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Joint ruin probability and risk contagion measure in a quota-share reinsurance risk model: An asymptotic approach*

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Abstract. Consider a quota-share reinsurance risk model with constant premiums and stochastic returns in which an insurer purchases fixed-proportion quota-share reinsurance for its business line. Both the insurer and reinsurer may engage in risk-free investments, with their nonnegative general (not necessarily Lévy) log-price processes following an arbitrary dependence structure. Under the assumption that claim sizes are pairwise strongly quasiasymptotically independent and follow heavy-tailed distributions, this paper derives asymptotic formulas for two types of finite-time joint ruin probabilities and for a risk contagion measure from the insurer to the reinsurer. Furthermore, we conduct numerical studies to verify the accuracy of the derived asymptotic results, using the Monte Carlo method combined with an explicit order-3.0 weak scheme.

Keywords: asymptotics, quota-share reinsurance risk model, finite-time joint ruin probability, risk contagion measure, explicit order-3.0 weak scheme.

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1 Introduction

Consider a continuous-time risk model in which an insurer purchases fixed-proportion quota-share reinsurance for its line of business. Specifically, claims X_i, $i \in \mathbb{N}$, where $\mathbb{N} = \{1, 2, \ldots\}$, arrive at times τ_i, $i \in \mathbb{N}$, forming a counting process

$$N(t) = \sup\{i \in \mathbb{N}\colon\ \tau_i \leqslant t\}, \quad t \geqslant 0, \tag{1}$$

where $\sup\emptyset = 0$ by convention, and the process has a finite mean function $\lambda(t) = \mathbf{E}[N(t)]$. Under the quota-share reinsurance strategy, for each claim X_i, $i \in \mathbb{N}$, the insurer retains a risk of ρX_i and transfers the remaining $(1-\rho)X_i$ to a reinsurer; here $\rho \in (0,1)$ denotes the quota-share retention level. Both the insurer and the reinsurer are supposed to engage in risk-free investments, giving rise to two general nonnegative stochastic processes $\{L_I(t),\ t \geqslant 0\}$ and $\{L_R(t),\ t \geqslant 0\}$, which represent the log-price processes of their respective investments. Typically, the claim sizes $\{X_i,\ i \in \mathbb{N}\}$ are assumed to be a sequence of identically distributed (though not necessarily independent) nonnegative random variables, with X as a generic representative and F as their common distribution function. Additionally, $\{X_i,\ i \in \mathbb{N}\}$, $\{N(t),\ t \geqslant 0\}$, and $\{(L_I(t), L_R(t))^\top,\ t \geqslant 0\}$ are mutually independent, while $\{L_I(t),\ t \geqslant 0\}$ and $\{L_R(t),\ t \geqslant 0\}$ may exhibit arbitrary dependence.

Under this modeling framework, the discounted aggregate claim amounts for the insurer and the reinsurer up to time $t \geqslant 0$ can be expressed as

$$D_I(t) = \rho \sum_{i=1}^{N(t)} X_i \mathrm{e}^{-L_I(\tau_i)} \quad \text{and} \quad D_R(t) = (1-\rho) \sum_{j=1}^{N(t)} X_j \mathrm{e}^{-L_R(\tau_j)}. \tag{2}$$

Furthermore, the discounted surplus processes of the insurer and the reinsurer with risk-free investments satisfy the following two stochastic differential equations (SDEs):

$$\mathrm{d}U_I(t) = c_I \mathrm{e}^{-L_I(t)}\,\mathrm{d}t - \mathrm{d}D_I(t), \qquad \mathrm{d}U_R(t) = c_R \mathrm{e}^{-L_R(t)}\,\mathrm{d}t - \mathrm{d}D_R(t), \tag{3}$$

with $U_I(0) = x$ and $U_R(0) = y$ denoting the initial surpluses of the insurer and the reinsurer, respectively, and $c_I > 0$ and $c_R > 0$ representing their corresponding constant premium rates. In general, the insurer is assumed to receive a total constant premium income of $c_T > 0$ per unit time. Of this total, ec_T is allocated to cover the insurer's operational expenses, with $e \in [0,1)$. For a given retention level $\rho \in (0,1)$, the insurer remits a premium of $(1-\rho)c_T$ per unit time to the reinsurer, with a proportional commission of $\delta(1-\rho)c_T$ deducted, where $\delta \in [0,1)$. Within this framework, the insurer's net retained premium rate and the reinsurer's premium rate can be explicitly specified as $c_I = (1-e)c_T - c_R$ and $c_R = (1-\delta)(1-\rho)c_T$, respectively. For additional details on quota-share reinsurance models, see, e.g., [13].

In the above model, the insurer and reinsurer may face joint ruin under various scenarios, leading to distinct types of joint ruin probabilities. For any time $t \geqslant 0$, two finite-time joint ruin probabilities can be defined as follows:

$$\psi_{\mathrm{sim}}(x,y;t) = \mathbf{P}\Big(\inf_{0 \leqslant s \leqslant t}\big\{U_I(s) \vee U_R(s)\big\} < 0 \,\Big|\, \big(U_I(0), U_R(0)\big)^\top = (x,y)^\top\Big),$$

which denotes the probability that the insurer and reinsurer experience joint ruin simultaneously over the time horizon $[0, t]$, where $U_1 \vee U_2 = \max\{U_1, U_2\}$; and

$$\psi_{\text{and}}(x, y; t) = \mathbf{P}\Big(\inf_{0 \leqslant s \leqslant t} U_I(s) < 0,\ \inf_{0 \leqslant s \leqslant t} U_R(s) < 0 \;\Big|\; \big(U_I(0), U_R(0)\big)^\top = (x, y)^\top\Big),$$

which represents the probability that both the insurer and reinsurer experience joint ruin, though not necessarily at the same time. Direct verification shows that $\psi_{\text{sim}}(x, y; t) \leqslant \psi_{\text{and}}(x, y; t)$ holds for all $t \geqslant 0$.

Although extensive research has focused on optimal reinsurance strategies (see, e.g., [3] and [9]), both insurers and reinsurers still suffer severe losses from catastrophic events, such as the 2022 Turkey–Syria earthquakes, the 2023 U.S. Hawaii wildfires, the 2023 Libya floods, and the 2024 U.S. Hurricane Helene, with some companies even becoming insolvent. To tackle the increasing extreme risks ahead, the insurance and reinsurance industry needs to reasonably assess the probabilities of joint ruin or survival by aligning the interests of both insurers and reinsurers. [12, 14, 15] developed a reinsurance risk model in which claims, originating from a portfolio of risks, arrive via a Poisson process. Within this framework, the insurer and reinsurer proportionally share both the liability of each individual claim and the total premium income. As a joint risk measure, [14] introduced the concept of finite-time joint survival probability for the insurer and reinsurer, and derived its closed-form expressions. For further advances in this area, see, e.g., [2, 7, 26]. In this paper, we focus on the aforementioned quota-share reinsurance risk model, aiming to conduct an asymptotic analysis of two types of finite-time joint ruin probabilities and a risk contagion measure between the insurer and the reinsurer in the context of large (i.e., heavy-tailed) claims. Similar to our model, other studies have also developed two-dimensional insurance risk models with investments; however, these models only consider insurers (without involving reinsurers) operating two lines of business. Under the assumption that the log-price process of investments for each line follows a Lévy process, several asymptotic formulas for different finite-time ruin probabilities have been derived for heavy-tailed claims. See, e.g., [10, 21, 22, 28, 31].

In the present paper, we continue to examine a quota-share reinsurance risk model where both the insurer and reinsurer invest in a risk-free market. Our model incorporates two fully general nonnegative stochastic processes as the log-price processes for the insurer and reinsurer, encompassing models such as the Vasicek model, the Cox–Ingersoll–Ross (CIR) model, and the Heston model. Notably, the two investment log-price processes may exhibit arbitrary dependence, while the claim sizes admit a specific dependence structure. Under the condition of heavy-tailed claim sizes, this paper aims to establish asymptotic formulas for the two aforementioned finite-time joint ruin probabilities, along with a risk contagion measure from the insurer to the reinsurer. We further conduct numerical studies to verify the accuracy of our derived asymptotic results and perform sensitivity analysis of the joint ruin probability with respect to key model parameters. However, in practical simulations, directly computing the values of the two general log-price processes at each time step is impractical, especially when these processes are governed by more complex models (e.g., the CIR model) defined by specific SDEs. To address this issue, we employ an explicit order-3.0 weak scheme, which is widely used for solving SDEs. One key advantage of this scheme is that it avoids derivatives of the drift and diffusion coefficients, enabling us to more conveniently approximate the values of the two log-price processes at discrete time points. For further details on the explicit order-3.0 weak scheme, we refer the reader to the monograph by [17].

