1(\xi^+)\bigr)\bigr)\bigr| \\
&\quad\leqslant 4\lambda\varpi^*\|x_2 - x_1\|_{PC} + p_0\|x_2 - x_1\|_{PC} + \frac{q_0}{1-k}\|x_2 - x_1\|_{PC} \\
&\quad= \left(4\lambda\varpi^* + p_0 + \frac{q_0}{1-k}\right)\|x_2 - x_1\|_{PC} < l\|x_2 - x_1\|_{PC}.
\end{aligned}$$

It means that $\|\mathfrak{T}x_2 - \mathfrak{T}x_1\|_{PC} \leqslant l\|x_2 - x_1\|_{PC}$, so $\mathfrak{T}$ is a contraction operator. According to the Banach fixed point theorem, $\mathfrak{T}$ has a unique fixed point x on $PC(I, \mathbb{R})$, which means that SYS (1) has a unique solution.

Second, we demonstrate the controllability of the system.

Suppose $x = x(t)$ is the solution of SYS (1), then for given $\eta \in [0,\xi) \cup (\xi,1]$ and $\overline{x} \in \mathbb{R}$, $x(\eta) = \overline{x}$, so

$$\begin{aligned}
\bar{x} &= x(\eta) = \mathfrak{T}x(\eta) \\
&= \int_0^{\xi} G(\eta,s)s^{\alpha-2}f\bigl(s,x(s),x(s+\sigma)\bigr)\,\mathrm{d}s + A\int_0^{\xi} G(\eta,s)s^{\alpha-2}u(s)\,\mathrm{d}s \\
&\quad + \int_{\xi}^{1} G(\eta,s)s^{\alpha-2}f\bigl(s,x(s),x(s-\sigma)\bigr)\,\mathrm{d}s + \varphi\bigl(\eta,P\bigl(x(\xi^-)\bigr),Q\bigl(x(\xi^+)\bigr)\bigr).
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathcal{W}u &= A\int_0^{\xi} G(\eta,s)s^{\alpha-2}u(s)\,\mathrm{d}s \\
&= \bar{x} - \Biggl(\int_0^{\xi} G(\eta,s)s^{\alpha-2}f\bigl(s,x(s),x(s+\sigma)\bigr)\,\mathrm{d}s \\
&\qquad + \int_{\xi}^{1} G(\eta,s)s^{\alpha-2}f\bigl(s,x(s),x(s-\sigma)\bigr)\,\mathrm{d}s + \varphi\bigl(\eta,P\bigl(x(\xi^-)\bigr),Q\bigl(x(\xi^+)\bigr)\bigr)\Biggr) \\
&:= \bar{x} - R\bigl(x(s),P\bigl(x(\xi^-)\bigr),Q\bigl(x(\xi^+)\bigr)\bigr).
\end{aligned} \tag{9}$$

From Lemma 5 it follows that $\mathcal{W}^{-1}$ exists. Thus, the control function of SYS (1) is as follows:

$$\begin{aligned} u &= \mathcal{W}^{-1}\Biggl(\bar{x} - \Biggl(\int_0^\xi G(\eta,s)s^{\alpha-2} f\bigl(s,x(s),x(s+\sigma)\bigr)\,\mathrm{d}s \\ &\quad + \int_\xi^1 G(\eta,s)(s-\xi)^{\alpha-2} f\bigl(s,x(s),x(s-\sigma)\bigr)\,\mathrm{d}s + \varphi\bigl(\eta,P\bigl(x(\xi^-)\bigr),Q\bigl(x(\xi^+)\bigr)\bigr)\Biggr)\Biggr) \\ &:= \mathcal{W}^{-1}\bigl(\bar{x} - R\bigl(x(s),P\bigl(x(\xi^-)\bigr),Q\bigl(x(\xi^+)\bigr)\bigr)\bigr). \end{aligned} \tag{10}$$

Define operator $\mathcal{M}: PC(I,\mathbb{R}) \to PC(I,\mathbb{R})$ for all $x \in PC(I,\mathbb{R})$,

$$\begin{aligned} \mathcal{M}x(t) &= \int_0^\xi G(t,s)s^{\alpha-2} f\bigl(s,x(s),x(s+\sigma)\bigr)\,\mathrm{d}s \\ &\quad + A\int_0^\xi G(t,s)s^{\alpha-2}\mathcal{W}^{-1}\bigl(\bar{x} - R\bigl(x(s),P\bigl(x(\xi^-)\bigr),Q\bigl(x(\xi^+)\bigr)\bigr)\bigr)\,\mathrm{d}s \\ &\quad + \int_\xi^1 G(t,s)(s-\xi)^{\alpha-2} f\bigl(s,x(s),x(s-\sigma)\bigr)\,\mathrm{d}s \\ &\quad + \varphi\bigl(t,P\bigl(x(\xi^-)\bigr),Q\bigl(x(\xi^+)\bigr)\bigr). \end{aligned} \tag{11}$$

Obviously, if operator $\mathcal{M}$ has a fixed point x on $PC(I,\mathbb{R})$, then x is a solution of SYS (1) and satisfies $x(\eta) = \bar{x}$, which means that SYS (1) is controllable.

