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Computational technique for robust dynamic optimization of fractional impulsive switched systems*

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Abstract. In this work, we study a robust dynamic optimization problem for nonlinear fractional impulsive switched systems (FISSs) with uncertain parameters. The main novelty lies in directly incorporating parameter sensitivity into the cost functional to enhance robustness against model uncertainty. To solve the resulting problem, the system sensitivity is first computed through an auxiliary FISS, and a time-scaling transformation is employed to reformulate the optimization over fixed switching instants. Tractable gradient expressions are then derived using a set of auxiliary systems. A gradient-based optimization method is subsequently developed to solve the transformed problem together with a numerical scheme tailored for the FISSs. Two numerical examples demonstrate that the proposed technique achieves effective optimization performance and improved robustness.

Keywords: fractional impulsive switched system, dynamic optimization, sensitivity, gradient computation, numerical technique.

1 Introduction

Fractional-order systems, which generalize classical integer-order systems by incorporating derivatives of noninteger order, have proven to be effective for modeling complex dynamics exhibiting memory and hereditary characteristics [13]. Impulse and switching are commonly used in systems where discontinuities or nonsmooth behaviors occur – either in time, in system dynamics, or in control strategies [4]. The integration of fractional calculus with impulse and switching has led to the emergence of fractional impulsive switched systems (FISSs) – a powerful modeling framework suitable for describing abrupt

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transitions, regime changes, and long-range dependencies in diverse applications such as robot control [1], neural networks [10], and biological systems [19].

Despite their rich modeling capabilities, optimization and control of FISSs remain a considerable challenge. Only a few theoretical and algorithmic results on optimization and control of FISSs have been reported. Priyadharsini et al. [14] investigated fractional stochastic impulsive systems, focusing on controllability and optimal control. Kasinathan et al. [3] derived sufficient conditions for the existence of mild solutions and confirmed the existence of optimal controls. In the context of FISSs, a controller that guaranteed finite-time stability under input saturation and switching was proposed in [16]. For the dynamic optimization of nonlinear FISSs, an effective optimize–then–discretize (OTD) strategy, in which the continuous gradients of the original problem are derived prior to discretization, was explored in [5]. Furthermore, this method has been extended in [7]. However, these methods are not applicable to the optimization and control of FISSs with uncertain parameters.

Parameter uncertainty is an inherent feature in most real-world systems, and neglecting it can significantly compromise the performance of optimal strategies [18]. Thus, it is crucial to find a robust optimal strategy against the parameter uncertainty. Rehbock et al. [15] developed a gradient-based computation approach to solve robust optimal control problems. Loxton et al. [9] showed that the system sensitivity [15] could be computed via an auxiliary initial value problem, allowing the reformulation of the original problem as a standard Mayer problem. Furthermore, this method has been extensively applied in a range of practical scenarios [2, 11]. Note that all these computational techniques are OTD-based strategies. More importantly, they focus on the optimal control of integer-order dynamical systems instead of fractional-order systems. As far as we are aware, only the paper [8] addressed the robust optimal control of nonlinear fractional-order systems. This method constructs a sensitivity system and then computes gradients using a discretize-then-optimize (DTO) strategy, that is, the system is first discretized, and the gradients are derived from the discrete formulation. It should be noted that this strategy yields gradients associated with the discretized problems rather than the original continuous fractional-order system. Moreover, the method developed in [8] is limited to address the robust optimal control problem involving fractional systems in a single stage.

Motivated by this, we investigate the robust dynamic optimization of a class of FISSs. The main contribution of our work is the integration of sensitivity-based robustness into dynamical optimization of FISSs together with an OTD-based gradient computation strategy, which fundamentally differs from the nonrobust formulation in [5, 7] and the DTO approach in [8]. Specifically, we incorporate the system sensitivity into the cost functional, where the switching times and impulsive amplitudes are decision variables. The system sensitivity is first computed through an auxiliary FISS, and a time-scaling transformation is employed to reformulate the optimization over fixed switching instants. Gradient formulas of the cost functional in the equivalent problem are then derived by solving a sequence of auxiliary systems. Finally, an optimization algorithm incorporating a tailored numerical scheme for FISS is developed to solve the transformed problem, and two numerical examples demonstrate that the proposed technique achieves effective optimization performance and improved robustness.

The remainder of the paper is structured as follows. Section 2 provides the problem formulation. Section 3 outlines the sensitivity computation and the time-scaling approach. Section 4 presents the gradient derivation. Section 5 develops the computational technique for our robust dynamical optimization of FISSs. Section 6 shows two numerical examples. Finally, Section 7 concludes the paper.

2 Problem formulation

Let $I_m := \{1, 2, \ldots, m\}$. Now, consider the following FISS with uncertain parameter vector:

$$ {}_{t_{i-1}}^{\;C}\mathrm{D}_t^{\alpha} x(t) = f^i\bigl(x(t), \zeta\bigr), \tag{1} $$

$t \in (t_{i-1}, t_i)$, $i \in I_m$, with the initial and impulsive conditions

$$ x\bigl(t_i^+\bigr) = \begin{cases} \varphi(\zeta) & \text{if } i = 0, \\ x(t_i^-) + h^i(x(t_i^-), \sigma) & \text{if } i \in I_{m-1}, \end{cases} \tag{2} $$

where $x(t) \in \mathbb{R}^{n_x}$ represents the state vector, $\zeta \in \mathbb{R}^{n_p}$ denotes the uncertain parameter vector, $\sigma \in \mathbb{R}^{n_c}$ is the impulsive control vector, t_i ($i \in I_{m-1}$) are the switching/impulsive times, t_0 and $t_m = T$ are given initial and terminal times, $x(t_i^{\pm})$ denote the values of $x(t)$ approached from the left and right at t_i, respectively. In the above system, the mappings $f^i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_p} \to \mathbb{R}^{n_x}$ ($i \in I_m$) are given functions in C^3, $\varphi : \mathbb{R}^{n_p} \to \mathbb{R}^{n_x}$ and $h^i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_c} \to \mathbb{R}^{n_x}$ ($i \in I_{m-1}$) are given functions in C^2. Note that these smoothness requirements are essential for our analysis in the sequel. The impulsive terms $h^i(x(t_i^-), \sigma)$ model sudden and instantaneous changes in the system state due to events such as switching actions, impacts, shocks, or abrupt control interventions. In (1), the fractional derivative ${}_{t_{i-1}}^{\;C}\mathrm{D}_t^{\alpha} x(t)$ with order vector α captures the memory effect and long-range temporal dependence typical in viscoelastic, diffusion, and anomalous dynamical processes. Furthermore, ${}_{t_{i-1}}^{\;C}\mathrm{D}_t^{\alpha} x(t) := ({}_{t_{i-1}}^{\;C}\mathrm{D}_t^{\alpha_1} x_1(t), \ldots, {}_{t_{i-1}}^{\;C}\mathrm{D}_t^{\alpha_{n_x}} x_{n_x}(t))^\top$, whose jth component ${}_{t_{i-1}}^{\;C}\mathrm{D}_t^{\alpha_j} x_j(t)$ denotes the Caputo's fractional derivative with fractional order $\alpha_j \in (0, 1]$ defined as

$$ {}_{t_{i-1}}^{\;C}\mathrm{D}_t^{\alpha_j} x_j(t) = \frac{1}{\Gamma(1 - \alpha_j)} \int_{t_{i-1}}^{t} \frac{\dot{x}_j(\tau)}{(t - \tau)^{\alpha_j}} \,\mathrm{d}\tau. \tag{3} $$

Here $\Gamma(\cdot)$ is the gamma function.

In FISS (1) and (2), the switching instants and impulsive control vector are to be optimally determined. Now, we define

$$ \mathcal{T} := \bigl\{ \nu = (t_1, t_2, \ldots, t_{m-1})^\top \bigm| t_i - t_{i-1} \geqslant \varDelta,\ i \in I_m \bigr\}, $$

where $\varDelta > 0$ is a given positive constant. Furthermore, we define

$$ \varSigma := \bigl\{ \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_{n_c})^\top \in \mathbb{R}^{n_c} \bigm| c_l \leqslant \sigma_l \leqslant d_l,\ l \in I_{n_c} \bigr\}, $$

where c_l and d_l are respectively the lower and upper bounds of σ_l. Any $(\nu, \sigma) \in \mathcal{T} \times \Sigma$ is referred to as a feasible pair. For each $(\nu, \sigma, \zeta) \in \mathcal{T} \times \Sigma \times \mathbb{R}^{n_p}$, let $x(\cdot \mid \nu, \sigma, \zeta)$ be the unique solution of FISS (1) and (2) as established in [13]. It is worth noting that a nominal vector ζ^* for the uncertain parameter vector ζ can be obtained by using parameter estimation method such as that described in [6]. Traditionally, the goal of optimizing $(\nu, \sigma) \in \mathcal{T} \times \Sigma$ is to minimize the following cost function under the nominal vector ζ^*:

$$J(\nu, \sigma \mid \zeta^*) = \sum_{i=1}^{m} \Phi_i\big(x(t_i^+ \mid \nu, \sigma, \zeta^*), x(t_i^- \mid \nu, \sigma, \zeta^*), \sigma\big), \tag{4}$$

where $\Phi_i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_c} \to \mathbb{R}$, $i \in I_m$, is in $\mathbb{C}^2$. However, in real applications, the values of uncertain parameter vector may fluctuate due to some unpredictable environmental and operational changes, which means that ζ may deviate slightly from its nominal parameter vector ζ^*. More importantly, these variations will affect the optimal solution of dynamic optimization problem with the cost function (4). Thus, a novel cost function involving the sensitivity term is introduced as follows:

$$\bar{J}^{\gamma}(\nu, \sigma \mid \zeta^*) = J(\nu, \sigma \mid \zeta^*) + \gamma \left(\frac{\partial J(\nu, \sigma|\zeta^*)}{\partial \zeta}\right)\left(\frac{\partial J(\nu, \sigma|\zeta^*)}{\partial \zeta}\right)^{\top}, \tag{5}$$

where $J(\nu, \sigma|\zeta^*)$ is as defined in (4), and $\gamma \geqslant 0$ is a given weighting factor. Note that minimizing $\bar{J}^{\gamma}$ encourages the quadratic sensitivity term to be as small as possible. Now, we state the following robust dynamic optimization of FISSs.

Problem Q1. Given the nominal parameter vector ζ^*, find a pair $(\nu, \sigma) \in \mathcal{T} \times \Sigma$ to minimize the cost function $\bar{J}^{\gamma}(\nu, \sigma \mid \zeta^*)$ defined in (5).

3 Problem transformation

Problem Q1 has two nonstandard characteristics: (i) the sensitivity term is involved, and (ii) the switching times are treated as optimization variables. These nonstandard characteristics pose challenges in numerically solving Problem Q1. To surmount these difficulties, we transform Problem Q1 into an equivalent standard dynamic optimization problem in this section.