The rest of the paper is organized as follows. Section 2 first reviews key concepts of heavy-tailed distributions and the strong quasiasymptotic independence structure, and then presents the main results on the asymptotic behavior of the two finite-time joint ruin probabilities for both the insurer and the reinsurer, as well as that of the risk contagion measure between them. Section 3 carries out numerical studies to verify the derived asymptotic results, using the Monte Carlo method combined with an explicit order-3.0 weak scheme. Section 4 proves the main results following a series of lemmas.

2 Preliminaries and main results

Throughout this paper, all limit relationships are understood to hold as $x \to \infty$ or $\min\{x, y\} \to \infty$ unless otherwise stated. For two positive functions f and g, we use the following standard notations: $f \sim g$ if $\lim f/g = 1$, $f = o(1)g$ if $\lim f/g = 0$, $f = O(1)g$ if $\limsup f/g < \infty$, $f \asymp g$ if $f = O(1)g$ and $g = O(1)f$, and $f \lesssim g$ if $\limsup f/g \leqslant 1$. For any $x, y \in \mathbb{R}$, we denote $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$, and $x^+ = x \vee 0$, $x^- = -(x \wedge 0)$. Furthermore, for two positive bivariate functions, relation $f(x, y; t) \sim g(x, y; t)$ holds uniformly for all t in a nonempty set A if

$$\lim_{x \wedge y \to \infty} \sup_{t \in A} \left| \frac{f(x, y; t)}{g(x, y; t)} - 1 \right| = 0;$$

and $f(x, y; t) \lesssim g(x, y; t)$ holds uniformly for all $t \in A$ if

$$\lim_{x \wedge y \to \infty} \sup_{t \in A} \frac{f(x, y; t)}{g(x, y; t)} \leqslant 1.$$

For any set A, we denote its indicator function by $\mathbf{1}_A$.

2.1 Heavy-tailed distributions and strong quasiasymptotic independence

In this paper, we restrict all claim sizes to follow heavy-tailed distributions. For a heavy-tailed distribution function F, its right tail $\overline{F}(x) = 1 - F(x)$ remains positive for all $x \in \mathbb{R}$. Among heavy-tailed distributions, consistently varying-tailed distributions are particularly important. A distribution F on $\mathbb{R}$ is said to be consistently varying-tailed, written as $F \in \mathcal{C}$, if $\lim_{y \downarrow 1} \overline{F}_*(y) = 1$ or $\lim_{y \uparrow 1} \overline{F}^*(y) = 1$, where $\overline{F}_*(y) = \liminf \overline{F}(xy)/\overline{F}(x)$ and $\overline{F}^*(y) = \limsup \overline{F}(xy)/\overline{F}(x)$ for any $y > 0$. In particular, a distribution F on $\mathbb{R}$ is said to be regularly varying-tailed, written as $F \in \mathcal{R}_{-\alpha}$ for some $0 \leqslant \alpha < \infty$, if $\overline{F}(xy) \sim y^{-\alpha}\overline{F}(x)$ for any fixed $y > 0$. Consistently and regularly varying-tailed distributions are widely used to model heavy-tailed phenomena in insurance and finance, see, e.g., [1] and [8].

Furthermore, for a distribution F on $\mathbb{R}$ with an ultimate right tail, its upper Matuszewska index is defined as

$$J_F^+ = -\lim_{y\to\infty} \frac{\log \overline{F}_*(y)}{\log y},$$

and it is closely related to consistently varying-tailed distributions. For $F \in \mathcal{C}$, we have $J_F^+ < \infty$. In particular, for $F \in \mathcal{R}_{-\alpha}$, $\alpha > 0$, this implies $J_F^+ = \alpha$.

We next introduce a dependence structure to model the relationship between any two claim sizes, which can be found in [11, 18, 24]. A sequence of nonnegative random variables $\{X_i,\ i \in \mathbb{N}\}$ is said to possess the pairwise *strong quasiasymptotic independence* structure if for any $i \neq j \in \mathbb{N}$,

$$\lim_{x\wedge y\to\infty} \mathbf{P}(X_i > x \mid X_j > y) = 0. \tag{4}$$

Note that the *strong quasiasymptotic independence* describes a type of weak dependence between two random variables and includes many commonly used copulas such as the Farlie–Gumbel–Morgenstern (FGM) copula, the Ali–Mikhail–Haq copula, and the Frank copula. Two similar concepts, termed *quasiasymptotic independence* and *strong asymptotic independence*, were formally proposed by [4] and [19], respectively. The former (quasiasymptotic independence) only requires setting $x = y$ in (4), making it slightly weaker than the aforementioned *strong quasiasymptotic independence*. In contrast, the latter (strong asymptotic independence) is more restrictive than the dependence structure we consider: it requires $\mathbf{P}(X_i > x,\ X_j > y) \sim C\mathbf{P}(X_i > x)\mathbf{P}(X_j > y)$ for some $C \geqslant 0$. For more applications of various weak dependence structures in finance and insurance, we refer the reader to [5, 20, 21, 32].

2.2 Main results

In our study, we restrict the time of interest to the set $\Lambda = \{t \geqslant 0\colon\ \lambda(t) > 0\}$.

Theorem 1. *Consider the quota-share reinsurance risk model* (3) *with a fixed retention level* $\rho \in (0,1)$. *Assume that the claim sizes* $\{X_i,\ i \in \mathbb{N}\}$ *form a sequence of pairwise strongly quasiasymptotically independent and identically distributed nonnegative random variables with a generic representative* X *and a common distribution* $F \in \mathcal{C}$; *the claim arrival process* $\{N(t),\ t \geqslant 0\}$ *is a general counting process with* $\lambda(t) = \mathbf{E}[N(t)]$*; and the investment log-price processes* $\{L_I(t),\ t \geqslant 0\}$ *and* $\{L_R(t),\ t \geqslant 0\}$ *are general nonnegative stochastic processes with arbitrary dependence and*

$$\sup_{0\leqslant s\leqslant t} L_I(s) + \sup_{0\leqslant s\leqslant t} L_R(s) < \infty$$

almost surely for any fixed $t \in \Lambda$. *Assume further that* $\{X_i,\ i \in \mathbb{N}\}$, $\{(L_I(t), L_R(t))^\top,\ t \geqslant 0\}$, *and* $\{N(t),\ t \geqslant 0\}$ *are mutually independent. If* $\mathbf{E}[(N(t))^{p+1}] < \infty$ *for some* $p > J_F^+$, *then it holds that for any fixed* $t \in \Lambda$,

$$\psi_{\mathrm{sim}}(x,y;t) \sim \psi_{\mathrm{and}}(x,y;t) \sim \int_{0^-}^{t} \mathbf{P}\biggl(X > \frac{x\mathrm{e}^{L_I(s)}}{\rho} \vee \frac{y\mathrm{e}^{L_R(s)}}{1-\rho}\biggr)\lambda(\mathrm{d}s). \tag{5}$$

Remark 1. In Theorem 1, the conditions imposed on the investment log-price processes of the insurer and reinsurer are mild. In particular, if $\{L_I(t),\, t \geqslant 0\}$ and $\{L_R(t),\, t \geqslant 0\}$ are nonnegative Lévy processes, all such conditions are naturally satisfied. Furthermore, these two nonnegative log-price processes are allowed to exhibit arbitrary dependence.

Corollary 1. *Under the conditions of Theorem* 1*, if* $F \in \mathcal{R}_{-\alpha}$ *for some* $\alpha > 0$*, then it holds that for any fixed* $t \in \Lambda$*,*

$$\begin{aligned}\psi_{\mathrm{sim}}(x, x; t) &\sim \psi_{\mathrm{and}}(x, x; t)\\ &\sim \overline{F}(x) \int_{0-}^{t} \mathbf{E}\big[\big(\rho^{\alpha} \mathrm{e}^{-\alpha L_I(s)}\big) \wedge \big((1-\rho)^{\alpha} \mathrm{e}^{-\alpha L_R(s)}\big)\big]\, \lambda(\mathrm{d}s). \qquad (6)\end{aligned}$$

Our main result regarding the joint ruin probability can be further extended to quantify the risk contagion from an insurer to a reinsurer. To this end, we define a risk contagion measure as

$$\begin{aligned}&\mathrm{Cont}(x; t)\\ &\quad = \mathbf{P}\Big(\inf_{0 \leqslant s \leqslant t} U_R(s) < 0 \,\Big|\, \inf_{0 \leqslant s \leqslant t} U_I(s) < 0,\ \big(U_I(0), U_R(0)\big)^{\top} = (x, x)^{\top}\Big), \qquad (7)\end{aligned}$$

which corresponds to the probability that the reinsurer is ruined, conditioned on the insurer having been ruined.