In fact, for any $x_1, x_2 \in PC(I,\mathbb{R})$ and all $t \in I$,

$$\begin{aligned} &\bigl|\mathcal{M}x_2(t) - \mathcal{M}x_1(t)\bigr| \\ &\quad\leqslant \Biggl|\int_0^\xi G(t,s)s^{\alpha-2} f\bigl(s,x_2(s),x_2(s+\sigma)\bigr)\,\mathrm{d}s \\ &\qquad - \int_0^\xi G(t,s)s^{\alpha-2} f\bigl(s,x_1(s),x_1(s+\sigma)\bigr)\,\mathrm{d}s\Biggr| \\ &\qquad + |A|\Biggl|\int_0^\xi G(t,s)s^{\alpha-2}\mathcal{W}^{-1}\bigl(\bar{x} - R\bigl(x_2(s),P\bigl(x_2(\xi^-)\bigr),Q\bigl(x_2(\xi^+)\bigr)\bigr)\bigr)\,\mathrm{d}s \\ &\qquad - \int_0^\xi G(t,s)s^{\alpha-2}\mathcal{W}^{-1}\bigl(\bar{x} - R\bigl(x_1(s),P\bigl(x_1(\xi^-)\bigr),Q\bigl(x_1(\xi^+)\bigr)\bigr)\bigr)\,\mathrm{d}s\Biggr| \end{aligned}$$

$$
\begin{aligned}
&+ \left| \int_\xi^1 G(t,s)(s-\xi)^{\alpha-2} f\big(s, x_2(s), x_2(s-\sigma)\big)\,\mathrm{d}s \right.\\
&\left. - \int_\xi^1 G(t,s)\big(s-\xi\big)^{\alpha-2} f\big(s, x_1(s), x_1(s-\sigma)\big)\,\mathrm{d}s \right|\\
&+ \big|\varphi\big(t, P\big(x_2(\xi^-)\big), Q\big(x_2(\xi^+)\big)\big) - \varphi\big(t, P\big(x_1(\xi^-)\big), Q\big(x_1(\xi^+)\big)\big)\big|\\
\leqslant{}& \left(4\lambda\varpi^* + p_0 + \frac{q_0}{1-k}\right)\|x_2 - x_1\|_{PC}\\
&+ \lambda|A|\big|\mathcal{W}^{-1}\big(R\big(x_1(s), P\big(x_1(\xi^-)\big), Q\big(x_1(\xi^+)\big)\big)\\
&- R\big(x_2(s), P\big(x_2(\xi^-)\big), Q\big(x_2(\xi^+)\big)\big)\big)\big|.
\end{aligned}
$$

Since,

$$
\begin{aligned}
&\big|\mathcal{W}^{-1}\big(R(x_1(s), P\big(x_1(\xi^-)\big), Q\big(x_1(\xi^+)\big)\big) - R\big(x_2(s), P\big(x_2(\xi^-)\big), Q\big(x_2(\xi^+)\big)\big)\big)\big|\\
&\quad = \left|\mathcal{W}^{-1}\left(\int_0^\xi G(\eta,s)s^{\alpha-2}\big(f\big(s, x_1(s), x_1(s+\sigma)\big) - f\big(s, x_2(s), x_2(s+\sigma)\big)\big)\,\mathrm{d}s\right)\right|\\
&\qquad + \left|\mathcal{W}^{-1}\left(\int_\xi^1 G(\eta,s)s^{\alpha-2}\big(f\big(s, x_1(s), x_1(s+\sigma)\big) - f\big(s, x_2(s), x_2(s+\sigma)\big)\big)\,\mathrm{d}s\right)\right|\\
&\qquad + \big|\mathcal{W}^{-1}\big(\varphi\big(\eta, P\big(x_1(\xi^-)\big), Q\big(x_1(\xi^+)\big)\big) - \varphi\big(\eta, P(x_2(\xi^-)), Q\big(x_2(\xi^+)\big)\big)\big)\big|\\
&\quad \leqslant |A\beta|^{-1}\left(4\lambda\varpi^* + p_0 + \frac{q_0}{1-k}\right)\|x_2 - x_1\|_{PC},
\end{aligned}
$$

we have

$$
\begin{aligned}
&\big|\mathcal{M}x_2(t) - \mathcal{M}x_1(t)\big|\\
&\quad \leqslant \left(4\lambda\varpi^* + p_0 + \frac{q_0}{1-k}\right)\|x_2 - x_1\|_{PC}\\
&\qquad + \lambda|A|\big|\mathcal{W}^{-1}\big(R\big(x_1(s), P\big(x_1(\xi^-)\big), Q\big(x_1(\xi^+)\big)\big)\\
&\qquad - R\big(x_2(s), P\big(x_2(\xi^-)\big), Q\big(x_2(\xi^+)\big)\big)\big)\big|\\
&\quad \leqslant \left(4\lambda\varpi^* + p_0 + \frac{q_0}{1-k}\right)\|x_2 - x_1\|_{PC}\\
&\qquad + \lambda|A||A\beta|^{-1}\left(4\lambda\varpi^* + p_0 + \frac{q_0}{1-k}\right)\|x_2 - x_1\|_{PC}\\
&\quad = \big(1 + \lambda\beta^{-1}\big)\left(4\lambda\varpi^* + p_0 + \frac{q_0}{1-k}\right)\|x_2 - x_1\|_{PC}.
\end{aligned}
$$