3.1 Computing the system sensitivity

For $(\nu, \sigma) \in \mathcal{T} \times \Sigma$ and $q \in I_{n_p}$, we consider the following auxiliary FISS (A):

$$ {}_{t_{i-1}}^{\;\;C}\mathrm{D}_t^{\alpha} \phi^q(t) = \frac{\partial f^i}{\partial x}\bigg|_{(x(t), \zeta^*)} \phi^q(t) + \frac{\partial f^i}{\partial \zeta_q}\bigg|_{(x(t), \zeta^*)},$$

$t \in (t_{i-1}, t_i)$, $i \in I_m$, with conditions

$$\phi^q(t_i^+) = \begin{cases} \frac{\partial \varphi(\zeta)}{\partial \zeta_q}\big|_{\zeta=\zeta^*} & \text{if } i = 0, \\ \phi^q(t_i^-) + \frac{\partial h^i(x(t_i^-), \sigma)}{\partial x^-} \phi^q(t_i^-) & \text{if } i \in I_{m-1}, \end{cases}$$

where ∂x^- represents an infinitesimal change in the variable $x(t_i^-)$. Given the vector of nominal parameter vector ζ^*, let $\phi^q(\cdot\,|\,\nu,\sigma,\zeta^*)$ be the solution of auxiliary FISS (A) for any $(\nu,\sigma)\in\mathcal{T}\times\Sigma$ [13]. We now present the following important result.

Theorem 1. *Given the nominal parameter vector ζ^*, it follows that, for each $(\nu,\sigma)\in\mathcal{T}\times\Sigma$,*

$$\frac{\partial x(t\,|\,\nu,\sigma,\zeta^*)}{\partial\zeta_q}=\phi^q(t\,|\,\nu,\sigma,\zeta^*),\tag{6}$$

$t\in[0,T]$, $q\in I_{n_p}$, *where $x(t\,|\,\nu,\sigma,\zeta^*)$ is the solution of FISS* (1) *and* (2)*.*

Proof. For each $t\in(t_{i-1},t_i)$, $i\in I_m$, $q\in I_{n_p}$, and $(\nu,\sigma)\in\mathcal{T}\times\Sigma$, we abbreviate the solution $x(t|\nu,\sigma,\zeta^*)$ of FISS (1) and (2) as $x(t)$. Furthermore, let $\epsilon>0$ be sufficiently small. Then the solution of the FISS defined by (1) and (2) corresponding to $\zeta^*+\epsilon e_q$ is denoted by $x^\epsilon(t)$, where $e_q\in\mathbb{R}^{n_p}$ is the qth unit vector. Thus, for $j\in I_{n_x}$, the following hold:

$$x_j(t)=x_j\bigl(t_{i-1}^+\bigr)+\frac{1}{\Gamma(\alpha_j)}\int\limits_{t_{i-1}}^{t}(t-\tau)^{\alpha_j-1}f_j^i\bigl(x(\tau),\zeta^*\bigr)\,\mathrm{d}\tau,$$

and

$$x_j^\epsilon(t)=x_j^\epsilon\bigl(t_{i-1}^+\bigr)+\frac{1}{\Gamma(\alpha_j)}\int\limits_{t_{i-1}}^{t}(t-\tau)^{\alpha_j-1}f_j^i\bigl(x^\epsilon(\tau),\,\zeta^*+\epsilon e_q\bigr)\,\mathrm{d}\tau.$$

Thus,

$$\begin{aligned}\frac{\partial x_j(t)}{\partial\zeta_q}&=\frac{\mathrm{d}x_j^\epsilon(t)}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0}\\&=\frac{\partial x_j(t_{i-1}^+)}{\partial\zeta_q}+\frac{\int_{t_{i-1}}^{t}(t-\tau)^{\alpha_j-1}\bigl(\frac{\partial f_j^i}{\partial x}|_{(x(\tau),\zeta^*)}\frac{\partial x(\tau)}{\partial\zeta_q}+\frac{\partial f_j^i}{\partial\zeta_q}|_{(x(\tau),\zeta^*)}\bigr)\,\mathrm{d}\tau}{\Gamma(\alpha_j)}.\end{aligned}\tag{7}$$

Furthermore, differentiating (2) with respect to ζ_q yields

$$\frac{\partial x(t_i^+)}{\partial\zeta_q}=\begin{cases}\frac{\partial\varphi(\zeta)}{\partial\zeta_q}|_{\zeta=\zeta^*} & \text{if } i=0,\\ \frac{\partial x(t_i^-)}{\partial\zeta_q}+\frac{\partial h^i(x(t_i^-),\sigma)}{\partial x^-}\frac{\partial x(t_i^-)}{\partial\zeta_q} & \text{if } i\in I_{m-1}.\end{cases}\tag{8}$$

Combining (7) with (8) gives

$${}_{t_{i-1}}^{\;\;C}\mathrm{D}_t^\alpha\left(\frac{\partial x(t)}{\partial\zeta_q}\right)=\frac{\partial f^i}{\partial x}\bigg|_{(x(t),\zeta^*)}\frac{\partial x(t)}{\partial\zeta_q}+\frac{\partial f^i}{\partial\zeta_q}\bigg|_{(x(t),\zeta^*)}\tag{9}$$

with conditions

$$\frac{\partial x(t_i^+)}{\partial\zeta_q}=\begin{cases}\frac{\partial\varphi(\zeta)}{\partial\zeta_q}|_{\zeta=\zeta^*} & \text{if } i=0,\\ \frac{\partial x(t_i^-)}{\partial\zeta_q}+\frac{\partial h^i(x(t_i^-),\sigma)}{\partial x^-}\frac{\partial x(t_i^-)}{\partial\zeta_q} & \text{if } i\in I_{m-1}.\end{cases}\tag{10}$$

Equations (9) and (10) indicate that $\partial x(t)/\partial\zeta_q$ also satisfies the auxiliary FISS (A). Thus, equality (6) holds due to the uniqueness of the solution to the auxiliary FISS (A). $\square$

Therefore, the sensitivity term in the new cost function (5) can be calculated as follows:

$$\frac{\partial J(\nu,\sigma|\zeta^*)}{\partial\zeta_q} = \sum_{i=1}^{m} \big(A_i^+(\vartheta,\sigma,\zeta^*)\phi^q(t_i^+\,|\,\nu,\sigma,\zeta^*) + A_i^-(\vartheta,\sigma,\zeta^*)\phi^q(t_i^-\,|\,\nu,\sigma,\zeta^*)\big),$$

where $q\in I_{n_p}$,

$$A_i^+(\nu,\sigma,\zeta^*) := \frac{\partial\Phi_i}{\partial x^+}\bigg|_{(x(t_i^+|\nu,\sigma,\zeta^*),\,x(t_i^-|\nu,\sigma,\zeta^*),\sigma)}, \tag{11}$$

and

$$A_i^-(\nu,\sigma,\zeta^*) := \frac{\partial\Phi_i}{\partial x^-}\bigg|_{(x(t_i^+|\nu,\sigma,\zeta^*),\,x(t_i^-|\nu,\sigma,\zeta^*),\sigma)}. \tag{12}$$

In (11) and (12), $\partial x^\pm$ denotes an infinitesimal change in the variable $x(t_i^\pm)$. Thus, the cost function (5) in Problem Q1 becomes

$$\begin{aligned}
&\bar J^\gamma(\nu,\sigma\,|\,\zeta^*)\\
&\quad= \sum_{i=1}^{m} \Phi_i\big(x(t_i^+\,|\,\nu,\sigma,\zeta^*),\,x(t_i^-\,|\,\nu,\sigma,\zeta^*),\,\sigma\big)\\
&\qquad+\gamma\sum_{q=1}^{n_p}\Bigg[\sum_{i=1}^{m}\big(A_i^+(\nu,\sigma,\zeta^*)\phi^q\big(t_i^+\,|\,\nu,\sigma,\zeta^*\big)+A_i^-(\nu,\sigma,\zeta^*)\phi^q\big(t_i^-\,|\,\nu,\sigma,\zeta^*\big)\big)\Bigg]^2.
\end{aligned} \tag{13}$$

So, Problem Q1 can be reformulated as the following equivalent problem.

Problem Q2. Given the nominal parameter vector ζ^*, find a pair $(\nu,\sigma)\in\mathcal{T}\times\Sigma$ to minimize the cost function (13).

3.2 Time-scaling transformation

The switching instants serve as optimization variables in Problem Q2, which are difficult to be optimized directly due to their iteration-dependent variability. To address this difficulty, we use a time-scaling transformation technique to reformulate the optimization over fixed switching instants.

Let

$$\Theta := \Bigg\{\vartheta = (\vartheta_1,\vartheta_2,\dots,\vartheta_m)^\top \;\Big|\; \sum_{i=1}^{m}\vartheta_i = T,\ \vartheta_i = t_i - t_{i-1}\geqslant \Delta,\ i\in I_m\Bigg\}.$$

Then, for each $\vartheta\in\Theta$, we define

$$t = \mu(s\,|\,\vartheta) = \begin{cases} \sum_{i=1}^{\lfloor s\rfloor}\vartheta_i + (s-\lfloor s\rfloor)\vartheta_{\lfloor s\rfloor+1} & \text{if } s\in[0,m),\\ T & \text{if } s=m. \end{cases} \tag{14}$$

Here $\lfloor\cdot\rfloor$ is the floor function. Let $\tilde{x}(s) = x(\mu(s|\vartheta)$. Then from the definition of Caputo derivative given in (3) it follows that, for $j \in I_{n_x}$,

$$\begin{aligned}
{}_{t_{i-1}}^{\ C}\mathrm{D}_t^{\alpha_j} x_j(t) &= \frac{1}{\Gamma(1-\alpha_j)} \int_{t_{i-1}}^{t} \frac{\dot{x}_j(\tau)}{(t-\tau)^{\alpha_j}}\,\mathrm{d}\tau \\
&= \frac{1}{\Gamma(1-\alpha_j)} \int_{\mu(i-1|\vartheta)}^{\mu(s|\vartheta)} \frac{\dot{x}_j(\mu(\eta|\vartheta))}{(\mu(s|\vartheta)-\mu(\eta|\vartheta))^{\alpha_j}}\,\mathrm{d}\mu(\eta|\vartheta) \\
&= \frac{1}{\vartheta_i^{\alpha_j}\Gamma(1-\alpha_j)} \int_{i-1}^{s} \frac{\dot{\tilde{x}}_j(\eta)}{(s-\eta)^{\alpha_j}}\,\mathrm{d}\eta \\
&= \frac{1}{\vartheta_i^{\alpha_j}}\ {}_{i-1}^{\ C}\mathrm{D}_s^{\alpha_j} \tilde{x}_j(s).
\end{aligned} \tag{15}$$

Consequently, applying transformation (14) along with the nominal parameter vector ζ^*, FISS (1) with (2) is transformed into FISS (B)

$$ {}_{i-1}^{\ C}\mathrm{D}_s^{\alpha_j} \tilde{x}_j(s) = \vartheta_i^{\alpha_j} f_j^i\bigl(\tilde{x}(s), \zeta^*\bigr), \tag{16}$$

$s \in (i-1,\, i)$, $i \in I_m$, $j \in I_{n_x}$, with conditions

$$\tilde{x}(i^+) = \begin{cases} \varphi(\zeta^*) & \text{if } i = 0, \\ \tilde{x}(i^-) + h^i(\tilde{x}(i^-), \sigma) & \text{if } i \in I_{m-1}. \end{cases} \tag{17}$$

Note that (15) ensures that FISS (B) is equivalent to FISS (1) and (2). Therefore, we can solve FISS (B) to obtain the solution of FISS (1) and (2).