Corollary 2. *Under the conditions of Corollary* 1*, it holds that for any fixed* $t \in \Lambda$*,*

$$\lim_{x \to \infty} \mathrm{Cont}(x; t) = \frac{\int_{0-}^{t} \mathbf{E}[(\mathrm{e}^{-\alpha L_I(s)}) \wedge ((\rho^{-1} - 1)^{\alpha} \mathrm{e}^{-\alpha L_R(s)})] \lambda(\mathrm{d}s)}{\int_{0-}^{t} \mathbf{E}[\mathrm{e}^{-\alpha L_I(s)}]\, \lambda(\mathrm{d}s)}. \qquad (8)$$

3 Numerical studies

In this section, we conduct numerical studies to verify the accuracy of the asymptotic formula derived from Corollary 1, specifically for the finite-time joint ruin probability $\psi_{\mathrm{and}}(x, x; t)$. To this end, we employ the Monte Carlo method combined with an explicit order-3.0 weak scheme. Additionally, we carry out a sensitivity analysis of $\psi_{\mathrm{and}}(x, x; t)$ with respect to the key parameters of the model and discuss the impact of the quota-share retention level on the risk contagion measure $\mathrm{Cont}(x; t)$.

3.1 Model setting and the explicit order-3.0 weak scheme

Assume that the claim sizes $\{X_i,\, i \in \mathbb{N}\}$ follow a common Pareto distribution

$$F(x) = 1 - \left(\frac{x_0}{x + x_0}\right)^{\alpha}, \quad x > 0, \qquad (9)$$

with parameters $\alpha > 0$ and $x_0 > 0$. We further assume that, for any fixed $d \in \mathbb{N}$, the dependence structure inherent in $(X_1, \dots, X_d)^\top$ is characterized by a Frank copula

$$C(\mathbf{u}) = -\frac{1}{\gamma} \ln\left(1 + \frac{\prod_{k=1}^{d}(\mathrm{e}^{-\gamma u_k} - 1)}{(\mathrm{e}^{-\gamma} - 1)^{d-1}}\right), \quad \mathbf{u} \in [0,1]^d, \tag{10}$$

with parameter $\gamma > 0$. Then it can be verified that $F \in \mathcal{R}_{-\alpha}$ and $\{X_i,\, i \in \mathbb{N}\}$ possess the pairwise strong quasiasymptotic independence structure (4); see, e.g., [4] and [23].

Assume that the inter-arrival times $\{\theta_i,\, i \in \mathbb{N}\}$ are independent and identically distributed nonnegative random variables following a common exponential distribution with intensity $\lambda > 0$. Then the claims occur at times $\tau_i = \sum_{j=1}^{i} \theta_j$, $i \in \mathbb{N}$, which constitute a Poisson renewal process $\{N(t),\, t \geqslant 0\}$.

For the investment log-price processes of the insurer and reinsurer, we assume that they can be respectively expressed as

$$L_I(t) = \int_0^t r_I(s)\,\mathrm{d}s \quad \text{and} \quad L_R(t) = \int_0^t r_R(s)\,\mathrm{d}s, \quad t \geqslant 0,$$

where $r_I(t)$ and $r_R(t)$ represent the two nonnegative stochastic short-rate processes both given by CIR models. We take the insurer's stochastic short-rate $r_I(t)$, $t \geqslant 0$, as an example. This short-rate process satisfies the following SDE:

$$\mathrm{d}r_I(t) = \kappa_I\big(\mu_I - r_I(t)\big)\,\mathrm{d}t + \delta_I \sqrt{r_I(t)}\,\mathrm{d}W(t), \tag{11}$$

where κ_I, μ_I, and δ_I are three positive constants satisfying the Feller condition $2\kappa_I\mu_I > \delta_I^2$ (see, e.g., [16, (7.40)]), and $W(t)$ denotes a standard Brownian motion. Clearly, (11) is a specific instance of the SDE

$$\mathrm{d}r_I(t) = b_I\big(r_I(t)\big)\,\mathrm{d}t + \sigma_I\big(r_I(t)\big)\,\mathrm{d}W(t) \tag{12}$$

with $b_I(r_I(t)) = \kappa_I(\mu_I - r_I(t))$ representing the drift coefficient and $\sigma_I(r_I(t)) = \delta_I\sqrt{r_I(t)}$ denoting the diffusion coefficient. Note that there exist three positive constants D_k, $k = 1, 2, 3$, such that for all $x \geqslant 0$ and $y \geqslant 0$,

$$\begin{aligned} \big|b_I(x)\big| + \big|\sigma_I(x)\big| &\leqslant D_1\big(1 + |x|\big), \\ \big|b_I(x) - b_I(y)\big| &\leqslant D_2|x - y|, \\ \big|\sigma_I(x) - \sigma_I(y)\big| &\leqslant D_3|x - y|^{1/2}. \end{aligned}$$

Then, by [25, Thm. 13.1] and [30, Thm. 1], each of the SDEs (12) and (11) admits a unique solution $r_I(t)$ for $t \in [0, T]$ with some finite $T > 0$. Furthermore, since the SDE (11) admits no analytical solution, we simulate $r_I(t)$ for any fixed $t > 0$ using an explicit weak order-3.0 scheme under the Feller condition, inspired by [17, Sect. 15.2].

We briefly introduce the explicit order-3.0 weak scheme in vector form. For any time $t > 0$, we divide $(0, t)$ into m_0 equal intervals with time nodes $t_n = nt/m_0$,

$n = 0, 1, \dots, m_0$, and the step size $\Delta(t) = t/m_0$. The scheme requires that

$$
\begin{aligned}
r_I(t_{n+1}) = {} & r_I(t_n) + b_I\big(r_I(t_n)\big)\Delta(t) + \sigma_I\big(r_I(t_n)\big)\Delta W \\
& + \frac{1}{2}H_{b_I}\Delta(t) + \frac{1}{\Delta(t)}H_{\sigma_I}\Delta Z \\
& + \sqrt{\frac{2}{\Delta(t)}}G_{b_I}\xi_I\Delta Z + \frac{1}{\sqrt{2\Delta(t)}}G_{\sigma_I}\xi_I\big((\Delta W)^2 - \Delta(t)\big) \\
& + \frac{1}{6}F_{b_I}^{++}\big(\Delta(t) + (\xi_I + \eta_I)\sqrt{\Delta(t)}\Delta W + \xi_I\eta_I\big((\Delta W)^2 - \Delta(t)\big)\big) \\
& + \frac{1}{24}\big(F_{\sigma_I}^{++} + F_{\sigma_I}^{-+} + F_{\sigma_I}^{+-} + F_{\sigma_I}^{--}\big)\Delta W \\
& + \frac{1}{24\sqrt{\Delta(t)}}\big(F_{\sigma_I}^{++} - F_{\sigma_I}^{-+} + F_{\sigma_I}^{+-} - F_{\sigma_I}^{--}\big)\big((\Delta W)^2 - \Delta(t)\big)\xi_I \\
& + \frac{1}{24\Delta(t)}\big(F_{\sigma_I}^{++} + F_{\sigma_I}^{--} - F_{\sigma_I}^{-+} - F_{\sigma_I}^{+-}\big)\big((\Delta W)^2 - 3\big)\Delta W\xi_I\eta_I \\
& + \frac{1}{24\sqrt{\Delta(t)}}\big(F_{\sigma_I}^{++} + F_{\sigma_I}^{-+} - F_{\sigma_I}^{+-} - F_{\sigma_I}^{--}\big)\big((\Delta W)^2 - \Delta(t)\big)\eta_I. \qquad (13)
\end{aligned}
$$