Thus,

$$\begin{aligned}\|\mathcal{M}x_2 - \mathcal{M}x_1\|_{PC} &\leqslant \left(1 + \lambda\beta^{-1}\right)\left(4\lambda\varpi^* + p_0 + \frac{q_0}{1-k}\right)\|x_2 - x_1\|_{PC} \\ &= l\|x_2 - x_1\|_{PC}.\end{aligned}$$

Since $0 < l < 1$, then $\mathcal{M}$ is a contraction operator.

In summary, according to the Banach fixed point theorem, $\mathcal{M}$ has a unique fixed point x in $PC(I, \mathbb{R})$, namely SYS (1) is controllable.

The proof is completed. □

Theorem 4. *Suppose* (H3) *and* (H4) *hold. If*

$$0 < \left(1 + \lambda\beta^{-1}\right)\left(2\lambda(\rho_2^* + \rho_3^*) + \frac{M_Q}{1-k} + M_P\right) < 1, \tag{12}$$

then SYS (1) *is controllable.*

Proof. From (10) the control function

$$u(t) = \mathcal{W}^{-1}\left(\bar{x} - R\left(x(s), P\left(x(\xi^-)\right), Q\left(x(\xi^+)\right)\right)\right)$$

and

$$\begin{aligned}\left|u(t)\right| &\leqslant \left\|\mathcal{W}^{-1}\right\|\left(|\bar{x}| + \left|\int_0^\xi G(\eta,s)s^{\alpha-2}f\left(s, x(s), x(s+\sigma)\right)\mathrm{d}s\right|\right. \\ &\quad + \left|\int_\xi^1 G(\eta,s)(s-\xi)^{\alpha-2}f\left(s, x(s), x(s-\sigma)\right)\mathrm{d}s\right| \\ &\quad \left. + \left|\varphi\left(\eta, P\left(x(\xi^-)\right), Q\left(x(\xi^+)\right)\right)\right|\right) \\ &\leqslant \left\|\mathcal{W}^{-1}\right\|\left(|\bar{x}| + \frac{1}{1-k}\left(\int_0^\xi s^{\alpha-2}\,\mathrm{d}s + \int_\xi^1 (s-\xi)^{\alpha-2}\,\mathrm{d}s\right)\left(\rho_1^* + \rho_2^*\|x\|_{PC}\right)\right. \\ &\quad \left. + \rho_3^*\|x\|_{PC} + M_P\|x\|_{PC} + N_P + \frac{1}{1-k}\left(M_Q\|x\|_{PC} + N_Q\right)\right) \\ &\leqslant |A\beta|^{-1}\left(|\bar{x}| + 2\lambda\left(\rho_1^* + \rho_2^*\|x\|_{PC} + \rho_3^*\|x\|_{PC}\right) + M_P\|x\|_{PC} + N_P\right. \\ &\quad \left. + \frac{1}{1-k}\left(M_Q\|x\|_{PC} + N_Q\right)\right) \\ &= |A\beta|^{-1}\left(|\bar{x}| + \left(2\lambda(\rho_2^* + \rho_3^*) + \frac{M_Q}{1-k} + M_P\right)\|x\|_{PC} + 2\lambda\rho_1^*\right. \\ &\quad \left. + \frac{N_Q}{1-k} + N_P\right).\end{aligned}$$

Since condition (12) holds, we can take

$$r = \frac{\lambda\beta^{-1}|\overline{x}| + (1+\lambda\beta^{-1})(2\lambda\rho_1^* + \frac{N_Q}{1-k} + N_P)}{1-(1+\lambda\beta^{-1})(2\lambda(\rho_2^*+\rho_3^*) + \frac{M_Q}{1-k} + M_P)},$$

Let $\Omega = \{x \in PC(I,\mathbb{R})\colon \|x\|_{PC} \leqslant r\}$. Then Ω is a nonempty bounded closed convex set in $PC(I,\mathbb{R})$.

Define operator $\mathcal{M}$ by (11). (H3) and (H4) indicate that $x \in PC(I,\mathbb{R})$ and for $t \in [0,1]$,

$$0 < \left|f\bigl(t, x(t), x(t\pm\sigma)\bigr)\right| \leqslant \rho_1^* + \rho_2^*\|x\|_{PC} + \rho_3^*\|x\|_{PC}$$

and

$$0 \leqslant \left|P(x)\right| \leqslant M_P\|x\|_{PC} + N_P, \qquad 0 \leqslant \left|Q(x)\right| \leqslant M_Q\|x\|_{PC} + N_Q.$$