Similarly, let $\tilde{\phi}^q(s) := \phi^q(\mu(s|\vartheta))$ for $q \in I_{n_p}$. Thus, the auxiliary FISS (A) is correspondingly transformed into FISS (C)

$$ {}_{i-1}^{\ C}\mathrm{D}_s^{\alpha_j} \tilde{\phi}_j^q(s) = \vartheta_i^{\alpha_j} \biggl(\frac{\partial f_j^i}{\partial \tilde{x}}\bigg|_{(\tilde{x}(s),\zeta^*)} \tilde{\phi}^q(s) + \frac{\partial f_j^i}{\partial \zeta_q}\bigg|_{(\tilde{x}(s),\zeta^*)} \biggr), \tag{18}$$

$s \in (i-1,\, i)$, $i \in I_m$, $j \in I_{n_x}$, with conditions

$$\tilde{\phi}^q(i^+) = \begin{cases} \frac{\partial \varphi(\zeta)}{\partial \zeta_q}|_{\zeta=\zeta^*} & \text{if } i = 0, \\ \tilde{\phi}^q(i^-) + \frac{\partial h^i(\tilde{x}(i^-),\sigma)}{\partial \tilde{x}^-}\tilde{\phi}^q(i^-) & \text{if } i \in I_{m-1}. \end{cases} \tag{19}$$

Note that both FISS (B) and auxiliary FISS (C) are defined over $[0, m]$, where the switching instants are fixed at $1, 2, \ldots,$ and $m-1$.

Now, given the nominal parameter vector ζ^*, we denote the solutions of FISS (B) and auxiliary FISS (C) by $\tilde{x}(\cdot \,|\, \vartheta, \sigma, \zeta^*)$ and $\tilde{\phi}^q(\cdot \,|\, \vartheta, \sigma, \zeta^*)$, respectively [13]. Then the cost function (13) takes the following form:

$$\begin{aligned}
&\tilde{J}^\gamma(\vartheta, \sigma \,|\, \zeta^*) \\
&\quad = \sum_{i=1}^{m} \Phi_i\big(\tilde{x}(i^+ \,|\, \vartheta, \sigma, \zeta^*),\, \tilde{x}(i^- \,|\, \vartheta, \sigma, \zeta^*), \sigma\big) \\
&\qquad + \gamma \sum_{q=1}^{n_p} \Bigg[\sum_{i=1}^{m} \big(\tilde{A}_i^+(\vartheta, \sigma, \zeta^*)\tilde{\phi}^q(i^+ \,|\, \vartheta, \sigma, \zeta^*) + \tilde{A}_i^-(\vartheta, \sigma, \zeta^*)\tilde{\phi}^q(i^- \,|\, \vartheta, \sigma, \zeta^*)\big)\Bigg]^2, \quad (20)
\end{aligned}$$

where

$$\tilde{A}_i^+(\vartheta, \sigma, \zeta^*) := \frac{\partial \Phi_i}{\partial \tilde{x}^+}\bigg|_{(\tilde{x}(i^+|\vartheta,\sigma,\zeta^*),\, \tilde{x}(i^-|\vartheta,\sigma,\zeta^*),\sigma)} \tag{21}$$

and

$$\tilde{A}_i^-(\vartheta, \sigma, \zeta^*) := \frac{\partial \Phi_i}{\partial \tilde{x}^-}\bigg|_{(\tilde{x}(i^+|\vartheta,\sigma,\zeta^*),\, \tilde{x}(i^-|\vartheta,\sigma,\zeta^*),\sigma)}. \tag{22}$$

As a result, Problem Q2 can be reformulated as the following equivalent problem by means of time-scaling transformation (14).

Problem Q3. Given the nominal vector ζ^*, find a pair $(\vartheta, \sigma) \in \Theta \times \Sigma$ to minimize the cost function (20).

4 Gradient computation

Problem Q3 is a dynamic optimization problem in which the decision variables consist of the duration vector between two adjacent switching times, denoted by ϑ, and impulsive control vector σ. Essentially, it is a mathematical programming problem, which is amenable to solution via gradient-based optimization techniques [17]. Next, we will derive the required gradients of the cost function (20) with respect to $(\vartheta, \sigma) \in \mathcal{T} \times \Sigma$.

For each $\iota \in I_m$, $q \in I_{n_p}$, and $j \in I_{n_x}$, we consider the auxiliary FISS (D)

$${}_{i-1}^{\;\;C}\mathrm{D}_s^{\alpha_j} \upsilon_j^\iota(s) = \begin{cases} 0 & \text{if } s \in (i-1,\, i),\ 1 \leqslant i < \iota, \\ \delta_{i\iota}\alpha_j \vartheta_i^{\alpha_j - 1} f_j^i(\tilde{x}(s), \zeta^*) & \\ \quad + \vartheta_i^{\alpha_j} \frac{\partial f_j^i}{\partial \tilde{x}}\big|_{(\tilde{x}(s),\zeta^*)} \upsilon^\iota(s) & \text{if } s \in (i-1,\, i),\ \iota \leqslant i \leqslant m \end{cases} \tag{23}$$

with

$$\upsilon^\iota(i^+) = \begin{cases} 0 & \text{if } i \in I_{\iota-1}, \\ \upsilon^\iota(i^-) + \frac{h^i(\tilde{x}(i^-),\sigma)}{\partial \tilde{x}^-} \upsilon^\iota(i^-) & \text{if } i \in \{\iota,\, \iota+1,\, \dots,\, m-1\} \end{cases} \tag{24}$$

and the auxiliary FISS (E)

$$
{}_{i-1}^{\ C}\mathrm{D}_s^{\alpha_j}\rho_j^{q,\iota}(s)=\begin{cases}0 & \text{if } s\in(i-1,\,i),\ 1\leqslant i<\iota,\\ \delta_{i\iota}\alpha_j\vartheta_i^{\alpha_j-1}\big(\frac{\partial f_j^i}{\partial\tilde{x}}|_{(\tilde{x}(s),\zeta^*)}\tilde{\phi}^q(s)+\frac{\partial f_j^i}{\partial\zeta_q}|_{(\tilde{x}(s),\zeta^*)}\big) & \\ \quad+\vartheta_i^{\alpha_j}\big((\upsilon^\iota(s))^\top\frac{\partial^2 f_j^i}{\partial\tilde{x}^2}|_{(\tilde{x}(s),\zeta^*)}\tilde{\phi}^q(s)+\frac{\partial f_j^i}{\partial\tilde{x}}|_{(\tilde{x}(s),\zeta^*)}\rho^{q,\iota}(s) & \\ \quad+\frac{\partial^2 f_j^i}{\partial\tilde{x}\partial\zeta_q}|_{(\tilde{x}(s),\zeta^*)}\upsilon^\iota(s)\big) & \text{if } s\in(i-1,\,i),\ \iota\leqslant i\leqslant m\end{cases}\tag{25}
$$

with

$$
\rho_j^{q,\iota}(i+)=\begin{cases}0 & \text{if } i\in I_{\iota-1},\\ \rho_j^{q,\iota}(i^-)+(\upsilon^\iota(i^-))^\top\frac{\partial^2 h_j^i(\tilde{x}(i^-),\sigma)}{\partial(\tilde{x}^-)^2}\tilde{\phi}^q(i^-) & \\ \quad+\frac{\partial h_j^i(\tilde{x}(i^-),\sigma)}{\partial\tilde{x}^-}\rho^{q,\iota}(i^-) & \text{if } i\in\{\iota,\dots,m-1\}.\end{cases}\tag{26}
$$

Here $\upsilon^\iota(s)=(\upsilon_1^\iota(s),\upsilon_2^\iota(s),\dots,\upsilon_{n_x}^\iota(s))^\top$, and $\delta_{i\iota}$ denotes the Kronecker delta defined as

$$
\delta_{i\iota}=\begin{cases}1 & \text{if } i=\iota,\\ 0 & \text{otherwise.}\end{cases}
$$

Given the nominal vector ζ^*, let $\upsilon^\iota(\cdot\,|\,\vartheta,\sigma,\zeta^*)$, and let $\rho^{q,\iota}(\cdot\,|\,\vartheta,\sigma,\zeta^*)$ $(q\in I_{n_p})$ be the unique solutions of the auxiliary FISSs (D) and (E), respectively. The following theorem describes the relationship between the solutions of the auxiliary FISSs (D) and (E) and the partial derivatives of the solutions $\tilde{x}(\cdot\,|\,\vartheta,\sigma,\zeta^*)$ and $\tilde{\phi}^q(\cdot\,|\,\vartheta,\sigma,\zeta^*)$ with respect to ϑ.

Theorem 2. *Given the nominal parameter vector ζ^*, it follows that, for each pair $(\vartheta,\sigma)\in\Theta\times\Sigma$, $\iota\in I_m$, and $q\in I_{n_p}$,*

$$
\frac{\partial\tilde{x}(s\,|\,\vartheta,\sigma,\zeta^*)}{\partial\vartheta_\iota}=\upsilon^\iota(s\,|\,\vartheta,\sigma,\zeta^*)
$$

and

$$
\frac{\partial\tilde{\phi}^q(s\,|\,\vartheta,\sigma,\zeta^*)}{\partial\vartheta_\iota}=\rho^{q,\iota}(s\,|\,\vartheta,\sigma,\zeta^*),
$$

where $s\in[0,m]$.

Proof. The proof proceeds using a method similar to that employed in Theorem 1. □

For each $l\in I_{n_c}$, $j\in I_{n_x}$, and $q\in I_{n_p}$, we further consider the auxiliary FISS (F)

$$
{}_{i-1}^{\ C}\mathrm{D}_s^{\alpha_j}\chi_j^l(s)=\vartheta_i^{\alpha_j}\frac{\partial f_j^i}{\partial\tilde{x}}\bigg|_{(\tilde{x}(s),\zeta^*)}\chi^l(s),\tag{27}
$$

$s\in(i-1,\,i)$, $i\in I_m$, with

$$
\chi^l(i^+)=\begin{cases}0 & \text{if } i=0,\\ \chi^l(i^-)+\frac{\partial h^i(\tilde{x}(i^-),\sigma)}{\partial\tilde{x}^-}\chi^l(i^-)+\frac{\partial h^i(x(i^-),\sigma)}{\partial\sigma_l} & \text{if } i\in I_{m-1}\end{cases}\tag{28}
$$

and the auxiliary FISS (G)

$$
{}_{i-1}^{\ \ C}\mathrm{D}_s^{\alpha}\lambda_j^{q,l}(s) = \vartheta_i^{\alpha_j}\Bigg(\big(\chi^l(s)\big)^\top \frac{\partial^2 f_j^i}{\partial \tilde{x}^2}\bigg|_{(\tilde{x}(s),\zeta^*)}\tilde{\phi}^q(s) + \frac{\partial f_j^i}{\partial \tilde{x}}\bigg|_{(\tilde{x}(s),\zeta^*)}\lambda^{q,l}(s) + \frac{\partial^2 f_j^i}{\partial \tilde{x}\partial\zeta_q}\bigg|_{(\tilde{x}(s),\zeta^*)}\chi^l(s)\Bigg), \tag{29}
$$

$s \in (i-1,\, i)$, $i \in I_m$, with

$$
\lambda_j^{q,l}(i^+) = \begin{cases} 0 & \text{if } i = 0, \\ \lambda_j^{q,l}(i^-) + \big(\chi^l(i^-)\big)^\top \frac{\partial^2 h_j^i(\tilde{x}(i^-),\sigma)}{(\partial\tilde{x}^-)^2}\tilde{\phi}^q(i^-) & \\ \quad + \frac{\partial h_j^i(\tilde{x}(i^-),\sigma)}{\partial\tilde{x}^-}\lambda^{q,l}(i^-) + \frac{\partial h_j^i(x(i^-),\sigma)}{\partial\sigma_l}\tilde{\phi}^q(i^-) & \text{if } i \in I_{m-1}. \end{cases} \tag{30}
$$

Here $\chi^l(s) = (\chi_1^l(s), \chi_2^l(s) \dots, \chi_{n_x}^l(s))^\top$. Let $\chi^l(\cdot \,|\, \vartheta, \sigma, \zeta^*)$ and $\lambda^{q,l}(\cdot \,|\, \vartheta, \sigma, \zeta^*)$ be the unique solutions of the auxiliary FISSs (F) and (G) for any $(\vartheta, \sigma) \in \Theta \times \Sigma$. Then the following result holds.