In the aforementioned (13),

$$
\begin{aligned}
H_g &= g^+ + g^- - \frac{3}{2}g\big(r_I(t_n)\big) - \frac{1}{4}\big(\tilde{g}^+ + \tilde{g}^-\big), \\
G_g &= \frac{1}{\sqrt{2}}\big(g^+ - g^-\big) - \frac{1}{4}\big(\tilde{g}^+ - \tilde{g}^-\big),
\end{aligned}
$$

and

$$
\begin{aligned}
F_g^{+\pm} = {} & g\big(r_I(t_n) + \big(b_I\big(r_I(t_n)\big) + b_I^+\big)\Delta(t) + \sigma\big(r_I(t_n)\big)\xi_I\sqrt{\Delta(t)} \pm \sigma_I^+\eta_I\sqrt{\Delta(t)}\,\big) \\
& - g^+ - g\big(r_I(t_n) + b_I\big(r_I(t_n)\big)\Delta(t) \pm \sigma_I\big(r_I(t_n)\big)\eta_I\sqrt{\Delta(t)}\big) + g\big(r_I(t_n)\big), \\
F_g^{-\pm} = {} & g\big(r_I(t_n) + \big(b_I\big(r_I(t_n)\big) + b_I^-\big)\Delta(t) - \sigma_I\big(r_I(t_n)\big)\xi_I\sqrt{\Delta(t)} \pm \sigma_I^-\eta_I\sqrt{\Delta(t)}\,\big) \\
& - g^- - g\big(r_I(t_n) + b_I\big(r_I(t_n)\big)\Delta(t) \pm \sigma_I\big(r_I(t_n)\big)\eta_I\sqrt{\Delta(t)}\big) + g\big(r_I(t_n)\big)
\end{aligned}
$$

with

$$
\begin{aligned}
g^\pm &= g\big(r_I(t_n) + b_I\big(r_I(t_n)\big)\Delta(t) \pm \sigma_I\big(r_I(t_n)\big)\xi_I\sqrt{\Delta(t)}\,\big), \\
\tilde{g}^\pm &= g\big(r_I(t_n) + 2b_I\big(r_I(t_n)\big)\Delta(t) \pm \sqrt{2}\sigma_I\big(r_I(t_n)\big)\xi_I\sqrt{\Delta(t)}\,\big),
\end{aligned}
$$

where $g(\cdot)$ denotes either $b_I(\cdot)$ or $\sigma_I(\cdot)$; ξ_I and η_I are independent random variables satisfying $\mathbf{P}(\xi_I = \pm 1) = \mathbf{P}(\eta_I = \pm 1) = 0.5$; and we introduce two correlated Gaussian random variables $\Delta W \sim N(0, \Delta(t))$ and $\Delta Z \sim N(0, (\Delta(t))^3/3)$ with covariance $\mathbf{E}[\Delta W \Delta Z] = (\Delta(t))^2/2$. From this we can readily derive that

$$
\Delta W = \zeta_1\big(\Delta(t)\big)^{1/2} \quad \text{and} \quad \Delta Z = \frac{1}{2}\left(\zeta_1 + \frac{\zeta_2}{\sqrt{3}}\right)\big(\Delta(t)\big)^{3/2}, \qquad (14)
$$

where ζ_1 and ζ_2 are two independent standard normal random variables.

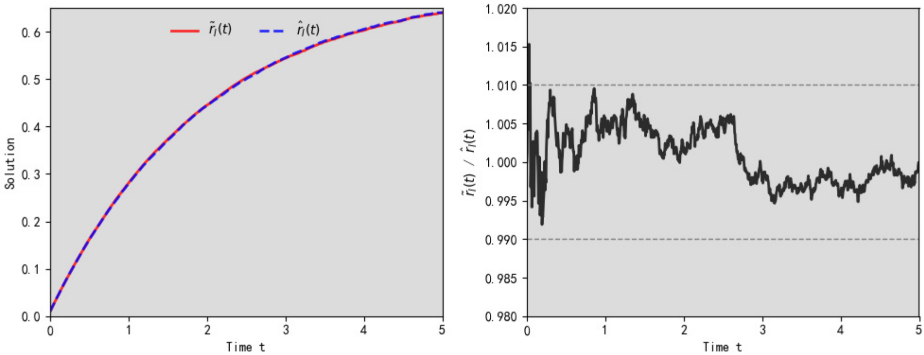

Figure 1. Comparison between the numerical solutions $\tilde{r}_I(t)$ and $\hat{r}_I(t)$ to the SDE (11) (left) and their ratio (right).

At the end of this subsection, we verify the effectiveness of the explicit order-3.0 weak scheme for solving SDEs by simulating the CIR model $r_I(t)$ defined by (11). The parameters for the solution $r_I(t)$ to the SDE (11) are set as follows: $\kappa_I = 0.5$, $\mu_I = 0.7$, $\delta_I = 0.025$, $r_I(0) = 0.01$. Since SDE (11) has no analytical solution, we assess its numerical accuracy by comparing two solutions, $\tilde{r}_I(t)$ and $\hat{r}_I(t)$, obtained using different step sizes. Specifically, we first use the explicit order-3.0 weak scheme to compute, for the jth sample path, $j = 1, \dots, m_1$, the numerical solutions $\tilde{r}_{Ij}(t)$ (with step size $\Delta(t) = t/(2m_0)$) and $\hat{r}_{Ij}(t)$ (with step size $\Delta(t) = t/m_0$). We then calculate the average numerical solutions $\tilde{r}_I(t)$ and $\hat{r}_I(t)$ of (11) as

$$\tilde{r}_I(t) = \frac{1}{m_1} \sum_{j=1}^{m_1} \tilde{r}_{Ij}(t) \quad \text{and} \quad \hat{r}_I(t) = \frac{1}{m_1} \sum_{j=1}^{m_1} \hat{r}_{Ij}(t).$$

Motivated by [33], the numerical solution $\tilde{r}_I(t)$ obtained with a smaller step size can be regarded as an approximation to the true solution of the CIR model to some extent. We set $m_0 = 1000$ and simulate $\tilde{r}_I(t)$ and $\hat{r}_I(t)$ for $m_1 = 100$ replicates. Figure 1 presents the average numerical solutions $\tilde{r}_I(t)$ and $\hat{r}_I(t)$ of SDE (11), as well as the ratio of these two solutions, for $t \in [0, 5]$. As shown in Fig. 1, the explicit order-3.0 weak scheme is an effective numerical method for simulating solutions to such SDEs.

3.2 Accuracy of the asymptotic estimates

We now turn to the simulation of $\psi_{\text{and}}(x, x; t)$ according to the following algorithm.

Step 1. For $t > 0$, generate $\hat{N}(t)$ numbers of exponentially distributed claim inter-arrival times $\hat{\theta}_i$ with parameter $\lambda > 0$, such that (1) is satisfied. Then, before time t, the claims arrive at times $\hat{\tau}_i = \hat{\theta}_1 + \dots + \hat{\theta}_i$, $i = 1, \dots, \hat{N}(t)$.

Step 2. Choose a positive integer m_2 and write $t'_i = it/m_2$, $i = 0, \dots, m_2$. Combine the above $\hat{\tau}_i$ and t'_i and sort them in ascending order, denoted by $t_0 \leqslant \dots \leqslant t_{\hat{N}(t)+m_2}$. Set $s_i = t_i - t_{i-1}$, $i = 1, \dots, \hat{N}(t) + m_2$.

Step 3. Choose a positive integer m_0 and write $t_l'' = lt_i/m_0$, $l = 0, \ldots, m_0$. Generate two standard normal random variables $\hat{\zeta}_1$ and $\hat{\zeta}_2$, use them and $\Delta(t_i)$ to simulate ΔW_i and ΔZ_i by (14). Simultaneously, generate four two-point distributed random variables $\hat{\xi}_I$, $\hat{\eta}_I$, $\hat{\xi}_R$, and $\hat{\eta}_R$. Generate m_1 interest rate paths $\hat{r}_{Ij}(t_i)$ and $\hat{r}_{Rj}(t_i)$ for $j = 1, \ldots, m_1$ via (13), using the pre-generated variables at time t_i.

Step 4. For $t > 0$, generate $(\hat{X}_1, \ldots, \hat{X}_{\hat{N}(t)})^\top$ with a common marginal Pareto distribution (9) and the Frank copula (10).

Step 5. For time $t > 0$ and retention level $\rho > 0$, the insurer's discounted surplus process may be expressed as

$$\hat{U}_I(t) = x + \frac{c_I}{2} \sum_{i=1}^{\hat{N}(t)+m_2} s_i \left(\mathrm{e}^{-\hat{L}_I(t_i)} + \mathrm{e}^{-\hat{L}_I(t_{i-1})}\right) - \rho \sum_{i=1}^{\hat{N}(t)} \hat{X}_i \mathrm{e}^{-\hat{L}_I(\hat{\tau}_i)},$$

where

$$\hat{L}_I(t_i) = \frac{\Delta(t_i)}{2m_1} \sum_{j=1}^{m_1} \sum_{l=1}^{m_0} \left(\hat{r}_{Ij}(t_l'') + \hat{r}_{Ij}(t_{l-1}'')\right).$$

The reinsurer's discounted surplus process $\hat{U}_R(t_i)$ can also be expressed similarly, using $\hat{r}_{Rj}(\cdot)$ and c_R.