Thus,

$$\begin{aligned}
&\left|\mathcal{M}x(t)\right| \\
&\quad\leqslant \left|\int_0^\xi G(t,s)s^{\alpha-2} f\bigl(s,x(s),x(s+\sigma)\bigr)\,\mathrm{d}s\right| \\
&\qquad + \left|A\int_0^\xi G(t,s)s^{\alpha-2}\mathcal{W}^{-1}\bigl(\bar{x} - R\bigl(x(s), P\bigl(x(\xi^-)\bigr), Q\bigl(x(\xi^+)\bigr)\bigr)\bigr)\,\mathrm{d}s\right| \\
&\qquad + \left|\int_\xi^1 G(t,s)\bigl(s-\xi\bigr)^{\alpha-2} f\bigl(s,x(s),x(s-\sigma)\bigr)\,\mathrm{d}s\right| + \left|\varphi\bigl(t, P\bigl(x(\xi^-)\bigr), Q\bigl(x(\xi^+)\bigr)\bigr)\right| \\
&\quad\leqslant 2\lambda\bigl(\rho_1^* + \rho_2^*\|x\|_{PC} + \rho_3^*\|x\|_{PC}\bigr) + M_P\|x\|_{PC} + N_P + \frac{1}{1-k}\bigl(M_Q\|x\|_{PC} + N_Q\bigr) \\
&\qquad + \lambda\beta^{-1}\left(|\bar{x}| + \left(2\lambda\bigl(\rho_2^*+\rho_3^*\bigr) + \frac{M_Q}{1-k} + M_P\right)\|x\|_{PC} + 2\lambda\rho_1^* + \frac{N_Q}{1-k} + N_P\right) \\
&\quad= \bigl(1+\lambda\beta^{-1}\bigr)\left(2\lambda(\rho_2^*+\rho_3^*) + \frac{M_Q}{1-k} + M_P\right)\|x\|_{PC} + \lambda\beta^{-1}|\bar{x}| \\
&\qquad + \bigl(1+\lambda\beta^{-1}\bigr)\left(2\lambda\rho_1^* + \frac{N_Q}{1-k} + N_P\right) \\
&\quad\leqslant r.
\end{aligned}$$

Therefore, $\|\mathcal{M}x\|_{PC} \leqslant r$, which implies that $\mathcal{M}(\Omega) \subseteq \Omega$. According to the continuity of function f, it is easy to obtain that operator $\mathcal{M}: \Omega \to \Omega$ is continuous.

Next, we prove that $\mathcal{M}(\Omega) \subseteq \Omega$ is a compact operator.

Based on Lemma 4, $G \in C(I\times I)$, so G is uniformly continuous on $I\times I$. That is, for any $\varepsilon > 0$, there exists $\delta_0 > 0$ such that $|G(t_2,s) - G(t_1,s)| < \varepsilon$ whenever $t_1, t_2, s \in I$

and $|t_1 - t_2| < \delta_0$. Let $\varepsilon > 0$, $\delta = \min\{\delta_0, \varepsilon\}$, and $x \in \Omega$. For $t_1, t_2 \in [0, \xi]$ and $|t_1 - t_2| < \delta$,

$$\begin{aligned}&\bigl|\varphi\bigl(t_2, P\bigl(x(\xi^-)\bigr), Q\bigl(x(\xi^+)\bigr)\bigr) - \varphi\bigl(t_1, P\bigl(x(\xi^-)\bigr), Q\bigl(x(\xi^+)\bigr)\bigr)\bigr| \\ &\quad\leqslant \frac{|t_2 - t_1|}{1-k}\bigl(M_Q\|x\|_{PC} + N_Q\bigr) \leqslant \frac{1}{1-k}\bigl(M_Q\|x\|_{PC} + N_Q\bigr)\varepsilon.\end{aligned}$$

For $t_1, t_2 \in (\xi, 1]$ and $|t_1 - t_2| < \delta$,

$$\begin{aligned}&\bigl|\varphi\bigl(t_2, P\bigl(x(\xi^-)\bigr), Q\bigl(x(\xi^+)\bigr)\bigr) - \varphi\bigl(t_1, P\bigl(x(\xi^-)\bigr), Q\bigl(x(\xi^+)\bigr)\bigr)\bigr| \\ &\quad\leqslant \frac{k|t_2 - t_1|}{1-k}\bigl(M_Q\|x\|_{PC} + N_Q\bigr) \leqslant \frac{1}{1-k}\bigl(M_Q\|x\|_{PC} + N_Q\bigr)\varepsilon.\end{aligned}$$