Theorem 3. *Given the nominal parameter vector ζ^*, it follows that, for each pair $(\vartheta, \sigma) \in \Theta \times \Sigma$, $q \in I_{n_p}$, and $l \in I_{n_c}$,*

$$
\frac{\partial \tilde{x}(s \,|\, \vartheta, \sigma, \zeta^*)}{\partial \sigma_l} = \chi^l(s \,|\, \vartheta, \sigma, \zeta^*)
$$

and

$$
I\frac{\partial \tilde{\phi}^q(s \,|\, \vartheta, \sigma, \zeta^*)}{\partial \sigma_l} = \lambda^{q,l}(s \,|\, \vartheta, \sigma, \zeta^*),
$$

where $s \in [0, m]$.

Proof. The proof proceeds using a method similar to that employed in Theorem 1. □

The following theorem is derived from the results of Theorems 2 and 3.

Theorem 4. *Given the nominal parameter vector ζ^* and $(\vartheta, \sigma) \in \Theta \times \Sigma$, the following holds*:

$$
\begin{aligned}
\frac{\partial \tilde{J}^\gamma(\vartheta, \sigma|\zeta^*)}{\partial \vartheta_\iota} = {} & \sum_{i=1}^{m}\Big(\tilde{A}_i^+(\vartheta, \sigma, \zeta^*)\upsilon^\iota(i^+) + \tilde{A}_i^-(\vartheta, \sigma, \zeta^*)\upsilon^\iota(i-)\Big) \\
& + \gamma\sum_{q=1}^{n_p}\Bigg\{2\sum_{i=1}^{m}\big(\tilde{A}_i^+(\vartheta, \sigma, \zeta^*)\tilde{\phi}^q(i^+) + \tilde{A}_i^-(\vartheta, \sigma, \zeta^*)\tilde{\phi}^q(i^-)\big) \\
& \times \sum_{i=1}^{m}\big[\big(\big(\upsilon^\iota(i^+)\big)^\top\tilde{A}_i^{++}(\vartheta, \sigma, \zeta^*) + \big(\upsilon^\iota(i^-)\big)^\top\tilde{A}_i^{-+}(\vartheta, \sigma, \zeta^*)\big)\tilde{\phi}^q(i^+) \\
& + \big(\big(\upsilon^\iota(i^+)\big)^\top\tilde{A}_i^{+-}(\vartheta, \sigma, \zeta^*) + \big(\upsilon^\iota(i^-)\big)^\top\tilde{A}_i^{--}(\vartheta, \sigma, \zeta^*)\big)\tilde{\phi}^q(i^-) \\
& + \tilde{A}_i^+(\vartheta, \sigma, \zeta^*)\rho^{q,\iota}(i^+) + \tilde{A}_i^-(\vartheta, \sigma, \zeta^*)\rho^{q,\iota}(i^-)\big]\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \tilde{J}^{\gamma}(\vartheta,\sigma\,|\,\zeta^*)}{\partial \sigma_l} &= \sum_{i=1}^{m}\biggl(\tilde{A}_i^{+}(\vartheta,\sigma,\zeta^*)\chi^l(i^+)+\tilde{A}_i^{-}(\vartheta,\sigma,\zeta^*)\chi^l(i^-)+\frac{\partial \Phi_i}{\partial \sigma_l}\bigg|_{(\tilde{x}(i^+),\tilde{x}(i^-),\sigma)}\biggr)\\
&\quad+\gamma\sum_{q=1}^{n_p}\biggl\{2\sum_{i=1}^{m}\Bigl(\tilde{A}_i^{+}(\vartheta,\sigma,\zeta^*)\tilde{\phi}^q(i^+)+\tilde{A}_i^{-}(\vartheta,\sigma,\zeta^*)\tilde{\phi}^q(i^-)\Bigr)\\
&\quad\times\sum_{i=1}^{m}\biggl[\biggl(\bigl(\chi^l(i^+)\bigr)^{\top}\tilde{A}_i^{++}(\vartheta,\sigma,\zeta^*)+\bigl(\chi^l(i^-)\bigr)^{\top}\tilde{A}_i^{-+}(\vartheta,\sigma,\zeta^*)\\
&\quad+\frac{\partial^2 \Phi_i}{\partial \sigma_l \partial \tilde{x}^+}\bigg|_{(\tilde{x}(i^+),\tilde{x}(i^-),\sigma)}\biggr)\tilde{\phi}^q(i^+)+\tilde{A}_i^{+}(\vartheta,\sigma,\zeta^*)\lambda^{q,l}(i^+)\\
&\quad+\biggl(\bigl(\chi^l(i^+)\bigr)^{\top}\tilde{A}_i^{+-}(\vartheta,\sigma,\zeta^*)+\bigl(\chi^l(i^-)\bigr)^{\top}\tilde{A}_i^{--}(\vartheta,\sigma,\zeta^*)\\
&\quad+\frac{\partial^2 \Phi_i}{\partial \sigma_l \partial \tilde{x}^-}\bigg|_{(\tilde{x}(i^+),\tilde{x}(i^-),\sigma)}\biggr)\tilde{\phi}^q(i^-)+\tilde{A}_i^{-}(\vartheta,\sigma,\zeta^*)\lambda^{q,l}(i^-)\biggr]\biggr\}.
\end{aligned}
$$

Here $\tilde{x}(\cdot) := \tilde{x}(\cdot\,|\,\vartheta,\sigma,\zeta^*)$, $\tilde{\phi}^q(\cdot) := \tilde{\phi}^q(\cdot\,|\,\vartheta,\sigma,\zeta^*)$, $\upsilon^{\iota}(\cdot) := \upsilon^{\iota}(\cdot\,|\,\vartheta,\sigma,\zeta^*)$, $\rho^{q,\iota}(\cdot) := \rho^{q,\iota}(\cdot\,|\,\vartheta,\sigma,\zeta^*)$, $\chi^l(\cdot) := \chi^l(\cdot\,|\,\vartheta,\sigma,\zeta^*)$, *and* $\lambda^{q,l}(\cdot) := \lambda^{q,l}(\cdot\,|\,\vartheta,\sigma,\zeta^*)$. $\tilde{A}_i^{+}(\vartheta,\sigma,\zeta^*)$ *and* $\tilde{A}_i^{-}(\vartheta,\sigma,\zeta^*)$ *are defined by* (21) *and* (22)*, and*

$$
\begin{aligned}
\tilde{A}_i^{++}(\vartheta,\sigma,\zeta^*) &:= \frac{\partial^2 \Phi_i}{\partial (\tilde{x}^+)^2}\bigg|_{(\tilde{x}(i^+),\tilde{x}(i^-),\sigma)},\\
\tilde{A}_i^{+-}(\vartheta,\sigma,\zeta^*) &:= \frac{\partial^2 \Phi_i}{\partial \tilde{x}^+ \partial \tilde{x}^-}\bigg|_{(\tilde{x}(i^+),\tilde{x}(i^-),\sigma)},\\
\tilde{A}_i^{-+}(\vartheta,\sigma,\zeta^*) &:= \frac{\partial^2 \Phi_i}{\partial \tilde{x}^- \partial \tilde{x}^+}\bigg|_{(\tilde{x}(i^+),\tilde{x}(i^-),\sigma)},\\
\tilde{A}_i^{--}(\vartheta,\sigma,\zeta^*) &:= \frac{\partial^2 \Phi_i}{\partial (\tilde{x}^-)^2}\bigg|_{(\tilde{x}(i^+),\tilde{x}(i^-),\sigma)}.
\end{aligned}
$$

Proof. The proof proceeds by differentiating (20) and then using the chain rule. □

5 Computational technique

5.1 Numerical scheme

From Theorem 4 it follows that $\partial \tilde{J}^{\gamma}(\vartheta,\sigma)/\partial\vartheta$ and $\partial \tilde{J}^{\gamma}(\vartheta,\sigma)/\partial\sigma$ can be obtained by solving the transformed FISS (B) along with the auxiliary FISSs (C)–(G) all forward in time. This section presents a numerical scheme tailored to these FISSs, upon which an optimization algorithm is subsequently developed to address Problem Q3.

For each $j \in I_{n_x}$, $q \in I_{n_p}$, $\iota \in I_m$, and $l \in I_{n_c}$, let

$$
g_j^i(\tilde{x}(s),\tilde{\phi}^q(s),\zeta^*) := \vartheta_i^{\alpha_j}\biggl(\frac{\partial f_j^i}{\partial \tilde{x}}\bigg|_{(\tilde{x}(s),\zeta^*)}\tilde{\phi}^q(s)+\frac{\partial f_j^i}{\partial \zeta_q}\bigg|_{(\tilde{x}(s),\zeta^*)}\biggr),
$$

$s \in (i-1,\, i)$, $i \in I_m$,

$$\tilde{h}_j^i(\tilde{x}(s), \upsilon^\iota(s), \zeta^*) := \begin{cases} 0 & \text{if } s \in (i-1,\, i),\ 1 \leqslant i < \iota, \\ \delta_{i\iota}\alpha_j \vartheta_i^{\alpha_j - 1} f_j^i(\tilde{x}(s), \zeta^*) + \vartheta_i^{\alpha_j} \frac{\partial f_j^i}{\partial \tilde{x}}|_{(\tilde{x}(s),\zeta^*)} \upsilon^\iota(s) & \text{if } s \in (i-1,\, i),\ \iota \leqslant i \leqslant m, \end{cases}$$

$$u_j^i(\tilde{x}(s), \tilde{\phi}^q(s), \upsilon^\iota(s), \rho^{q,\iota}(s), \zeta^*) := \begin{cases} 0 & \text{if } s \in (i-1,\, i),\ 1 \leqslant i < \iota, \\ \delta_{i\iota}\alpha_j \vartheta_i^{\alpha_j - 1} \big(\frac{\partial f_j^i}{\partial \tilde{x}}|_{(\tilde{x}(s),\zeta^*)} \tilde{\phi}^q(s) + \frac{\partial f_j^i}{\partial \zeta_q}|_{(\tilde{x}(s),\zeta^*)} \big) & \\ \quad + \vartheta_i^{\alpha_j} \big((\upsilon^\iota(s))^\top \frac{\partial^2 f_j^i}{\partial \tilde{x}^2}|_{(\tilde{x}(s),\zeta^*)} \tilde{\phi}^q(s) & \\ \quad + \frac{\partial f_j^i}{\partial \tilde{x}}|_{(\tilde{x}(s),\zeta^*)} \rho^{q,\iota}(s) + \frac{\partial^2 f_j^i}{\partial \tilde{x} \partial \zeta_q}|_{(\tilde{x}(s),\zeta^*)} \upsilon^\iota(s) \big) & \text{if } s \in (i-1,\, i),\ \iota \leqslant i \leqslant m, \end{cases}$$

and for $s \in (i-1,\, i)$, $i \in I_m$,

$$v_j^i\big(\tilde{x}(s), \chi^l(s), \zeta^*\big) := \vartheta_i^{\alpha_j} \frac{\partial f_j^i}{\partial \tilde{x}}\bigg|_{(\tilde{x}(s),\zeta^*)} \chi^l(s),$$

$$\begin{aligned} &w_j^i\big(\tilde{x}(s), \tilde{\phi}^q(s), \chi^l(s), \lambda^{q,l}(s), \zeta^*\big) \\ &\quad := \vartheta_i^{\alpha_j} \Bigg(\big(\chi^l(s)\big)^\top \frac{\partial^2 f_j^i}{\partial \tilde{x}^2}\bigg|_{(\tilde{x}(s),\zeta^*)} \tilde{\phi}^q(s) + \frac{\partial f_j^i}{\partial \tilde{x}}\bigg|_{(\tilde{x}(s),\zeta^*)} \lambda^{q,l}(s) \\ &\qquad + \frac{\partial^2 f_j^i}{\partial \tilde{x} \partial \zeta_q}\bigg|_{(\tilde{x}(s),\zeta^*)} \chi^l(s) \Bigg). \end{aligned}$$