Step 6. Repeat Steps 1–5 M_1 times. Select the samples satisfying

$$\left(\bigwedge_{i=0}^{\hat{N}(t)+m_2} \hat{U}_I(t_i)\right) \vee \left(\bigwedge_{i=0}^{\hat{N}(t)+m_2} \hat{U}_R(t_i)\right) < 0$$

and record their number as M_2. In this way, $\psi_{\mathrm{and}}(x, x; t)$ can be simulated by

$$\widehat{\psi}_{\mathrm{and}}(x, x; t) = \frac{M_2}{M_1}.$$

Next, we seek to compute the asymptotic estimate for the finite-time joint ruin probability $\psi_{\mathrm{and}}(x, x; t)$. Note that the integral in (6) is computationally intractable; we specifically simulate $L_I(t_i)$ and $L_R(t_i)$, $i = 0, \ldots, \hat{N}(t) + m_2$, by repeating Steps 3 and 5 outlined above, and we can thereby estimate the integral on the right-hand side of (6) as

$$\begin{aligned}\frac{1}{2m_1} \sum_{i=1}^{\hat{N}(t)+m_2} \sum_{j=1}^{m_1} s_i \Big(& \left(\rho^\alpha \mathrm{e}^{-\alpha \hat{L}_{Ij}(t_i)}\right) \wedge \left((1-\rho)^\alpha \mathrm{e}^{-\alpha \hat{L}_{Rj}(t_i)}\right) \\ & + \left(\rho^\alpha \mathrm{e}^{-\alpha \hat{L}_{Ij}(t_{i-1})}\right) \wedge \left((1-\rho)^\alpha \mathrm{e}^{-\alpha \hat{L}_{Rj}(t_{i-1})}\right)\Big).\end{aligned}$$

Model specifications and parameters are listed in Table 1. Figure 2 (left) presents the simulated values of the finite-time joint ruin probability $\psi_{\mathrm{and}}(x, x; t)$ alongside their asymptotic estimates from relation (6). Figure 2 (right) presents the ratios of these two sets of values with initial wealth x varying from 400 to 800 in steps of 20. As shown in

Table 1. Specified settings and parameters in Corollary 1.

On the quota-share reinsurance risk model:
The premium rates of the insurer and the reinsurer $c_I = c_R = 0.15$
The quota-share retention level $\rho = 0.5$
The claim-arrival process is a Poisson process with intensity $\lambda = 6$
Time $t = 5$
On the claim-size:
The claim sizes are identically distributed according to a common Pareto distribution with regular variation index $\alpha = 1.5$ and location parameter $x_0 = 3$ in (9)
The dependence structure among claim sizes is governed by a Frank copula with dependence parameter $\gamma = 0.5$ in (10)
On the log-price process of the investment:
The stochastic short-rate processes for the insurer and the reinsurer follow two CIR models with parameters $\kappa_I = 0.5$, $\kappa_R = 0.4$, $\mu_I = 0.7$, $\mu_R = 0.6$, $\delta_I = 0.025$, $\delta_R = 0.027$ in (11)
The initial values of the short-rates $r_I(0) = r_R(0) = 0.01$
$m_2 = 100$ in algorithm Step 2
$m_0 = 1000$ and $m_1 = 100$ in algorithm Step 3
Sample size $M_1 = 5 \times 10^5$

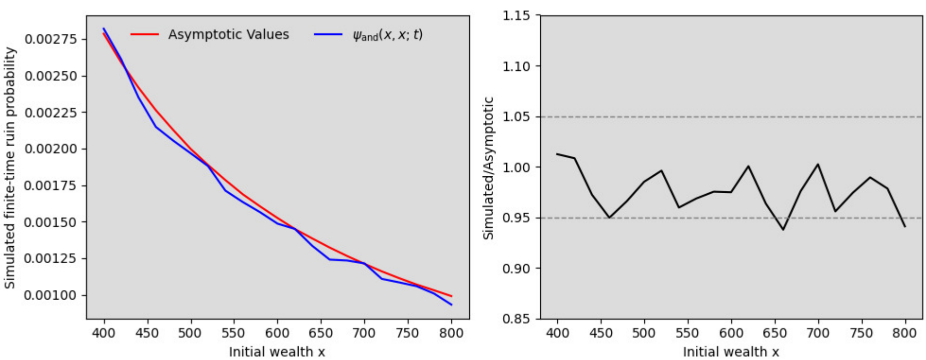

Figure 2. Comparison between the simulated and the asymptotic values of $\psi_{\text{and}}(x, x; t)$ (left) and their ratio (right) via the Frank copula.

Fig. 2, for large initial wealth x of both the insurer and reinsurer, the simulated values of $\psi_{\text{and}}(x, x; t)$ cluster around the asymptotic values. Additionally, nearly all ratios of the simulated to asymptotic values of $\psi_{\text{and}}(x, x; t)$ lie within the interval $[0.95, 1.05]$. This confirms the reasonableness of our asymptotic result, Corollary 1, with fluctuations attributed to the inherent limitations of the Monte Carlo method. The high accuracy of the asymptotic result for $\psi_{\text{sim}}(x, x; t)$ can be verified similarly.

3.3 Sensitivity analysis

In this subsection, we conduct a sensitivity analysis on the finite-time joint ruin probability $\psi_{\text{and}}(x, x; t)$ to assess the impact of five key model parameters. These parameters include the regular variation index α and the location parameter x_0 in (9) related to claim sizes, the intensity λ of the claim-arrival Poisson process $N(t)$, and the long-term mean parameters μ_I and μ_R in (11) derived from the CIR model. As established in Corollary 1, the strong quasiasymptotic independence between claim sizes does not affect the asymptotic estimates of the finite-time joint ruin probability; thus, we exclude the Frank copula parameter γ in (10). Given the high-accuracy asymptotic estimates verified in Subsection 3.2, we base our sensitivity analysis on the right-hand side of (6). As a benchmark, we set $\alpha = 1.5$, $x_0 = 3$, $\lambda = 6$, $\mu_I = 0.7$, and $\mu_R = 0.6$.

We introduce small percentage changes to α, x_0, λ, μ_I, μ_R, then record the corresponding changes in $\psi_{\text{and}}(x, x; t)$ via the right-hand side of (6). Table 2 summarizes the percentage changes in the asymptotic values of $\psi_{\text{and}}(x, x; t)$ for different large initial wealth values x. As expected, the table indicates that the joint ruin probability increases under the following conditions: when α decreases (or x_0 increases), which corresponds to heavier-tailed claim sizes; or when λ increases, which corresponds to more claims occurring within $[0, t]$. Conversely, the joint ruin probability decreases when μ_I and μ_R increases, which corresponds to higher long-term mean investment return rates. Notably, Table 2 reveals that the finite-time joint ruin probability is more sensitive to the claim-size parameters than to other model parameters; in particular, the heavy-tailedness of claim sizes is of primary importance.

Table 2. Sensitivity analysis of $\psi_{\text{and}}(x, x; 5)$ with respect to model parameters.

Model parameters	$\psi_{\text{and}}(x, x; 5)$		
	$x = 500$	$x = 600$	$x = 700$
% change in α			
+15%	−74.875%	−75.972%	−76.761%
+10%	−60.848%	−61.430%	−62.493%
+5%	−37.711%	−37.128%	−38.734%
($\alpha = 1.5$)	(0.001998)	(0.001525)	(0.001211)
−5%	+58.314%	+60.657%	+65.028%
−10%	+152.581%	+162.242%	+168.291%
−15%	+304.661%	+326.800%	+339.589%
% change in x_0			
+15%	+22.292%	+22.132%	+23.450%
+10%	+15.457%	+15.441%	+15.286%
+5%	+6.742%	+6.692%	+10.237%
($x_0 = 3$)	(0.001998)	(0.001525)	(0.001211)
−5%	−6.830%	−6.235%	−6.255%
−10%	−15.154%	−14.249%	−14.471%
−15%	−20.413%	−21.156%	−21.566%
% change in λ			
+15%	+14.125%	+14.046%	+16.773%
+10%	+9.122%	+8.927%	+12.430%
+5%	+4.239%	+6.379%	+4.980%
($\lambda = 6$)	(0.001998)	(0.001525)	(0.001211)
−5%	−3.478%	−5.703%	−3.508%
−10%	−10.647%	−9.291%	−9.962%
−15%	−13.499%	−15.704%	−14.863%

Continued on next page

Table 2 (*continued from previous page*)

Model parameters	$\psi_{\text{and}}(x, x; 5)$		
	$x = 500$	$x = 600$	$x = 700$
% change in μ_I and μ_R			
$+15\%$	-8.670%	-7.395%	-7.938%
$+10\%$	-6.208%	-6.167%	-2.941%
$+5\%$	-1.373%	-3.621%	-2.675%
$((\mu_I, \mu_R) = (0.7, 0.6))$	(0.001998)	(0.001525)	(0.001211)
-5%	$+2.334\%$	$+3.314\%$	$+3.223\%$
-10%	$+5.416\%$	$+5.422\%$	$+7.819\%$
-15%	$+10.903\%$	$+8.848\%$	$+9.773\%$

Table 3. Sensitivity analysis of $\mathrm{Cont}(x; 5)$ with respect to model parameters.