As a result, for $t_1, t_2 \in [0, \xi]$ or $t_1, t_2 \in (\xi, 1]$ and $|t_1 - t_2| < \delta$,

$$\begin{aligned}&\bigl|\mathcal{M}x(t_2) - \mathcal{M}x(t_1)\bigr| \\ &\quad\leqslant \int_0^\xi \bigl|G(t_2, s) - G(t_1, s)\bigr| s^{\alpha-2} \bigl|f\bigl(s, x(s), x(s+\sigma)\bigr)\bigr| \,\mathrm{d}s \\ &\qquad + |A| \int_0^\xi \bigl|G(t_2, s) - G(t_1, s)\bigr| s^{\alpha-2} \bigl|\mathcal{W}^{-1}\bigl(\bar{x} - R\bigl(x(s), P\bigl(x(\xi^-)\bigr), Q\bigl(x(\xi^+)\bigr)\bigr)\bigr)\bigr| \,\mathrm{d}s \\ &\qquad + \int_\xi^1 \bigl|G(t_2, s) - G(t_1, s)\bigr| (s-\xi)^{\alpha-2} \bigl|f\bigl(s, x(s), x(s-\sigma)\bigr)\bigr| \,\mathrm{d}s \\ &\qquad + \bigl|\varphi\bigl(t_2, P\bigl(x(\xi^-)\bigr), Q\bigl(x(\xi^+)\bigr)\bigr) - \varphi\bigl(t_1, P\bigl(x(\xi^-)\bigr), Q\bigl(x(\xi^+)\bigr)\bigr)\bigr| \\ &\quad\leqslant \frac{\varepsilon}{1-k}\left(\int_0^\xi s^{\alpha-2}\,\mathrm{d}s + \int_\xi^1 (s-\xi)^{\alpha-2}\,\mathrm{d}s\right)\bigl(\rho_1^* + \rho_2^*\|x\|_{PC} + \rho_3^*\|x\|_{PC}\bigr) \\ &\qquad + \frac{\varepsilon}{1-k}\bigl(M_Q\|x\|_{PC} + N_Q\bigr) \\ &\qquad + \lambda\beta^{-1}\left(|\bar{x}| + \left(2\lambda(\rho_2^* + \rho_3^*) + \frac{M_Q}{1-k} + M_P\right)\|x\|_{PC} + 2\lambda\rho_1^* + \frac{N_Q}{1-k} + N_P\right)\varepsilon \\ &\quad\leqslant \left(2\lambda\bigl(\rho_1^* + \rho_2^* r + \rho_3^* r\bigr) + \frac{M_Q r + N_Q}{1-k}\right. \\ &\qquad \left. + \lambda\beta^{-1}\left(|\bar{x}| + \left(2\lambda(\rho_2^* + \rho_3^*) + \frac{M_Q}{1-k} + M_P\right)r + 2\lambda\rho_1^* + \frac{N_Q}{1-k} + N_P\right)\right)\varepsilon.\end{aligned}$$

Hence, $\mathcal{M}$ is equicontinuous, and $\mathcal{M}(\Omega) \subseteq \Omega$. By Arzelà–Ascoli theorem, the operator $\mathcal{M}$ is compact. Thus, $\mathcal{M}$ is completely continuous.

In conclusion, by the Schauder fixed point theorem, the operator $\mathcal{M}$ has a fixed point in $PC(I, \mathbb{R})$. Therefore, SYS (1) is controllable.

The proof is completed. □

4 Ulam–Hyers stability analysis

In this section, we discuss the Ulam–Hyers stability for SYS (1). First, we present the definitions of Ulam–Hyers stability for this system as follows.

Definition 5. (See [13].) SYS (1) is said to be Ulam–Hyers stable if there exists a constant $c > 0$ such that for any $\varepsilon > 0$ and any function $z \in PC(I, \mathbb{R})$ satisfying the inequalities system

$$\begin{aligned} &\bigl|\mathfrak{D}_0^\alpha z(t) + f\bigl(t, z(t), z(t+\sigma)\bigr) + Au(t)\bigr| < \varepsilon, \quad t \in (0, \xi), \\ &\bigl|\mathfrak{D}_\xi^\alpha z(t) + f\bigl(t, z(t), z(t-\sigma)\bigr)\bigr| < \varepsilon, \quad t \in (\xi, 1) \\ &\Delta z|_{t=\xi} = P\bigl(z(\xi^-)\bigr), \qquad \Delta z'|_{t=\xi} = Q\bigl(z(\xi^+)\bigr), \\ &z(0) = 0, \qquad z'(1) = kz'(0), \end{aligned} \tag{13}$$

there exists a unique solution $x \in PC(I, \mathbb{R})$ of SYS (1) such that

$$\|z - x\|_{PC} \leqslant c\varepsilon.$$

Theorem 5. *Assume that all the conditions of Theorem* 3 *are satisfied. Then SYS* (1) *is Ulam–Hyers stable.*

Proof. Let functions $\phi_1, \phi_2 \in PC(I, \mathbb{R})$ and $|\phi_1(t)| \leqslant \varepsilon$, $|\phi_2(t)| \leqslant \varepsilon$. Consider the following system:

$$\begin{aligned} &\mathfrak{D}_0^\alpha z(t) + f\bigl(t, z(t), z(t+\sigma)\bigr) + Au(t) + \phi_1(t) = 0, \quad t \in (0, \xi), \\ &\mathfrak{D}_\xi^\alpha z(t) + f\bigl(t, z(t), z(t-\sigma)\bigr) + \phi_2(t) = 0, \quad t \in (\xi, 1), \\ &\Delta z|_{t=\xi} = P\bigl(z(\xi^-)\bigr), \qquad \Delta z'|_{t=\xi} = Q\bigl(z(\xi^+)\bigr), \\ &z(0) = 0, \qquad z'(1) = kz'(0). \end{aligned} \tag{14}$$