Then, by using Riemann–Liouville integration [12], Eqs. (16), (18), (23), (25), (27), and (29) are equivalent to

$$\tilde{x}_j(s) = \tilde{x}_j\big((i-1)^+\big) + \frac{1}{\Gamma(\alpha_j)} \int_{i-1}^{s} (s-r)^{\alpha_j - 1} \vartheta_i^{\alpha_j} f_j^i\big(\tilde{x}(r), \zeta^*\big)\, \mathrm{d}r, \tag{31}$$

$$\tilde{\phi}_j^q(s) = \tilde{\phi}_j^q\big((i-1)^+\big) + \frac{1}{\Gamma(\alpha_j)} \int_{i-1}^{s} (s-r)^{\alpha_j - 1} g_j^i\big(\tilde{x}(r), \tilde{\phi}^q(r), \zeta^*\big)\, \mathrm{d}r, \tag{32}$$

$$\upsilon_j^\iota(s) = \upsilon_j^\iota\big((i-1)^+\big) + \frac{1}{\Gamma(\alpha_j)} \int_{i-1}^{s} (s-r)^{\alpha_j - 1} \tilde{h}_j^i\big(\tilde{x}(r), \upsilon^\iota(r), \zeta^*\big)\, \mathrm{d}r, \tag{33}$$

$$\begin{aligned} \rho_j^{q,\iota}(s) &= \rho_j^{q,\iota}\big((i-1)^+\big) \\ &\quad + \frac{1}{\Gamma(\alpha_j)} \int_{i-1}^{s} (s-r)^{\alpha_j - 1} u_j^i\big(\tilde{x}(r), \tilde{\phi}^q(r), \upsilon^\iota(r), \rho^{q,\iota}(r), \zeta^*\big)\, \mathrm{d}r, \end{aligned} \tag{34}$$

$$\chi_j^l(s) = \chi_j^l\big((i-1)^+\big) + \frac{1}{\Gamma(\alpha_j)} \int_{i-1}^{s} (s-r)^{\alpha_j - 1} v_j^i\big(\tilde{x}(r), \chi^l(r), \zeta^*\big)\, \mathrm{d}r, \tag{35}$$

$$\lambda_j^{q,l}(s) = \lambda_j^{q,l}\big((i-1)^+\big) + \frac{1}{\Gamma(\alpha_j)} \int\limits_{i-1}^{s} (s-r)^{\alpha_j - 1} w_j^i\big(\tilde{x}(r), \tilde{\phi}^q(r), \chi^l(r), \lambda^{q,l}(r), \zeta^*\big)\,\mathrm{d}r \tag{36}$$

for $s \in (i-1, i)$ and $i \in I_m$. Now, we uniformly divide the interval $[i-1, i]$ into N^i subintervals with the partition points $s_p^i = i - 1 + p/N^i$, $p = 1, 2, \dots, N^i$. Then Eqs. (31)–(36) can be written as

$$\tilde{x}_j^{i,p} = \tilde{x}_j^{i,0} + \frac{1}{\Gamma(\alpha_j)} \sum_{z=1}^{p} \int\limits_{s_{z-1}^i}^{s_z^i} (s_z^i - r)^{\alpha_j - 1} \vartheta_i^{\alpha_j} f_j^i\big(\tilde{x}(r), \zeta^*\big)\,\mathrm{d}r, \tag{37}$$

$$\tilde{\phi}_j^{q,i,p} = \tilde{\phi}_j^{q,i,0} + \frac{1}{\Gamma(\alpha_j)} \sum_{z=1}^{p} \int\limits_{s_{z-1}^i}^{s_z^i} (s_z^i - r)^{\alpha_j - 1} g_j^i\big(\tilde{x}(r), \tilde{\phi}^q(r), \zeta^*\big)\,\mathrm{d}r, \tag{38}$$

$$\upsilon_j^{\iota,i,p} = \upsilon_j^{\iota,i,0} + \frac{1}{\Gamma(\alpha_j)} \sum_{z=1}^{p} \int\limits_{s_{z-1}^i}^{s_z^i} (s_z^i - r)^{\alpha_j - 1} \tilde{h}_j^i\big(\tilde{x}(r), \upsilon^\iota(r), \zeta^*\big)\,\mathrm{d}r, \tag{39}$$

$$\rho_j^{q,\iota,i,p} = \rho_j^{q,\iota,i,0} + \frac{1}{\Gamma(\alpha_j)} \sum_{z=1}^{p} \int\limits_{s_{z-1}^i}^{s_z^i} (s_z^i - r)^{\alpha_j - 1} u_j^i\big(\tilde{x}(r), \tilde{\phi}^q(r), \upsilon^\iota(r), \rho^{q,\iota}(r), \zeta^*\big)\,\mathrm{d}r, \tag{40}$$

$$\chi_j^{l,i,p} = \chi_j^{l,i,0} + \frac{1}{\Gamma(\alpha_j)} \sum_{z=1}^{p} \int\limits_{s_{z-1}^i}^{s_z^i} (s_z^i - r)^{\alpha_j - 1} v_j^i\big(\tilde{x}(r), \chi^l(r), \zeta^*\big)\,\mathrm{d}r, \tag{41}$$

$$\lambda_j^{q,l,i,p} = \lambda_j^{q,l,i,0} + \frac{1}{\Gamma(\alpha_j)} \sum_{z=1}^{p} \int\limits_{s_{z-1}^i}^{s_z^i} (s_z^i - r)^{\alpha_j - 1} w_j^i\big(\tilde{x}(r), \tilde{\phi}^q(r), \chi^l(r), \lambda^{q,l}(r), \zeta^*\big)\,\mathrm{d}r. \tag{42}$$

Here $\tilde{x}_j^{i,p} = \tilde{x}_j(s_p^i)$, $\tilde{\phi}_j^{q,i,p} = \tilde{\phi}_j^q(s_p^i)$, $\rho_j^{q,\iota,i,p} = \rho_j^{q,\iota}(s_p^i)$, $\chi_j^{l,i,p} = \chi_j^l(s_p^i)$, and $\lambda_j^{q,l,i,p} = \lambda_j^{q,l}(s_p^i)$. $\tilde{x}_j^{i,0}$, $\tilde{\phi}_j^{q,i,0}$, $\rho_j^{q,\iota,i,0}$, $\chi_j^{l,i,0}$, and $\lambda_j^{q,l,i,0}$ represent the right limits of the corresponding states at s_0^i. Furthermore, according to [5], we expand the integrands of Eqs. (37)–(42) at point

$$r_{j,z}^{i,p} = i - 1 + \frac{[(e_z^p + 1)^{\alpha_j + 1} - (e_z^p)^{\alpha_j + 1}] + (\alpha_j + 1)[(e_z^p + 1)^{\alpha_j}(z-1) - (e_z^p)^{\alpha_j} z]}{N^i(\alpha_j + 1)[(e_z^p + 1)^{\alpha_j} - (e_z^p)^{\alpha_j}]},$$

where $e_z^p = p - z$. Thus, Eqs. (37)–(42) become

$$\tilde{x}_j^{i,p} = \tilde{x}_j^{i,0} + \beta_j^i \sum_{z=1}^{p} \kappa_{j,z}^p \vartheta_i^{\alpha_j} f_j^i\big(\tilde{x}(r_{j,z}^{i,p}), \zeta^*\big), \tag{43}$$

$$\tilde{\phi}_j^{q,i,p} = \tilde{\phi}_j^{q,i,0} + \beta_j^i \sum_{z=1}^{p} \kappa_{j,z}^p g_j^i\big(\tilde{x}\big(r_{j,z}^{i,p}\big), \tilde{\phi}^q\big(r_{j,z}^{i,p}\big), \zeta^*\big), \tag{44}$$

$$\upsilon_j^{\iota,i,p} = \upsilon_j^{\iota,i,0} + \beta_j^i \sum_{z=1}^{p} \kappa_{j,z}^p \tilde{h}_j^i\big(\tilde{x}\big(r_{j,z}^{i,p}\big), \upsilon\big(r_{j,z}^{i,p}\big), \zeta^*\big), \tag{45}$$

$$\rho_j^{q,\iota,i,p} = \rho_j^{q,\iota,i,0} + \beta_j^i \sum_{z=1}^{p} \kappa_{j,z}^p u_j^i\big(\tilde{x}\big(r_{j,z}^{i,p}\big), \tilde{\phi}^q\big(r_{j,z}^{i,p}\big), \upsilon\big(r_{j,z}^{i,p}\big), \rho^{q,\iota}\big(r_{j,z}^{i,p}\big), \zeta^*\big), \tag{46}$$

$$\chi_j^{l,i,p} = \chi_j^{l,i,0} + \beta_j^i \sum_{z=1}^{p} \kappa_{j,z}^p v_j^i\big(\tilde{x}\big(r_{j,z}^{i,p}\big), \chi^l\big(r_{j,z}^{i,p}\big), \zeta^*\big), \tag{47}$$

$$\lambda_j^{q,l,i,p} = \lambda_j^{q,l,i,0} + \beta_j^i \sum_{z=1}^{p} \kappa_{j,z}^p w_j^i\big(\tilde{x}\big(r_{j,z}^{i,p}\big), \chi^l\big(r_{j,z}^{i,p}\big), \lambda^{q,l}\big(r_{j,z}^{i,p}\big), \zeta^*\big), \tag{48}$$

where $\beta_j^i = 1/((N^i)^{\alpha_j}\Gamma(\alpha_j+1))$ and $\kappa_{j,z}^p = (e_z^p+1)^{\alpha_j} - (e_z^p)^{\alpha_j}$. Note that $\tilde{x}(r_{j,z}^{i,p})$, $\tilde{\phi}^q(r_{j,z}^{i,p})$, $\tau^\iota(r_{j,z}^{i,p})$, $\rho^{q,\iota}(r_{j,z}^{i,p})$, $\chi^l(r_{j,z}^{i,p})$, and $\lambda^{q,l}(r_{j,z}^{i,p})$ are unavailable directly, which can be approximated by linear interpolation using two nearest points as follows:

$$\check{x}\big(r_{j,z}^{i,p}\big) = \tilde{x}^{i,z-1} + \pi_{j,z}^{i,p}\big(\tilde{x}^{i,z} - \tilde{x}^{i,\,z-1}\big), \tag{49}$$

$$\check{\phi}^q\big(r_{j,z}^{i,p}\big) = \tilde{\phi}^{q,i,z-1} + \pi_{j,z}^{i,p}\big(\tilde{\phi}^{q,i,z} - \tilde{\phi}^{q,i,\,z-1}\big), \tag{50}$$

$$\check{\tau}^\iota\big(r_{j,z}^{i,p}\big) = \tau^{\iota,i,z-1} + \pi_{j,z}^{i,p}\big(\tau^{\iota,i,z} - \tau^{\iota,i,\,z-1}\big), \tag{51}$$

$$\check{\rho}^{q,\iota}\big(r_{j,z}^{i,p}\big) = \rho^{q,\iota,i,z-1} + \pi_{j,z}^{i,p}\big(\rho^{q,\iota,i,z} - \rho^{q,\iota,i,\,z-1}\big), \tag{52}$$

$$\check{\chi}^l\big(r_{j,z}^{i,p}\big) = \chi^{l,i,z-1} + \pi_{j,z}^{i,p}\big(\chi_{j,z}^{l,i,z} - \chi_{j,z}^{l,i,\,z-1}\big), \tag{53}$$

$$\check{\lambda}^{q,l}\big(r_{j,z}^{i,p}\big) = \lambda^{q,l,i,z-1} + \pi_{j,z}^{i,p}\big(\lambda^{q,l,i,z} - \lambda^{q,l,i,\,z-1}\big), \tag{54}$$

where $\pi_{j,z}^{i,p} = N^i(r_{j,z}^{i,p} - s_{z-1}^i)$.