Model parameters	$\mathrm{Cont}(x; 5)$
% change in ρ	
$+15\%$	-24.917%
$+10\%$	-15.225%
$+5\%$	-4.180%
$(\rho = 0.5)$	(0.987396)
-5%	$+1.291\%$
-10%	-0.035%
-15%	$+1.204\%$
% change in $(\alpha, x_0, \lambda, \mu_I, \mu_R)$	
$+15\%$	$+0.642\%$
$+10\%$	$+1.190\%$
$+5\%$	$+1.212\%$
$(\alpha, x_0, \lambda, \mu_I, \mu_R) = (1.5, 3, 6, 0.7, 0.6)$	(0.987396)
-5%	$+0.944\%$
-10%	$+0.390\%$
-15%	$+1.199\%$

3.4 A discussion on the quota-share retention level

In this subsection, we conduct a further sensitivity analysis of the risk contagion measure $\mathrm{Cont}(x; t)$ in (7) by adjusting two sets of parameters: the quota-share retention level ρ and the five key model parameters introduced in Subsection 3.3. We retain the settings and parameters from the previous subsections and base our analysis on the right-hand side of (8).

As in Subsection 3.3, Table 3 documents the percentage changes in the asymptotic values of $\mathrm{Cont}(x; t)$ when small perturbations are applied to the quota-share retention level ρ and the other five key parameters $(\alpha, x_0, \lambda, \mu_I, \mu_R)$. From the right-hand side of (8) we note that for large $\rho \in (0, 1)$, a larger ρ leads to a smaller value of the risk contagion measure. This is because a larger ρ means the insurer retains a greater proportion of claim sizes relative to the reinsurer. Notably, Table 3 indicates that the risk contagion measure is more sensitive to the quota-share retention level (particularly when ρ is relatively large) than to the other model parameters. Thus, the retention level is a key driving factor for analyzing the risk contagion from the insurer to the reinsurer.

4 Proofs of the main results

We begin this section with several lemmas. The first lemma is derived from [6, Thm. 3.3].

Lemma 1. *Let ξ be a real-valued random variable with distribution $V \in \mathcal{C}$, and let η be a nonnegative random variable that is nondegenerate at zero and independent of ξ. If $\mathbf{E}[\eta^p] < \infty$ for some $p > J_V^+$, then $\mathbf{P}(\xi\eta > x) \asymp \overline{V}(x)$.*

Lemma 2. *Under the conditions of Theorem 1, it holds that for any fixed $n \in \mathbb{N}$, any fixed $t \in \Lambda$, and uniformly for all $(t_1, \dots, t_n) \in [0, t]^n$,*

$$\mathbf{P}\Biggl(\sum_{i=1}^{n} X_i \mathrm{e}^{-L_I(t_i)} > \frac{x}{\rho},\ \sum_{j=1}^{n} X_j \mathrm{e}^{-L_R(t_j)} > \frac{y}{1-\rho}\Biggr) \sim \sum_{i=1}^{n} \mathbf{P}\biggl(X_i > \frac{x\mathrm{e}^{L_I(t_i)}}{\rho} \vee \frac{y\mathrm{e}^{L_R(t_i)}}{1-\rho}\biggr). \tag{15}$$

Proof. For notational simplicity, we write

$$S_n^I = \sum_{i=1}^{n} X_i \mathrm{e}^{-L_I(t_i)} \quad \text{and} \quad S_n^R = \sum_{j=1}^{n} X_j \mathrm{e}^{-L_R(t_j)}.$$

We first estimate the upper bound of the joint tail probability on the left-hand side of (15). Clearly, we have that for any $\varepsilon \in (0,1)$ and all $(t_1, \dots, t_n) \in [0, t]^n$,

$$\begin{aligned}
&\mathbf{P}\biggl(S_n^I > \frac{x}{\rho},\ S_n^R > \frac{y}{1-\rho}\biggr) \\
&\quad\leqslant \mathbf{P}\Biggl(S_n^I > \frac{x}{\rho},\ S_n^R > \frac{y}{1-\rho}, \\
&\qquad\quad \bigcup_{i=1}^{n}\biggl(X_i \mathrm{e}^{-L_I(t_i)} > \frac{(1-\varepsilon)x}{\rho}\biggr),\ \bigcup_{j=1}^{n}\biggl(X_j \mathrm{e}^{-L_R(t_j)} > \frac{(1-\varepsilon)y}{1-\rho}\biggr)\Biggr) \\
&\qquad + \mathbf{P}\Biggl(S_n^I > \frac{x}{\rho},\ S_n^R > \frac{y}{1-\rho},\ \bigcap_{i=1}^{n}\biggl(X_i \mathrm{e}^{-L_I(t_i)} \leqslant \frac{(1-\varepsilon)x}{\rho}\biggr)\Biggr) \\
&\qquad + \mathbf{P}\Biggl(S_n^I > \frac{x}{\rho},\ S_n^R > \frac{y}{1-\rho},\ \bigcap_{j=1}^{n}\biggl(X_j \mathrm{e}^{-L_R(t_j)} \leqslant \frac{(1-\varepsilon)y}{1-\rho}\biggr)\Biggr) \\
&\quad =: I_1 + I_2 + I_3.
\end{aligned} \tag{16}$$

As for I_1, we have that

$$I_1 \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{P}\biggl(X_i \mathrm{e}^{-L_I(t_i)} > \frac{(1-\varepsilon)x}{\rho},\ X_j \mathrm{e}^{-L_R(t_j)} > \frac{(1-\varepsilon)y}{1-\rho}\biggr)$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \mathbf{P}\biggl(X_i > (1-\varepsilon)\biggl(\frac{x\mathrm{e}^{L_I(t_i)}}{\rho} \vee \frac{y\mathrm{e}^{L_R(t_i)}}{1-\rho}\biggr)\biggr) \\
&\quad + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \mathbf{P}\biggl(X_i\mathrm{e}^{-L_I(t_i)} > \frac{(1-\varepsilon)x}{\rho},\ X_j\mathrm{e}^{-L_R(t_j)} > \frac{(1-\varepsilon)y}{1-\rho}\biggr) \\
&=: I_{11} + I_{12}.
\end{aligned} \tag{17}
$$

On the one hand, since $\{L_I(t),\, t \geqslant 0\}$ and $\{L_R(t),\, t \geqslant 0\}$ are both nonnegative and $\{X_i,\, i \in \mathbb{N}\}$ are pairwise strongly quasiasymptotically independent, by $F \in \mathcal{C}$ we have that uniformly for all $(t_1, \dots, t_n) \in [0, t]^n$,

$$
\begin{aligned}
I_{12} &\leqslant \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \mathbf{P}\bigl(X_i > (1-\varepsilon)x,\ X_j > (1-\varepsilon)y\bigr) \\
&\leqslant \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \mathbf{P}\bigl(\bigl(X_i > (1-\varepsilon)(x\vee y),\ X_j > (1-\varepsilon)y\bigr) \\
&\qquad\qquad \cup \bigl(X_i > (1-\varepsilon)x,\ X_j > (1-\varepsilon)(x\vee y)\bigr)\bigr) \\
&\leqslant \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \mathbf{P}\bigl(X_i > (1-\varepsilon)(x\vee y),\ X_j > (1-\varepsilon)y\bigr) \\
&\quad + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \mathbf{P}\bigl(X_i > (1-\varepsilon)x,\ X_j > (1-\varepsilon)(x\vee y)\bigr) \\
&= o(1)\overline{F}(x\vee y).
\end{aligned} \tag{18}
$$