Recall formula (6), and let the solution of SYS (14) be

$$\begin{aligned} z &= \mathfrak{T}z(t) \\ &= \int_0^\xi G(t,s)s^{\alpha-2}\bigl(f\bigl(s, z(s), z(s+\sigma)\bigr) + \phi_1(s)\bigr)\,\mathrm{d}s \\ &\quad + \int_\xi^1 G(t,s)(s-\xi)^{\alpha-2}\bigl(f\bigl(s, z(s), z(s-\sigma)\bigr) + \phi_2(s)\bigr)\,\mathrm{d}s \\ &\quad + A\int_0^\xi G(t,s)s^{\alpha-2}u_z(s)\,\mathrm{d}s + \varphi\bigl(t, P\bigl(z(\xi^-)\bigr), Q\bigl(z(\xi^+)\bigr)\bigr). \end{aligned}$$

We know from the assumptions that SYS (14) is controllable, and it's control function is as follows:

$$\begin{aligned} u_z(t) = \mathcal{W}^{-1}\Bigg(\bar{x} - \Bigg(&\int_0^{\xi} G(\eta,s)s^{\alpha-2}\big(f\big(s,z(s),z(s+\sigma)\big)+\phi_1(s)\big)\,\mathrm{d}s \\ &+ \int_{\xi}^{1} G(\eta,s)(s-\xi)^{\alpha-2}\big(f\big(s,z(s),z(s-\sigma)\big)+\phi_2(s)\big)\,\mathrm{d}s \\ &+ \varphi\big(\eta,P\big(z(\xi^-)\big),Q\big(z(\xi^+)\big)\big)\Bigg)\Bigg) \\ := \mathcal{W}^{-1}&\big(\bar{x} - R\big(z(s),P\big(z(\xi^-)\big),Q\big(z(\xi^+)\big)\big)\big). \end{aligned}$$

From formula (10) the control function of SYS (1) is

$$\begin{aligned} u_x(t) = \mathcal{W}^{-1}\Bigg(\bar{x} - \Bigg(&\int_0^{\xi} G(\eta,s)s^{\alpha-2}f\big(s,x(s),x(s+\sigma)\big)\,\mathrm{d}s \\ &+ \int_{\xi}^{1} G(\eta,s)(s-\xi)^{\alpha-2}f\big(s,x(s),x(s-\sigma)\big)\,\mathrm{d}s + \varphi\big(\eta,P\big(x(\xi^-)\big),Q\big(x(\xi^+)\big)\big)\Bigg)\Bigg) \\ := \mathcal{W}^{-1}&\big(\bar{x} - R\big(x(s),P\big(x(\xi^-)\big),Q\big(x(\xi^+)\big)\big)\big). \end{aligned}$$

Thus,

$$\begin{aligned} |u_z - u_x| &= \big|\mathcal{W}^{-1}\big(\bar{x} - R\big(z(s),P\big(z(\xi^-)\big),Q\big(z(\xi^+)\big)\big)\big) \\ &\quad - \mathcal{W}^{-1}\big(\bar{x} - R\big(x(s),P\big(x(\xi^-)\big),Q\big(x(\xi^+)\big)\big)\big)\big| \\ &= \bigg|\mathcal{W}^{-1}\Bigg(\bar{x} - \Bigg(\int_0^{\xi} G(\eta,s)s^{\alpha-2}\big(f\big(s,z(s),z(s+\sigma)\big)+\phi_1(s)\big)\,\mathrm{d}s \\ &\quad + \int_{\xi}^{1} G(\eta,s)(s-\xi)^{\alpha-2}\big(f\big(s,z(s),z(s-\sigma)\big)+\phi_2(s)\big)\,\mathrm{d}s \\ &\quad + \varphi\big(\eta,P\big(z(\xi^-)\big),Q\big(z(\xi^+)\big)\big)\Bigg)\Bigg) \\ &\quad - \mathcal{W}^{-1}\Bigg(\bar{x} - \Bigg(\int_0^{\xi} G(\eta,s)s^{\alpha-2}f\big(s,x(s),x(s+\sigma)\big)\,\mathrm{d}s \\ &\quad + \int_{\xi}^{1} G(\eta,s)(s-\xi)^{\alpha-2}f\big(s,x(s),x(s-\sigma)\big)\,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&+ \varphi\bigl(\eta, P\bigl(x(\xi^-)\bigr), Q\bigl(x(\xi^+)\bigr)\bigr)\Bigr)\Bigr)\Bigr| \\
&\leqslant |A\beta|^{-1}\frac{2\varpi^*}{1-k}\Biggl(\int_0^\xi s^{\alpha-2}\,\mathrm{d}s + \int_\xi^1 (s-\xi)^{\alpha-2}\,\mathrm{d}s\Biggr)\|z-x\|_{PC} \\
&+ |A\beta|^{-1}\biggl(p_0 + \frac{q_0}{1-k}\biggr)\|z-x\|_{PC} \\
&+ |A\beta|^{-1}\frac{1}{1-k}\Biggl(\int_0^\xi s^{\alpha-2}\bigl|\phi_1(s)\bigr|\,\mathrm{d}s + \int_\xi^1 s^{\alpha-2}\bigl|\phi_2(s)\bigr|\,\mathrm{d}s\Biggr) \\
&\leqslant |A\beta|^{-1}\biggl(4\lambda\varpi^* + p_0 + \frac{q_0}{1-k}\biggr)\|z-x\|_{PC} + 2\lambda|A\beta|^{-1}\varepsilon.
\end{aligned}
$$