By substituting (49) into (43), we obtain

$$\begin{aligned}\tilde{x}_j^{i,p} = \tilde{x}_j^{i,0} &+ \beta_j^i \sum_{z=1}^{p-1} \kappa_{j,z}^p \vartheta_i^{\alpha_j} f_j^i\big(\check{x}\big(r_{j,z}^{i,p}\big), \zeta^*\big) \\ &+ \beta_j^i \vartheta_i^{\alpha_j} f_j^i\big(\tilde{x}^{i,\,p-1} + \pi_{j,p}^{i,p}\big(\tilde{x}^{i,p} - \tilde{x}^{i,\,p-1}\big),\, \zeta^*\big).\end{aligned} \tag{55}$$

Obviously, (55) is implicit with respect to $\tilde{x}^{i,p}$. To obtain an explicit form of (55), the last term at the right-hand side is further expanded using Taylor's expansion and subsequently rearranged, resulting in

$$\big(E - \bar{A}^{i,p}\big)\tilde{x}^{i,p} = \bar{\alpha}^{i,p}, \tag{56}$$

where E is the $n_x \times n_x$ identity matrix, $\bar{A}^{i,p}$ is an $n_x \times n_x$ matrix whose jth row $\bar{A}_j^{i,p}$ is defined by

$$\bar{A}_j^{i,p} = \beta_j^i \vartheta_i^{\alpha_j} \pi_{j,p}^{i,p} \frac{\partial f_j^i}{\partial x}\bigg|_{(\tilde{x}^{i,p-1},\zeta^*)},$$

and $\bar{\alpha}^{i,p}$ is an n_x-dimensional column vector whose jth element $\bar{\alpha}_j^{i,p}$ is defined by

$$\bar{\alpha}_j^{i,p} = \tilde{x}_j^{i,0} + \beta_j^i \sum_{z=1}^{p-1} \kappa_{j,z}^p \vartheta_i^{\alpha_j} f_j^i\big(\check{x}(r_{j,z}^{i,p}), \zeta^*\big) + \beta_j^i \vartheta_i^{\alpha_j} f_j^i\big(\tilde{x}^{i,\,p-1}, \zeta^*\big) - \bar{A}_j^{i,p} \tilde{x}^{i,\,p-1}.$$

By solving (56), we can obtain $\tilde{x}^{i,p}$. Furthermore, all $\tilde{x}(r_{j,z}^{i,p})$ can also be computed by (49). Then substituting (50) into (44) gives

$$\tilde{\phi}_j^{q,i,p} = \tilde{\phi}_j^{q,i,0} + \beta_j^i \sum_{z=1}^{p-1} \kappa_{j,z}^p g_j^i\big(\check{x}\big(r_{j,z}^{i,p}\big), \check{\phi}^q\big(r_{j,z}^{i,p}\big), \zeta^*\big) + \beta_j^i g_j^i\big(\check{x}\big(r_{j,p}^{i,p}\big), \tilde{\phi}^{q,i,p-1} + \pi_{j,p}^{i,p}\big(\tilde{\phi}^{q,i,p} - \tilde{\phi}^{q,i,p-1}\big), \zeta^*\big). \tag{57}$$

To obtain an explicit form of (57), we further expand the last term at the right-hand side around $\tilde{\phi}^{q,i,p-1}$, which results in

$$\big(E - \bar{B}^{q,i,p}\big)\tilde{\phi}^{q,i,p} = \bar{\beta}^{q,i,p}, \tag{58}$$

where $\bar{B}^{q,i,p}$ is an $n_x \times n_x$ matrix whose jth row $\bar{B}_j^{q,i,p}$ is given by

$$\bar{B}_j^{q,i,p} = \beta_j^i \pi_{j,p}^{i,p} \frac{\partial g_j^i}{\partial \phi}\bigg|_{(\check{x}(r_{j,z}^{i,p}),\tilde{\phi}^{q,i,p-1},\zeta^*)},$$

and $\bar{\beta}^{q,i,p}$ is an n_x-dimensional column vector, whose jth element $\bar{\beta}_j^{i,p}$ is given by

$$\bar{\beta}_j^{q,i,p} = \tilde{\phi}_j^{i,0} + \beta_j^i \sum_{z=1}^{p-1} \kappa_{j,z}^p g_j^{q,i}\big(\check{x}\big(r_{j,z}^{i,p}\big), \check{\phi}^q\big(r_{j,z}^{i,p}\big), \zeta^*\big) + \beta_j^i g_j^{q,i}\big(\check{x}\big(r_{j,p}^{i,p}\big), \tilde{\phi}^{q,i,p-1}, \zeta^*\big) - \bar{B}_j^{i,p} \tilde{\phi}^{q,i,p-1}.$$

Similarly, substituting Eqs. (51)–(54) into (45)–(48) and expanding the last term yield

$$\big(E - \bar{C}^{\iota,i,p}\big)\upsilon^{\iota,i,p} = \bar{\gamma}^{\iota,i,p}, \tag{59}$$

$$\big(E - \bar{D}^{q,\iota,i,p}\big)\rho^{q,\iota,i,p} = \bar{\xi}^{q,\iota,i,p}, \tag{60}$$

$$\big(E - \bar{F}^{l,i,p}\big)\chi^{l,i,p} = \bar{\eta}^{l,i,p}, \tag{61}$$

$$\big(E - \bar{G}^{q,l,i,p}\big)\lambda^{q,l,i,p} = \bar{\vartheta}^{q,l,i,p}, \tag{62}$$

where $\bar{C}^{\iota,i,p}$, $\bar{D}^{q,\iota,i,p}$, $\bar{F}^{l,i,p}$, and $\bar{G}^{q,l,i,p}$ are all $n_x \times n_x$ matrices, whose jth rows are respectively denoted by $\bar{C}_j^{\iota,i,p}$, $\bar{D}_j^{q,\iota,i,p}$, $\bar{F}_j^{l,i,p}$, and $\bar{G}_j^{q,l,i,p}$. Furthermore, $\bar{\gamma}^{\iota,i,p}$, $\bar{\xi}^{q,\iota,i,p}$,

$\bar{\eta}^{l,i,p}$, and $\bar{\vartheta}^{q,l,i,p}$ are n_x-dimensional column vectors, whose jth elements are respectively denoted by $\bar{\gamma}_j^{\iota,i,p}$, $\bar{\zeta}_j^{q,\iota,i,p}$, $\bar{\eta}_j^{l,i,p}$, and $\bar{\vartheta}_j^{q,l,i,p}$. These notations are defined by

$$
\begin{aligned}
\bar{C}_j^{\iota,i,p} &= \beta_j^i \pi_{j,p}^{i,p} \frac{\partial \tilde{h}_j^i}{\partial \upsilon}\bigg|_{(\check{x}(r_{j,p}^{i,p}),\, \upsilon^{\iota,i,p-1},\, \zeta^*)}, \\
\bar{D}_j^{q,\iota,i,p} &= \beta_j^i \pi_{j,p}^{i,p} \frac{\partial u_j^i}{\partial \rho}\bigg|_{(\check{x}(r_{j,p}^{i,p}),\, \check{\phi}^q(r_{j,p}^{i,p}),\, \check{\upsilon}^\iota(r_{j,p}^{i,p}),\, \rho^{q,\iota,i,p-1},\, \zeta^*)}, \\
\bar{F}_j^{l,i,p} &= \beta_j^i \pi_{j,p}^{i,p} \frac{\partial v_j^i}{\partial \chi}\bigg|_{(\check{x}(r_{j,p}^{i,p}),\, \chi^{\iota,i,p-1},\, \zeta^*)}, \\
\bar{G}_j^{q,l,i,p} &= \beta_j^i \pi_{j,p}^{i,p} \frac{\partial w_j^i}{\partial \lambda}\bigg|_{(\check{x}(r_{j,p}^{i,p}),\, \check{\chi}^l(r_{j,p}^{i,p}),\, \lambda^{q,l,i,p-1},\, \zeta^*)}, \\
\bar{\gamma}_j^{\iota,i,p} &= \upsilon_j^{\iota,i,0} + \beta_j^i \sum_{z=1}^{p-1} \kappa_{j,z}^p \tilde{h}_j^i\big(\check{x}\big(r_{j,z}^{i,p}\big), \check{\upsilon}\big(r_{j,z}^{i,p}\big), \zeta^*\big) + \beta_j^i \tilde{h}_j^i\big(\check{x}\big(r_{j,p}^{i,p}\big), \upsilon^{\iota,i,p-1}, \zeta^*\big) \\
&\quad - \bar{C}_j^{\iota,i,p} \upsilon^{\iota,i,p-1}, \\
\bar{\xi}_j^{q,\iota,i,p} &= \rho_j^{q,\iota,i,0} + \beta_j^i \sum_{z=1}^{p-1} \kappa_{j,z}^p u_j^i\big(\check{x}\big(r_{j,z}^{i,p}\big), \check{\phi}^q\big(r_{j,z}^{i,p}\big), \check{\upsilon}\big(r_{j,z}^{i,p}\big), \check{\rho}^{q,\iota}\big(r_{j,z}^{i,p}\big), \zeta^*\big) \\
&\quad + \beta_j^i u_j^i\big(\check{x}\big(r_{j,p}^{i,p}\big), \check{\phi}^q\big(r_{j,p}^{i,p}\big), \check{\upsilon}\big(r_{j,p}^{i,p}\big), \rho^{q,\iota,i,p-1}, \zeta^*\big) - \bar{D}_j^{q,\iota,i,p} \rho^{q,\iota,i,p-1}, \\
\bar{\eta}_j^{l,i,p} &= \chi_j^{l,i,0} + \beta_j^i \sum_{z=1}^{p-1} \kappa_{j,z}^p v_j^i\big(\check{x}\big(r_{j,z}^{i,p}\big), \check{\chi}^l\big(r_{j,z}^{i,p}\big), \zeta^*\big) + \beta_j^i v_j^i\big(\check{x}\big(r_{j,p}^{i,p}\big), \chi^{l,i,p-1}, \zeta^*\big) \\
&\quad - \bar{F}_j^{l,i,p} \chi^{l,i,p-1}, \\
\bar{\vartheta}_j^{q,l,i,p} &= \lambda_j^{q,l,i,0} + \beta_j^i \sum_{z=1}^{p-1} \kappa_{j,z}^p w_j^i\big(\check{x}\big(r_{j,z}^{i,p}\big), \check{\chi}^l\big(r_{j,z}^{i,p}\big), \check{\lambda}^{q,l}\big(r_{j,z}^{i,p}\big), \zeta^*\big) \\
&\quad + \beta_j^i w_j^i\big(\check{x}\big(r_{j,p}^{i,p}\big), \check{\chi}^l\big(r_{j,p}^{i,p}\big), \lambda^{q,l,i,p-1}, \zeta^*\big) - \bar{G}_j^{q,l,i,p} \lambda^{q,l,i,p-1}.
\end{aligned}
$$