On the other hand, again by the nonnegativity of $\{L_I(t),\, t \geqslant 0\}$ and $\{L_R(t),\, t \geqslant 0\}$, it follows from Lemma 1 that uniformly for all $(t_1, \dots, t_n) \in [0, t]^n$,

$$
\begin{aligned}
I_{11} &\geqslant \sum_{i=1}^{n} \mathbf{P}\biggl(X_i > \frac{x\mathrm{e}^{\sup_{0\leqslant s\leqslant t} L_I(s)}}{\rho} \vee \frac{y\mathrm{e}^{\sup_{0\leqslant s\leqslant t} L_R(s)}}{1-\rho}\biggr) \\
&\geqslant \sum_{i=1}^{n} \mathbf{P}\bigl(X_i\bigl(\bigl(\rho\mathrm{e}^{-\sup_{0\leqslant s\leqslant t} L_I(s)}\bigr) \wedge \bigl((1-\rho)\mathrm{e}^{-\sup_{0\leqslant s\leqslant t} L_R(s)}\bigr)\bigr) > x\vee y\bigr) \\
&\asymp \overline{F}(x\vee y),
\end{aligned} \tag{19}
$$

which implies that uniformly for all $(t_1, \dots, t_n) \in [0, t]^n$,

$$
I_{12} = o(1) I_{11}. \tag{20}
$$

Plugging (20) into (17) yields, uniformly for all $(t_1,\dots,t_n)\in[0,t]^n$,

$$I_1 \lesssim \sum_{i=1}^n \iint_{[1,\infty)^2} \overline{F}\biggl((1-\varepsilon)\biggl(\frac{xu}{\rho}\vee\frac{yv}{1-\rho}\biggr)\biggr)\mathbf{P}\bigl(\mathrm{e}^{L_I(t_i)}\in \mathrm{d}u,\ \mathrm{e}^{L_R(t_i)}\in \mathrm{d}v\bigr)$$

$$\lesssim \overline{F}^*(1-\varepsilon)\sum_{i=1}^n \mathbf{P}\biggl(X_i>\frac{x\mathrm{e}^{L_I(t_i)}}{\rho}\vee\frac{y\mathrm{e}^{L_R(t_i)}}{1-\rho}\biggr). \tag{21}$$

As for I_2, by reasoning similar to that for (18), we obtain, uniformly for all $(t_1,\dots,t_n)\in[0,t]^n$, that

$$\begin{aligned} I_2 &\leqslant \sum_{j=1}^n \mathbf{P}\biggl(S_n^I>\frac{x}{\rho},\ X_j>\frac{y}{n},\ X_j\mathrm{e}^{-L_I(t_j)}\leqslant\frac{1-\varepsilon}{\rho}x\biggr)\\ &\leqslant \sum_{j=1}^n \mathbf{P}\biggl(\sum_{\substack{i=1\\ i\neq j}}^n X_i>\varepsilon x,\ X_j>\frac{y}{n}\biggr)\leqslant \sum_{j=1}^n\sum_{\substack{i=1\\ i\neq j}}^n \mathbf{P}\biggl(X_i>\frac{\varepsilon x}{n},\ X_j>\frac{\varepsilon y}{n}\biggr)\\ &= o(1)\overline{F}(x\vee y)\\ &= o(1)\sum_{i=1}^n \mathbf{P}\biggl(X_i>\frac{x\mathrm{e}^{L_I(t_i)}}{\rho}\vee\frac{y\mathrm{e}^{L_R(t_i)}}{1-\rho}\biggr), \end{aligned} \tag{22}$$

where we used the spirit of (19) in the last step. We can also derive the same result for I_3 as in (22). By combining (16), (21), and (22), letting $x\wedge y\to\infty$ first and then $\varepsilon\downarrow 0$, and using the fact that $F\in\mathcal{C}$, we conclude that the asymptotic upper bound of (15) holds uniformly for all $(t_1,\dots,t_n)\in[0,t]^n$.

We next consider the lower bound of the joint tail probability on the left-hand side of (15). By Bonferroni's inequality, we have that for all $(t_1,\dots,t_n)\in[0,t]^n$,

$$\begin{aligned} &\mathbf{P}\biggl(S_n^I>\frac{x}{\rho},\ S_n^R>\frac{y}{1-\rho}\biggr)\\ &\quad\geqslant \mathbf{P}\biggl(\bigcup_{i=1}^n\biggl(X_i\mathrm{e}^{-L_I(t_i)}>\frac{x}{\rho}\biggr),\ \bigcup_{j=1}^n\biggl(X_j\mathrm{e}^{-L_R(t_j)}>\frac{y}{1-\rho}\biggr)\biggr)\\ &\quad\geqslant \mathbf{P}\biggl(\bigcup_{i=1}^n\biggl(X_i\mathrm{e}^{-L_I(t_i)}>\frac{x}{\rho},\ X_i\mathrm{e}^{-L_R(t_i)}>\frac{y}{1-\rho}\biggr)\biggr)\\ &\quad\geqslant \sum_{i=1}^n \mathbf{P}\biggl(X_i>\frac{x\mathrm{e}^{L_I(t_i)}}{\rho}\vee\frac{y\mathrm{e}^{L_R(t_i)}}{1-\rho}\biggr)-\sum_{1\leqslant i<j\leqslant n}\mathbf{P}(X_i>x,\ X_j>y)\\ &\quad=: \sum_{i=1}^n \mathbf{P}\biggl(X_i>\frac{x\mathrm{e}^{L_I(t_i)}}{\rho}\vee\frac{y\mathrm{e}^{L_R(t_i)}}{1-\rho}\biggr)-I_4. \end{aligned} \tag{23}$$

As for I_4, reasoning similar to that for (20) yields, uniformly for all $(t_1,\dots,t_n)\in[0,t]^n$,

$$I_4 = o(1)\sum_{i=1}^{n}\mathbf{P}\biggl(X_i > \frac{x\mathrm{e}^{L_I(t_i)}}{\rho}\vee\frac{y\mathrm{e}^{L_R(t_i)}}{1-\rho}\biggr). \tag{24}$$

Plugging (24) into (23) leads to the lower bound of (15) holding uniformly for all $(t_1,\dots,t_n)\in[0,t]^n$. □

The following lemma gives an explicit expression for the asymptotic joint tail behavior of the insurer's and reinsurer's discounted aggregate claim amounts defined in (2). This result is not only crucial for proving Theorem 1 but also has independent significance.

Lemma 3. *Under the conditions of Theorem* 1*, it holds that for any fixed* $t\in\Lambda$*,*

$$\mathbf{P}\bigl(D_I(t)>x,\ D_R(t)>y\bigr)\sim\varphi(x,y;t), \tag{25}$$

where

$$\varphi(x,y;t)=\int_{0^-}^{t}\mathbf{P}\biggl(X>\frac{x\mathrm{e}^{L_I(s)}}{\rho}\vee\frac{y\mathrm{e}^{L_R(s)}}{1-\rho}\biggr)\,\lambda(\mathrm{d}s). \tag{26}$$

Proof. For any large integer M and any fixed $t\in\Lambda$, we split the joint tail probability on the left-hand side of (25) into two parts:

$$\begin{aligned}
&\mathbf{P}\bigl(D_I(t)>x,\ D_R(t)>y\bigr)\\
&\quad=\Biggl(\sum_{n=1}^{M}+\sum_{n=M+1}^{\infty}\Biggr)\mathbf{P}\Biggl(\sum_{i=1}^{n}X_i\mathrm{e}^{-L_I(\tau_i)}>\frac{x}{\rho},\ \sum_{j=1}^{n}X_j\mathrm{e}^{-L_R(\tau_j)}>\frac{y}{1-\rho},\ N(t)=n\Biggr)\\
&\quad=: J_1+J_2.
\end{aligned} \tag{27}$$