Sequentially,

$$
\begin{aligned}
\bigl|z(t)-x(t)\bigr| \leqslant{}& \Biggl|\int_0^\xi G(t,s)s^{\alpha-2}\bigl(f\bigl(s,z(s),z(s+\sigma)\bigr) + \phi_1(s)\bigr)\,\mathrm{d}s \\
&- \int_0^\xi G(t,s)s^{\alpha-2} f\bigl(s,x(s),x(s+\sigma)\bigr)\,\mathrm{d}s\Biggr| \\
&+ \Biggl|A\int_0^\xi G(t,s)s^{\alpha-2}u_z(s)\,\mathrm{d}s - A\int_0^\xi G(t,s)s^{\alpha-2}u_x(s)\,\mathrm{d}s\Biggr| \\
&+ \Biggl|\int_\xi^1 G(t,s)(s-\xi)^{\alpha-2}\bigl(f\bigl(s,z(s),z(s-\sigma)\bigr) + \phi_2(s)\bigr)\,\mathrm{d}s \\
&- \int_\xi^1 G(t,s)(s-\xi)^{\alpha-2} f\bigl(s,x(s),x(s-\sigma)\bigr)\,\mathrm{d}s\Biggr| \\
&+ \bigl|\varphi\bigl(t, P\bigl(z(\xi^-)\bigr), Q\bigl(z(\xi^+)\bigr)\bigr) - \varphi\bigl(t, P\bigl(x(\xi^-)\bigr), Q\bigl(x(\xi^+)\bigr)\bigr)\bigr| \\
\leqslant{}& \bigl(1+\lambda\beta^{-1}\bigr)\biggl(4\lambda\varpi^* + p_0 + \frac{q_0}{1-k}\biggr)\|z-x\|_{PC} + 2\lambda\bigl(1+\lambda\beta^{-1}\bigr)\varepsilon.
\end{aligned}
$$

Because inequality (8) holds,

$$
\|z-x\| < \frac{2\lambda(1+\lambda\beta^{-1})}{1-(1+\lambda\beta^{-1})(4\lambda\varpi^* + p_0 + \frac{q_0}{1-k})}\varepsilon.
$$

Let

$$
c := \frac{2\lambda(1+\lambda\beta^{-1})}{1-(1+\lambda\beta^{-1})(4\lambda\varpi^* + p_0 + \frac{q_0}{1-k})},
$$

hence, for $t \in I$,

$$\left|z(t) - x(t)\right| \leqslant c\varepsilon.$$

The proof is completed. □

5 Example

In this section, we consider the following system:

$$\begin{aligned}
&\mathfrak{D}_0^{6/5} x(t) + \frac{t}{3} + \frac{x(t)}{1000+t} + \frac{x(t+\frac{1}{5})}{1000} + 7u(t) = 0, \quad t \in \left(0, \frac{1}{4}\right),\\
&\mathfrak{D}_{1/4}^{6/5} x(t) + \frac{t}{3} + \frac{x(t)}{1000+t} + \frac{x(t-\frac{1}{5})}{1000} = 0, \quad t \in \left(\frac{1}{4}, 1\right),\\
&\Delta x|_{t=1/4} = \frac{x(\frac{1}{4}^-)}{200 + x(\frac{1}{4}^-)}, \qquad \Delta x'|_{t=1/4} = \frac{-x(\frac{1}{4}^+)}{100 + x(\frac{1}{4}^+)},\\
&x(0) = 0, \qquad x'(1) = \frac{1}{2} x'(0),
\end{aligned} \tag{15}$$

where $\alpha = 6/5$, $t_0 = 0$, $\xi = 1/4$, $\sigma = 1/5$, $A = 7$, $k = 1/2$, and $\tau = 3/4$,

$$f\big(t, x(t), x(t+\sigma)\big) = \frac{t}{3} + \frac{x(t)}{1000+t} \frac{x(t+\frac{1}{5})}{1000},$$

$$f\big(t, x(t), x(t-\sigma)\big) = \frac{t}{3} + \frac{x(t)}{1000+t} + \frac{x(t-\frac{1}{5})}{1000},$$

$$P\left(x\left(\frac{1}{4}^-\right)\right) = \frac{x(\frac{1}{4}^-)}{200 + x(\frac{1}{4}^-)}, \qquad Q\left(x\left(\frac{1}{4}^+\right)\right) = \frac{-x(\frac{1}{4}^+)}{100 + x(\frac{1}{4}^+)}.$$