Furthermore, let

$$
B_i^- := \frac{\partial h_j^i}{\partial \tilde{x}^-}\bigg|_{(\tilde{x}^{i-1,N^{i-1}},\sigma)}, \qquad B_i^{\sigma_l} := \frac{\partial h_j^i}{\partial \sigma_l}\bigg|_{(\tilde{x}^{i-1,N^{i-1}},\sigma)},
$$
$$
H^{i,j} := \frac{\partial^2 h_j^i}{\partial (\tilde{x}^-)^2}\bigg|_{(\tilde{x}^{i-1,N^{i-1}},\sigma)}.
$$

Then the initial and impulsive conditions (17), (19), (24), (26), (28), and (30) become

$$
\tilde{x}^{i,0} = \begin{cases} \varphi(\zeta^*) & \text{if } i = 0, \\ \tilde{x}^{i-1,N^{i-1}} + h^i(\tilde{x}^{i-1,N^{i-1}}, \sigma) & \text{if } i \in I_{m-1}, \end{cases} \tag{63}
$$

$$\tilde{\phi}^{q,i,0} = \begin{cases} \frac{\partial \varphi(\zeta)}{\partial \zeta_q}\big|_{\zeta=\zeta^*} & \text{if } i = 0, \\ \tilde{\phi}^{q,i-1,N^{i-1}} + B_i^- \tilde{\phi}^{q,i-1,N^{i-1}} & \text{if } i \in I_{m-1}, \end{cases} \tag{64}$$

$$\upsilon^{\iota,i,0} = \begin{cases} 0 & \text{if } i \in I_{\iota-1}, \\ \upsilon^{\iota,i-1,N^{i-1}} + B_i^- \upsilon^{\iota,i-1,N^{i-1}} & \text{if } i \in \{\iota, \dots, m-1\}, \end{cases} \tag{65}$$

$$\rho_j^{q,\iota,i,0} = \begin{cases} 0, & \text{if } i \in I_{\iota-1}, \\ \rho_j^{q,\iota,i-1,N^{i-1}} & \\ \quad + \left(\upsilon^{\iota,i-1,N^{i-1}}\right)^\top H^{i,j} \tilde{\phi}^{q,i-1,N^{i-1}} & \\ \quad + B_i^- \rho^{q,\iota,i-1,N^{i-1}} & \text{if } i \in \{\iota, \dots, m-1\}, \end{cases} \tag{66}$$

$$\chi^{l,i,0} = \begin{cases} 0 & \text{if } i = 0, \\ \chi^{l,i-1,N^{i-1}} + B_i^- \chi^{l,i-1,N^{i-1}} + B_i^{\sigma_l} & \text{if } i \in I_{m-1}, \end{cases} \tag{67}$$

$$\lambda_j^{q,l,i,0} = \begin{cases} 0 & \text{if } i = 0, \\ \lambda_j^{q,l,i-1,N^{i-1}} + \left(\chi^{l,i-1,N^{i-1}}\right)^\top H^{i,j} \tilde{\phi}^{q,i-1,N^{i-1}} & \\ \quad + B_i^- \lambda^{q,l,i-1,N^{i-1}} & \text{if } i \in I_{m-1}. \end{cases} \tag{68}$$

Thus, we can obtain the solutions of the FISSs (B)–(G) by solving the linear equations (56), (58), (59)–(62) with conditions (63)–(68). In addition, we can show that the above numerical solution method exhibits second-order convergence, with a proof similar to that of Theorem 5.1 in [5].

5.2 Optimization algorithm

Based on the numerical technique tailored for FISSs, a gradient-based optimization algorithm for solving Problem Q3 is outlined as Algorithm 1. It should be noted that, in Algorithm 1, Steps 5–7 can be implemented by classical gradient-based optimization techniques, e.g., sequential quadratic programming method [17].

Algorithm 1. Optimization algorithm for solving Problem Q3

Step 1. Initialize the iteration counter $\varpi = 1$ and the decision vector $(\vartheta_\varpi, \sigma_\varpi) \in \Theta \times \Sigma$.

Step 2. Solve the FISS (B) and auxiliary FISSs (C)–(G) using the numerical technique described above. This yields the variables $\tilde{x}(\cdot \mid \zeta^*, \vartheta_\varpi, \sigma_\varpi)$, $\tilde{\phi}^q(\cdot \mid \zeta^*, \vartheta_\varpi, \sigma_\varpi)$, $\upsilon^\iota(\cdot \mid \zeta^*, \vartheta_\varpi, \sigma_\varpi)$, $\rho^{q,\iota}(\cdot \mid \zeta^*, \vartheta_\varpi, \sigma_\varpi)$, $\mu^l(\cdot \mid \zeta^*, \vartheta_\varpi, \sigma_\varpi)$, and $\lambda^{q,l}(\cdot \mid \zeta^*, \vartheta_\varpi, \sigma_\varpi)$ for each $q \in I_{n_p}$, $\iota \in I_m$, and $l \in I_{n_c}$.

Step 3. Compute the cost function $\tilde{J}^\gamma(\vartheta_\varpi, \sigma_\varpi \mid \zeta^*)$ as defined in (20) using $\tilde{x}(\cdot \mid \zeta^*, \vartheta_\varpi, \sigma_\varpi)$ and $\tilde{\phi}^q(\cdot \mid \zeta^*, \vartheta_\varpi, \sigma_\varpi)$.

Step 4. Compute the gradients $\partial \tilde{J}^\gamma(\vartheta_\varpi, \sigma_\varpi \mid \zeta^*)/\partial \vartheta_\iota$ and $\partial \tilde{J}^\gamma(\vartheta_\varpi, \sigma_\varpi \mid \zeta^*)/\partial \sigma_l$ using $\tilde{x}(\cdot \mid \zeta^*, \vartheta_\varpi, \sigma_\varpi)$, $\tilde{\phi}^q(\cdot \mid \zeta^*, \vartheta_\varpi, \sigma_\varpi)$, $\upsilon^\iota(\cdot \mid \zeta^*, \vartheta_\varpi, \sigma_\varpi)$, $\rho^{q,\iota}(\cdot \mid \zeta^*, \vartheta_\varpi, \sigma_\varpi)$, $\mu^l(\cdot \mid \zeta^*, \vartheta_\varpi, \sigma_\varpi)$, and $\lambda^{q,l}(\cdot \mid \zeta^*, \vartheta_\varpi, \sigma_\varpi)$, as specified in Theorem 4, for each $\iota \in I_m$ and $l \in I_{n_c}$.

Step 5. If $(\vartheta_\varpi, \sigma_\varpi)$ is a locally optimal solution, then output $(\vartheta_\varpi, \sigma_\varpi)$ and terminate the algorithm. Otherwise, proceed to Step 6.

Step 6. Determine a new search direction and update the decision vector $(\vartheta_{\varpi+1}, \sigma_{\varpi+1})$ using a line search strategy.

Step 7. Increment the iteration counter: $\varpi \leftarrow \varpi + 1$. Then return to Step 2.

6 Numerical examples

Two numerical examples are solved by using Algorithm 1. Example 1 is a synthetic numerical case, while Example 2 addresses a practical application in shrimp harvesting. All computations have been performed under MATLAB environment on a PC with a 12th Gen GHz Intel Core i7-12700 CPU and 32.0 GB RAM.

Example 1. Consider the following dynamic optimization of a FISS:

$$\text{Minimize} \quad J(t_1,\sigma) = 10^6 x_1^2(2) + 10^6 x_2^2(2) \tag{69}$$

subject to fractional switched system

$$\begin{aligned} {}_0^C\mathrm{D}_t^{\alpha_1} x_1(t) &= x_1(t) + 0.5\sin\bigl(x_2(t)\bigr), \\ {}_0^C\mathrm{D}_t^{\alpha_2} x_2(t) &= -0.5\cos\bigl(x_1(t)\bigr) - x_2(t), \end{aligned} \qquad t \in (0, t_1),$$

and

$$\begin{aligned} {}_{t_1}^C\mathrm{D}_t^{\alpha_1} x_1(t) &= 0.3\sin\bigl(x_1(t)\bigr) + 0.5x_2(t), \\ {}_{t_1}^C\mathrm{D}_t^{\alpha_2} x_2(t) &= -0.5x_1(t) + 0.3\cos\bigl(x_2(t)\bigr), \end{aligned} \qquad t \in (t_1, 2],$$

with conditions

$$x(0) = (1+\zeta,\, 3)^\top, \qquad x(t_1^+) = \bigl(x(t_1^-) + \sigma_1, x_2(t_1^-) + \sigma_2\bigr)^\top.$$

Here ζ is the uncertain parameter, t_1 denotes the switching time, $\sigma = (\sigma_1, \sigma_2)^\top$ is the impulsive vector, whose components are the impulsive amplitudes with $-5 \leqslant \sigma_1, \sigma_2 \leqslant 5$.

Let the nominal value $\zeta^* = 0$. Thus, the robust cost function is as follows:

$$\bar{J}^\gamma(t_1, \sigma \,|\, \zeta^*) = J(t_1, \sigma \,|\, \zeta^*) + \gamma\left(\frac{\partial J(t_1, \sigma|\zeta^*)}{\partial \zeta}\right)^2, \tag{70}$$

where $\gamma \geqslant 0$ is a weight.

Using the proposed algorithm and three sets of fractional orders, we solve Example 1 for the weights $\gamma = 0$ and $\gamma = 9 \cdot 10^{-7}$. The numerical results are presented in Table 1.

Table 1. Numerical results for Example 1.