We first consider J_1. For $n\in\mathbb{N}$, denote by $H(t_1,\dots,t_{n+1})$ the joint distribution of $(\tau_1,\dots,\tau_{n+1})^\top$ and write $\Omega=\{(t_1,\dots,t_{n+1})\colon 0\leqslant t_1\leqslant\cdots\leqslant t_n\leqslant t<t_{n+1}\}$. By Lemma 2, for any fixed $n=1,\dots,M$, the summand in (27) can be estimated by

$$\begin{aligned}
&\int_{\Omega}\mathbf{P}\Biggl(\sum_{i=1}^{n}X_i\mathrm{e}^{-L_I(t_i)}>\frac{x}{\rho},\ \sum_{j=1}^{n}X_j\mathrm{e}^{-L_R(t_j)}>\frac{y}{1-\rho}\Biggr)\mathrm{d}H(t_1,\dots,t_{n+1})\\
&\quad\sim\int_{\Omega}\sum_{i=1}^{n}\mathbf{P}\biggl(X_i>\frac{x\mathrm{e}^{L_I(t_i)}}{\rho}\vee\frac{y\mathrm{e}^{L_R(t_i)}}{1-\rho}\biggr)\,\mathrm{d}H(t_1,\dots,t_{n+1})\\
&\quad=\sum_{i=1}^{n}\mathbf{P}\biggl(X_i>\frac{x\mathrm{e}^{L_I(\tau_i)}}{\rho}\vee\frac{y\mathrm{e}^{L_R(\tau_i)}}{1-\rho},\ N(t)=n\biggr),
\end{aligned}$$

which implies that

$$\begin{aligned}
J_1&\sim\Biggl(\sum_{n=1}^{\infty}-\sum_{n=M+1}^{\infty}\Biggr)\sum_{i=1}^{n}\mathbf{P}\biggl(X_i>\frac{x\mathrm{e}^{L_I(\tau_i)}}{\rho}\vee\frac{y\mathrm{e}^{L_R(\tau_i)}}{1-\rho},\ N(t)=n\biggr)\\
&=: J_{11}-J_{12}.
\end{aligned} \tag{28}$$

By using Fubini's theorem, we have that

$$J_{11} = \sum_{i=1}^{\infty} \mathbf{P}\biggl(X_i > \frac{x\mathrm{e}^{L_I(\tau_i)}}{\rho} \vee \frac{y\mathrm{e}^{L_R(\tau_i)}}{1-\rho},\ \tau_i \leqslant t\biggr) = \varphi(x,y;t). \tag{29}$$

As for J_{12}, by the nonnegativity of $\{L_I(t), t \geqslant 0\}$ and $\{L_R(t), t \geqslant 0\}$ we have that

$$J_{12} \leqslant \overline{F}(x \vee y) \sum_{n=M+1}^{\infty} n\mathbf{P}\bigl(N(t)=n\bigr) = \overline{F}(x \vee y)\,\mathbf{E}\bigl[N(t)\bigl(\mathbf{1}_{(N(t)>M)}\bigr)\bigr].$$

Similarly to (19), it follows from Lemma 1 and $\lambda(t) < \infty$ that

$$\begin{aligned} &\varphi(x,y;t) \\ &\quad \geqslant \mathbf{P}\bigl(X\bigl(\bigl(\rho\mathrm{e}^{-\sup_{0\leqslant s\leqslant t} L_I(s)}\bigr) \wedge \bigl((1-\rho)\mathrm{e}^{-\sup_{0\leqslant s\leqslant t} L_R(s)}\bigr)\bigr) > x \vee y\bigr)\lambda(t) \\ &\quad \asymp \overline{F}(x \vee y). \end{aligned} \tag{30}$$

The above two estimates and $\lambda(t) < \infty$ imply that

$$\lim_{M\to\infty} \lim_{x\wedge y\to\infty} \frac{J_{12}}{\varphi(x,y;t)} = 0. \tag{31}$$

Plugging (29) and (31) into (28) yields

$$\lim_{M\to\infty} \lim_{x\wedge y\to\infty} \frac{J_1}{\varphi(x,y;t)} = 1. \tag{32}$$

We next deal with J_2. Since $\{L_I(t),\ t \geqslant 0\}$ and $\{L_R(t),\ t \geqslant 0\}$ are both nonnegative, by [29, Lemma 2.4] there exists a large constant C such that for all $x, y \geqslant 0$,

$$\begin{aligned} J_2 &\leqslant \sum_{n=M+1}^{\infty} \mathbf{P}\Biggl(\sum_{i=1}^{n} X_i > x \vee y\Biggr)\mathbf{P}\bigl(N(t)=n\bigr) \\ &\leqslant C\overline{F}(x \vee y) \sum_{n=M+1}^{\infty} n^{p+1}\mathbf{P}\bigl(N(t)=n\bigr) \\ &= C\overline{F}(x \vee y)\mathbf{E}\bigl[\bigl(N(t)\bigr)^{p+1}\mathbf{1}_{(N(t)>M)}\bigr]. \end{aligned}$$

Then it follows from (30) and the condition $\mathbf{E}[(N(t))^{p+1}] < \infty$, $p > J_F^+$, that

$$\lim_{M\to\infty} \lim_{x\wedge y\to\infty} \frac{J_2}{\varphi(x,y;t)} = 0. \tag{33}$$

Therefore, the desired relation (25) follows from (27), (32), and (33). □

We are ready to prove our main results.

Proof of Theorem 1. Clearly, for any fixed $t \in \Lambda$, we have that

$$\mathbf{P}\Biggl(D_I(t) > x + c_I \int_0^t \mathrm{e}^{-L_I(s)}\,\mathrm{d}s,\ D_R(t) > y + c_R \int_0^t \mathrm{e}^{-L_R(s)}\,\mathrm{d}s\Biggr) \leqslant \psi_{\mathrm{sim}}(x, y; t) \leqslant \psi_{\mathrm{and}}(x, y; t) \leqslant \mathbf{P}\bigl(D_I(t) > x,\ D_R(t) > y\bigr).$$

By Lemma 3, to prove (5), it suffices to prove

$$\mathbf{P}\Biggl(D_I(t) > x + c_I \int_0^t \mathrm{e}^{-L_I(s)}\,\mathrm{d}s,\ D_R(t) > y + c_R \int_0^t \mathrm{e}^{-L_R(s)}\,\mathrm{d}s\Biggr) \gtrsim \varphi(x, y; t),$$

where $\varphi(x, y; t)$ is defined by (26). Indeed, for any $0 < \varepsilon < 1$, since $\{L_I(t),\ t \geqslant 0\}$ and $\{L_R(t),\ t \geqslant 0\}$ are both nonnegative, as done in (21), we obtain that for any fixed $t \in \Lambda$ and sufficiently large x, y,

$$\begin{aligned}
&\mathbf{P}\Biggl(D_I(t) > x + c_I \int_0^t \mathrm{e}^{-L_I(s)}\,\mathrm{d}s,\ D_R(t) > y + c_R \int_0^t \mathrm{e}^{-L_R(s)}\,\mathrm{d}s\Biggr)\\
&\geqslant \mathbf{P}\bigl(D_I(t) > x + c_I t,\ D_R(t) > y + c_R t\bigr)\\
&\geqslant \mathbf{P}\bigl(D_I(t) > (1+\varepsilon)x,\ D_R(t) > (1+\varepsilon)y\bigr)\\
&\sim \int_{0^-}^t \iint_{[1,\infty)^2} \overline{F}\biggl((1+\varepsilon)\biggl(\frac{xu}{\rho} \vee \frac{yv}{1-\rho}\biggr)\biggr)\mathbf{P}\bigl(\mathrm{e}^{L_I(s)} \in \mathrm{d}u,\ \mathrm{e}^{L_R(s)} \in \mathrm{d}v\bigr)\,\lambda(\mathrm{d}s)\\
&\gtrsim \overline{F}_*(1+\varepsilon)\varphi(x, y; t) \sim \varphi(x, y; t) \quad \text{as } \varepsilon \downarrow 0,
\end{aligned}$$

where the third step follows from Lemma 3, and the last step follows from $F \in \mathcal{C}$. This completes the proof of Theorem 1. □

Proof of Corollary 1. We remark that $\{L_I(t),\ t \geqslant 0\}$ and $\{L_R(t),\ t \geqslant 0\}$ are nonnegative and $\sup_{0 \leqslant s \leqslant t} L_I(s) < \infty$ and $\sup_{0 \leqslant s \leqslant t} L_R(s) < \infty$ hold almost surely for any fixed $t \in \Lambda$. The desired relation follows from Theorem 1 via [27, Lemma 5.3]. □

Proof of Corollary 2. We employ a similar yet simpler argument to that used in the proof of Theorem 1. Specifically, by replacing Lemma 2 with Theorem 2.3 of [18], we obtain that for any fixed $t \in \Lambda$,

$$\mathbf{P}\Bigl(\inf_{0 \leqslant s \leqslant t} U_I(s) < 0 \Bigm| U_I(0) = x\Bigr) \sim \int_{0^-}^t \mathbf{P}\biggl(X > \frac{x\mathrm{e}^{L_I(s)}}{\rho}\biggr)\,\lambda(\mathrm{d}s) \sim \rho^\alpha \overline{F}(x) \int_{0^-}^t \mathbf{E}\bigl[\mathrm{e}^{-\alpha L_I(s)}\bigr]\,\lambda(\mathrm{d}s),$$

where the second step again uses [27, Lemma 5.3]. The desired result thus follows directly from the above relation and Corollary 1. □

Conflicts of interest. The authors declare no conflicts of interest.

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