It is readily found that the control function satisfying $x(1/6) = 2$ is given by

$$\begin{aligned}
u = \mathcal{W}^{-1}\Bigg(2 - \Bigg(&\int_0^{1/4} G\left(\frac{1}{6}, s\right) s^{\alpha-2} \left(\frac{s}{3} + \frac{x(s)}{1+s} + x\left(s + \frac{1}{5}\right)\right)\Bigg)\Bigg)\,\mathrm{d}s\\
&+ \int_{1/4}^{1} G\left(\frac{1}{6}, s\right)\left(s - \frac{1}{4}\right)^{\alpha-2}\left(\frac{s}{3} + \frac{x(s)}{1+s} + x\left(s - \frac{1}{5}\right)\right)\mathrm{d}s\\
&+ \varphi\left(\frac{1}{6}, \frac{x(\frac{1}{4}^-)}{200 + x(\frac{1}{4}^-)}, \frac{-x(\frac{1}{4}^+)}{100 + x(\frac{1}{4}^+)}\right),
\end{aligned}$$

and

$$\beta = \int_0^{1/4} G\left(\frac{1}{6}, s\right) s^{\alpha-2}\,\mathrm{d}s = \frac{35}{36 \cdot 6^{1/5}} + \frac{1}{3}\left(\frac{5}{2^{2/5}} - \frac{5}{6^{1/5}}\right) \approx 0.77780.$$

Thus,

$$\lambda = \frac{\tau^{\alpha-1}}{(1-k)(\alpha-1)} = \frac{(\frac{3}{4})^{6/5-1}}{(1-\frac{1}{2})(\frac{6}{5}-1)} = 5 \cdot 2^{3/5} \cdot 3^{1/5} \approx 9.44088,$$

and for any $t \in I, x_1, x_2, y_1, y_2 \in \mathbb{R}^+$,

$$0 \leqslant \left|f(t, x_2, y_2) - f(t, x_1, y_1)\right| \leqslant \frac{1}{1000}\left(|x_2 - x_1| + |y_2 - y_1|\right), \qquad \varpi^* = \frac{1}{1000}.$$

Therefore, (H1) is satisfied. For any $x_1, x_2 \in \mathbb{R}^+$,

$$0 \leqslant \left|P(x_2) - P(x_1)\right| \leqslant \frac{1}{200}|x_2 - x_1|, \qquad 0 \leqslant \left|Q(x_2) - Q(x_1)\right| \leqslant \frac{1}{100}|x_2 - x_1|,$$
$$p_0 = \frac{1}{200}, \qquad q_0 = \frac{1}{100}.$$

Consequently, (H2) is satisfied.

Then

$$0 < \left(1 + \lambda\beta^{-1}\right)\left(4\lambda\varpi^* + p_0 + \frac{q_0}{1-k}\right) = l \approx 0.82458 < 1.$$

According to Theorems 3 and 5, SYS (15) is controllable and Ulam–Hyers stable.

On the other hand, for any $t \in I, x, y \in \mathbb{R}^+$,

$$0 < \left|f(t, x, y)\right| \leqslant \frac{t}{3} + \frac{1}{1000+t}|x| + \frac{1}{1000}|y| \leqslant \frac{1}{3} + \frac{1}{1000}|x| + \frac{1}{1000}|y|,$$
$$\rho_2^* = \rho_3^* = \frac{1}{1000},$$

and

$$0 \leqslant \left|P(x)\right| = \left|\frac{x}{200+x}\right| \leqslant \frac{1}{200}|x|, \qquad 0 \leqslant \left|Q(x)\right| = \left|\frac{-x}{100+x}\right| \leqslant \frac{1}{100}|x|,$$
$$M_P = \frac{1}{200}, \qquad M_Q = \frac{1}{100}.$$

So, (H3) and (H4) are satisfied.

At this point,

$$\left(1 + \lambda\beta^{-1}\right)\left(2\lambda(\rho_2^* + \rho_3^*) + \frac{M_Q}{1-k} + M_P\right) \approx 0.82458 < 1.$$

According to Theorem 4, SYS (15) is controllable.

6 Conclusions

We study an impulsive fractional-order differential system with delay, which exhibits different dynamic behaviors in two distinct stages. Under the constraints of boundary conditions, we investigate the controllability problem of the system (i.e., the system passing through a given point) and prove the system's controllability. We apply the classical Banach fixed point theorem and Schauder fixed point theorem to the analysis of this model, combined with analytical techniques.

Overall, the research methodologies we have employed hold profound application value and practical implications. In our prior work, as detailed in [28], we delved into the Ulam–Hyers stability of piecewise conformable systems. Building upon this solid foundation, the present paper undertakes an in-depth investigation into the controllability and stability of SYS (1). Looking ahead to the subsequent research phase, our focus will be directed towards systems incorporating ABC fractional derivatives. We aim to comprehensively explore the aspects of controllability and stability within these systems, thereby contributing to the expansion of knowledge in this specialized domain.

In the long-term research trajectory, we are committed to conducting a thorough theoretical analysis and a painstakingly detailed simulation of the actual model. This dual-pronged approach is designed to not only rigorously validate our existing research outcomes but also to significantly broaden the scope of practical applications of these findings, thus maximizing their potential impact across various relevant fields.

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