	$\alpha = (1.0, 1.0)^\top$	$\alpha = (0.95, 0.95)^\top$	$\alpha = (0.95, 0.90)^\top$
		$\gamma = 0$	
J	6.3287E−09	4.8341E−10	2.7684E−10
$[\partial J/\partial\zeta]^2$	0.4900	0.3411	0.4512
t_1^*	1.3546	1.1493	0.7561
σ^*	$(-4.9878, -1.1241)^\top$	$(-4.1806, -1.4108)^\top$	$(-2.5977, -1.8176)^\top$
CPU [s]	5.7120	6.8260	7.1610
		$\gamma = 9\text{E}{-}07$	
J	8.2379E−09	1.4915E−09	3.5768E−09
$[\partial J/\partial\zeta]^2$	2.3439E−04	2.1050E−04	3.9808E−04
t_1^*	0.2273	0.6853	0.0126
σ^*	$(-1.1294, -2.8018)^\top$	$(-2.3529, -1.8975)^\top$	$(-0.8089, -3.4274)^\top$
CPU [s]	5.6209	6.0120	9.2289

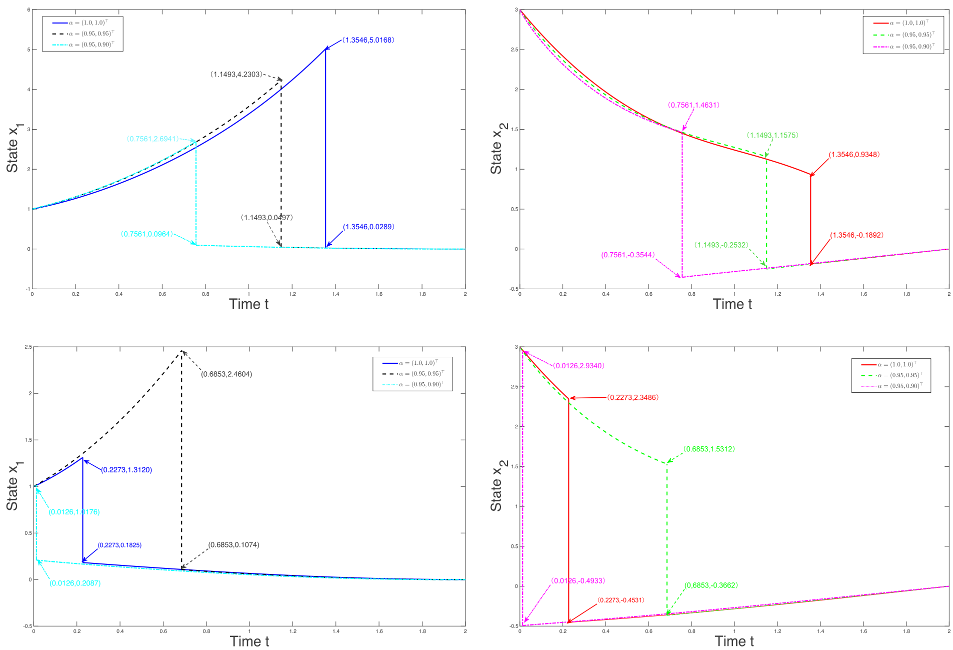

Figure 1. Optimal states and controls for Example 1.

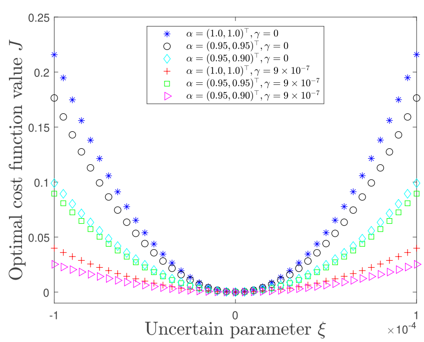

Figure 2. The system costs for different fractional orders α and weights γ in Example 1.

From Table 1 it is clear that the sensitivity of the cost function (69) to the uncertain parameter ζ are significantly reduced for the weight $\gamma = 9 \cdot 10^{-7}$, compared with the case $\gamma = 0$ in (70). Therefore, as shown in Table 1, the robust optimal impulsive control problem with $\gamma = 9 \cdot 10^{-7}$ enhances the robustness of the computed optimal control strategies. Furthermore, the optimal states corresponding to the optimal impulsive control strategies are illustrated in Fig. 1.

To demonstrate the robustness of the optimal impulsive control strategies listed in Table 1, we evaluate the corresponding cost function values under various disturbances of the nominal parameter ζ^*. Figure 2 shows the variations in the cost function due to these disturbances. As shown in Fig. 2, the optimal impulsive control strategies obtained with $\gamma = 9 \times \cdot 10^{-7}$ exhibit strong robustness against parameter uncertainties.

Example 2. Consider the following shrimp harvesting problem in [19]:

$$\text{Maximize} \quad J(\nu, \sigma) = \sum_{i=1}^{3} \left[0.008\sigma_i x_1(t_i^-) x_2(t_i^-) - 50\right] \tag{71}$$

subject to FISS

$$\begin{aligned} {}_{t_{i-1}}^{\ C}\mathrm{D}_t^{\alpha_1} x_1(t) &= -0.03 x_1(t), \\ {}_{t_{i-1}}^{\ C}\mathrm{D}_t^{\alpha_2} x_2(t) &= 3.5 - 0.00001 x_1(t) x_2(t) \end{aligned} \tag{72}$$

$t \in (t_{i-1}, t_i)$, $i \in I_3$, with conditions

$$x(t_i^+) = \begin{cases} (40000, 1 + \zeta)^\top & \text{if } i = 0, \\ (x_1(t_i^-) - \sigma_i x_1(t_i^-), x_2(t_i^-))^\top & \text{if } i \in I_2. \end{cases} \tag{73}$$

Here t denotes the time in weeks, $x_1(t)$ and $x_2(t)$ represent the number of the shrimp and the average weight of a shrimp. The initial and final times are $t_0 = 0$ and $t_3 = 13.2$, respectively. The vector of harvesting time is given by $\nu = (t_1, t_2)^\top$, and the corresponding vector of harvesting fractions is $\sigma = (\sigma_1, \sigma_2)^\top$ with $\sigma_3 = 1$. In addition, the switching times and harvesting fractions are subject to the following bound constraints [5]:

$$t_i - t_{i-1} \geqslant 0.001, \quad i \in I_3, \tag{74}$$

and

$$0.01 \leqslant \sigma_i \leqslant 1, \quad i \in I_2. \tag{75}$$

Assume that the nominal value of uncertain parameter is $\zeta^* = 0$ [5]. Then the robust optimal impulsive control problem is to choose (ν, σ) subject to fractional impulsive system (72)–(73) and constraints (74), (75) in order to minimize the following cost function:

$$\bar{J}^\gamma(\nu, \sigma | \zeta^*) = -J(\nu, \sigma | \zeta^*) + \gamma \left(\frac{\partial J(\nu, \sigma | \zeta^*)}{\partial \zeta} \right)^2,$$

where $\gamma \geqslant 0$ is a weighting coefficient.

We solve Example 2 for three different sets of fractional orders and two weights by using our proposed algorithm. The computed numerical results are listed in Table 2. As observed from Table 2, all the sensitivities of objective function (71) for $\gamma = 5$ are significantly reduced compared to those for $\gamma = 0$ although the values of the objective function are decreased slightly. Notably, the computed optimal objective value $3.1889 \cdot 10^3$ for $\alpha = (1.0, 1.0)^\top$ and $\gamma = 0$ is consistent with the value reported in [5]. Furthermore, the optimal states corresponding the optimal impulsive control strategies in Table 2 are plotted in Fig. 3.

Table 2. Numerical results for Example 2.

	$\alpha = (1.0, 1.0)^\top$	$\alpha = (0.99, 0.99)^\top$	$\alpha = (0.90, 0.99)^\top$
		$\gamma = 0$	
J	3.1889E+03	3.1742E+03	3.2369E+03
$[\partial J/\partial \zeta]^2$	1.0987E+03	1.1575E+03	1.1602E+03
ν^*	$(4.2551, 7.7972)^\top$	$(4.2806, 7.8238)^\top$	$(4.3058, 7.8003)^\top$
σ^*	$(0.3874, 0.4539)^\top$	$(0.3853, 0.4527)^\top$	$(0.3857, 0.4501)^\top$
CPU [s]	16.3010	18.2030	18.6899
		$\gamma = 5$	
J	2.8559E+03	2.8160E+03	2.8816E+03
$[\partial J/\partial \zeta]^2$	39.2719	46.2644	45.9046
ν^*	$(9.2702, 10.9972)^\top$	$(9.1762, 10.9342)^\top$	$(9.3938, 11.0539)^\top$
σ^*	$(0.2851, 0.3219)^\top$	$(0.1982, 0.3439)^\top$	$(0.2440, 0.3299)^\top$
CPU [s]	16.5199	20.5050	38.0319

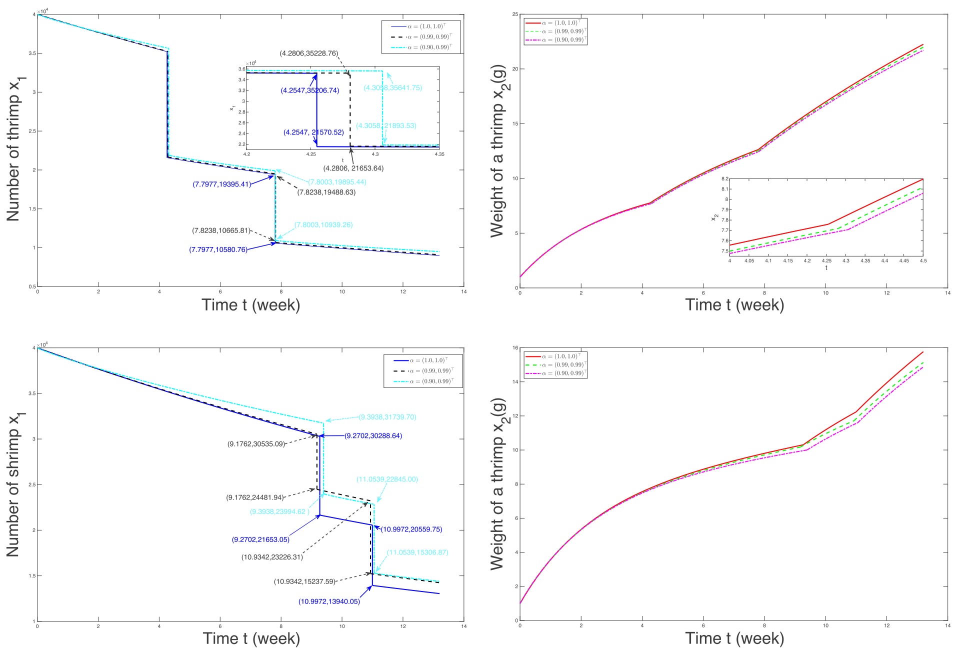

Figure 3. Optimal states and controls for Example 2.

We further evaluate the robustness of the optimal impulsive control strategies in Table 2 by introducing the various disturbances to the nominal parameter ζ^*. The resulting variations in the objective function value are illustrated in Fig. 4. From Fig. 4 it can be seen that the changes of the objective function values for $\gamma = 5$ are smaller than those for $\gamma = 0$. This indicates that the obtained optimal impulsive control strategies for $\gamma = 5$ exhibit greater robustness against parameter disturbances.

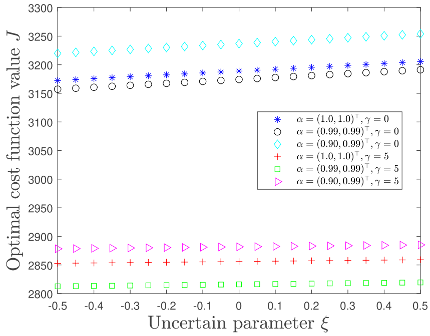

Figure 4. The system costs for different fractional orders α and weights γ in Example 2.

7 Conclusions

In this paper, we proposed a computational technique for robust dynamic optimization of FISS subject to parameter uncertainties. The key contributions of this study are: (i) a sensitivity computational method tailored to FISS that explicitly quantifies the impact of the parameter uncertainty; (ii) a time-scaling transformation approach to effectively handle variable switching instants; (iii) the derivation of gradient formulas required for efficient optimization; and (iv) the development of a gradient-based optimization algorithm integrated with a customized numerical scheme for solving the FISSs. The effectiveness of the proposed method were demonstrated through two numerical examples, which showed significant improvements in control sensitivity and overall system performance.

Conflicts of interest. The authors declare no conflicts of interest.

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